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SIMPLIFIED PRODUCTION-INVENTORY CONTROL CHART

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This paper is concerned with a problem in which a firm is engaged in manufacturing a number of types and sizes of goods at equal intervals, e. g. monthly, and the production is forced to fulfill the customer's demand immediately.

In general the products are subject to considerable variations in demands. Let us assume that the amount demanded is uncertain but has a certain probability distribution.

If inventory remains after the orders have been filled in any period, it is carried over into the next period. But, in case where the order can not be delivered from available inventory, the difference should not be filled in by the production of the next period because of the needs of immediate delivery.

In this situation, it may be appropriate to consider only the shortage cost, and neglect the other costs. The shortage cost, however, may be obscure in most cases, then we discuss the probability which the shortage occurs.

The model used in this paper is a production process with a lag time. We define a lag time as the time between the issuance of production order and the arrival of finished goods into stock. We assume that the lag time is equal to the production period. Then the production schedules are to be made for the future production periods.

The list of symbols used in this paper is:

$F(q)$ = the probability distribution function for the sales demand in each period.

q_k = the actual demand in the k-th period.

x_k = the initial stock at the beginning of the k-th period.

y_k = the size of the scheduled production at the beginning of the k-

th period, which arrives at the beginning of the $(k+1)$-th period.

s_k = the starting stock at the beginning of the k-th period, which is the sum of x_k and y_{k-1}.

r_k = the difference between s_k and q_k.

If the actual demand q_k is less than the starting stock s_k, then there must be a stock r_k available at the end of this period, and this stock $r_k = s_k - q_k$ is equal to x_{k+1}. On the other hand, if q_k is greater than s_k, then the shortage must occur.

The larger the starting stock s_k is, however, the smaller the risk of the shortage becomes, and vice versa. The occurrence of the shortage is dependent on the actual demand q_k and the starting stock s_k in this period. Hence the shortage is dependent on the initial stock in the planning period and the scheduled production in the preceeding period. The initial stock x_k, however, is given and not within the control.

Then the production should be planned for the shortage of the next period. Let the starting stock and the scheduled production be s_k and y_k in the k-th period respectively, and the actual demands of the successive two periods be q and q' respectively. (See Fig. 1.)

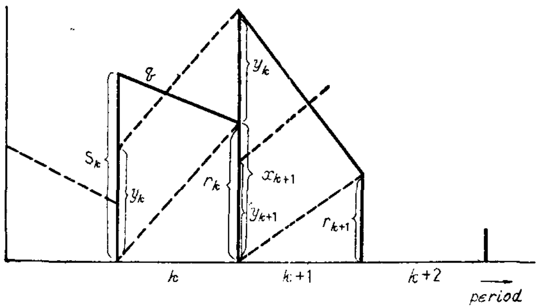

Fig. 1.

If $s_k > q$, then the initial stock x_{k+1} of the $(k+1)$-th period is $s_k - q$, and the starting stock s_{k+1} is equal to $s_k - q + y_k$. Furthermore, if $s_k - q + y_k \geq q'$, then there must be the available stock at the end of the $(k+1)$-

th period. On the other hand, if $s_k<q$, then x_{k+1} must vanish by the previous assumption of the immediate delivery. In this case, the starting stock of the $(k+1)$-th period is y_k only. Thus, according as $y_k>q'$ or $y_k<q'$, the surplus or the shortage will occur respectively.

Let $P(s,y)$ be the probability that the shortage occurs at the end of the next period when in the planning period the starting stock and the scheduled production are s and y respectively. Then,

$$P(s,y)=\int_0^s dF(q)\int_{s+y-q}^{\infty} dF(q')+\int_s^{\infty} dF(q)\int_y^{\infty} dF(q'). \tag{1}$$

(See Fig. 2.)

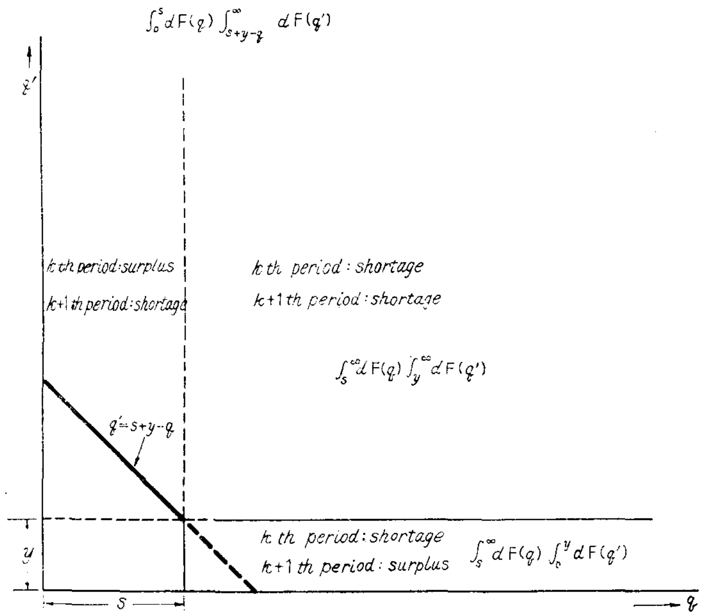

Fig. 2.

The policy by which a production schedule is to be made is to reduce the probability for the occurrence of the shortage to a certain small value, e.g. 5 per cent. On the basis of this criterion, we can construct the following rule.

Assume that there is a stock value s for which

$$P(s,0)=\int_{0}^{s}dF(q)\int_{s-q}^{\infty}dF(q')+\int_{s}^{\infty}dF(q')\leq 0.05. \qquad (2)$$

This relation indicates that the scheduled production value in eq. (1) is equal to zero. Thus, if we start with such a stock value, then there is no need to produce in the planning period.

We can next normalize the mean value of the demand distribution $F(q)$, using the proper unit. Thus ws set y equal to unity in eq. (1). If there is a value s for which

$$P(s,1)=\int_{0}^{s}dF(q)\int_{s+1-q}^{\infty}dF(q')+\int_{s}^{\infty}dF(q)\int_{1}^{\infty}dF(q')\leq 0.05, \qquad (3)$$

then it is sufficient to produce one unit in the planning period.

Solving eqs. (2) and (3) under the equality sign, we obtain the solutions $s=\bar{s}$ and $s=s'$ respectively. Fig. 3 shows a simplified production inventory control chart. The ordinate represents the starting stock s and the abscissa the period number. The line $s=\bar{s}$ corresponds to the upper control limit. If the starting stock s_k is plotted above the line $s=\bar{s}$ in any period, then there must be no need for the production. If the starting stock falls between the lines $s=\bar{s}$ and $s=s'$, it is decided to produce one unit. If the starting stock falls below the line $s=s'$, it is necessary to produce more than one unit.

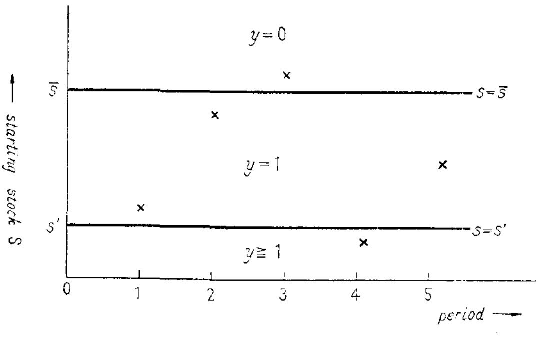

Fig. 3.

To determine the control limits in more detail, we should calculate 5 per cent points for the various values of y by using the equation

P(s, y) =0.05.

In the following example, normal distribution is assumed for the demand as usual. Assuming that the mean and the standard deviation are 1 and σ respectively, the expression (1) can be written as follows,

$$P(s,y)=\frac{1}{2\pi\sigma^2}\int_0^s dq\ exp\left\{-\frac{(q-1)^2}{2\sigma^2}\right\}\int_{s+y-q}^{\infty} dq'\ exp\left\{-\frac{(q'-1)^2}{2\sigma^2}\right\}$$
$$+\frac{1}{2\pi\sigma^2}\int_s^{\infty} dq\ exp\left\{-\frac{(q-1)^2}{2\sigma^2}\right\}\int_y^{\infty} dq'\ exp\left\{-\frac{(q'-1)^2}{2\sigma^2}\right\}, \tag{4}$$

where $\sigma \ll 1$.

Setting $s=1+h\sigma$ and $y=1+k\sigma$, we can write eq. (4) in the following form:

$$P(s,y)=p(h,k)=\frac{1}{\sqrt{2\pi}}\int_{-1/\sigma}^{h} exp\left\{-\frac{x^2}{2}\right\}\Phi(k+h-x)\,dx+\Phi(h)\Phi(k), \tag{5}$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_x^{\infty} exp\left(-\frac{x^2}{2}\right)dx. \tag{6}$$

We first assume that h is large in order to obtain the upper control limit. In this case, the upper and lower limits of the integral in eq. (5) can be taken to be infinity, and the second term becomes negligibly small. As a result, eq. (5) reduces to the following form:

$$p(h,k)\approx\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2}\right)\Phi(h+k-x)\,dx=\Phi\left(\frac{k+h}{\sqrt{2}}\right). \tag{7}$$

Thus we get the relation $\Phi\left(\frac{k+h}{\sqrt{2}}\right)\doteq 0.05$. Using the table of the normal distribution, it is seen that

$$k+h=2.33. \tag{8}$$

This relation is valid for $h>3.4$ and $\sigma<0.3$. To determine the upper control limit, we set $y=0$, *i.e.* $k=-1/\sigma$. Then we get $s=1+(2.33+1/\sigma)\sigma=2+2.33\sigma$. It is readily seen that eq. (8) gives $h=3.96$ and $h=3.50$ for $k=-1.63$ and $k=-1.17$ respectively.

Next we consider the case where $s=1$, *i.e.* $h=0$. Using eq. (5), it can be shown that

$$\frac{1}{2}\Phi(k)+\frac{1}{2}\Phi(k/\sqrt{2})+\frac{1}{2}\left\{1-\Phi(k/\sqrt{2})\right\}^2-\frac{1}{8}=0.05. \tag{9}$$

Using the table, we get $k=1.45$ for $h=0$.

Similary setting $y=1$, i.e. $k=0$, it is seen that

$$\frac{1}{2}\Phi(h/\sqrt{2})-\frac{1}{2}\left\{1-\Phi(h/\sqrt{2})\right\}^2+\frac{1}{8}=0.05. \tag{10}$$

We can solve this second order algebraic equation with respect to $\Phi(h/\sqrt{2})$. Thus,

$$\Phi(h/\sqrt{2})=0.45-\sqrt{19.4}.$$

Hence we get $h=2.31$ for $k=0$.

We can next consider the lower control limit. Let s be equal to zero, i.e. $h=-1/\sigma$. Then, in eq. (5), the first term vanishes, and the first factor of the second term becomes nearly equal to 1. Therefore, the result of eq. (5) is

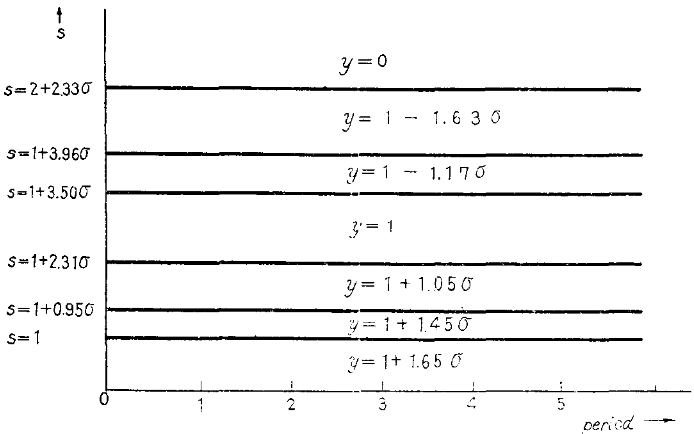

Fig. 4.

$$\Phi(k) = 0.05.$$

Thus we get $k=1.645$, *i.e.* $y=1+1.645\sigma$ for $s=0$. In general, it is difficult to determine analyticaly the other values of h and k, so that we need to carry out the numerical caluculation for eq. (5). For example, we obtain $h=0.95$ for $k=1.05$.

The results of the above calculations are shown in Fig. 4. When a plot for the starting stock s goes over the upper limit $s=2+2.33\sigma$, there is no need for the production. If a point falls between the lines $s=2+2.33\sigma$ and $s=1+3.96\sigma$, it is decided to produce $1-1.63\sigma$ units. Similarly, if a point falls between the line $s=1+3.96\sigma$ and $s=1+3.50\sigma$, it is decided to produce $1-1.17\sigma$ units, and so on.

The control device we have obtained is the result of only preliminary study. In most cases, however, we have a number of types and sizes of goods which have the same probability distribution for demand, and thus it is convenient to use the same sheet of this simplified production inventory control chart by using the proper unit for demand.