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THREE-PERSON COOPERATIVE GAMES

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SUMMARY

Economic positions of each players in some three-person cooperative team are variously described in game-theoretic terms. The notions of equilibrium points of Nash [2] and Farquharson [1] in the non-cooperative game theory are used.

INTRODUCTION

The chief problem in the theory of the general n-person game seems to be that of determining the proper definition of a solution for them. The cooperative theories in which the players are expected to form coalitions, developed by von Neumann-Morgenstern, Shapley and others, have the shortcoming that the notion of solution does not give much insight into how the games should be played.

The cooperative game discussed below is rather special. The players are supposed to be able to discuss the situation and agree on a rational joint plan of action, an agreement that should be assumed to be enforceable.

Consider now a three-person game, *i. e.*, a set of three players, each with an associated finite set of pure strategies, and with a payoff function, p_i, which maps the set of all n-tuples of pure strategies into the real numbers. The mixed strategies for each players are allowed and the payoff function p_i has a unique extension to the n-tuples of mixed strategies. This extension we shall also denote by p_i, writing $p_i(s_1, s_2, s_3)$. Negotiated cooperation for the three players will naturally be to choice s_1, s_2, and s_3 such that it results

$$M \equiv \max_{s_1, s_2, s_3} \sum_{i=1}^{3} p_i\ (s_1,\ s_2,\ s_3)$$

and to distribute this amount among the three players in accordance with their "economic positions" in the team of the players.

Our concept of a reasonable distribution is based on the following two principles.

(i) A rational player should never accept a final payment less than the amount he is certain he can obtain if the players do not form coalition and compete with each other.

(ii) The players select "threat" strategies independently which each will use if negotiated cooperation with the other players is not possible. If a player refuses his final payment his opponents play non-cooperatively choosing their threat strategies and thus he relatively inflict a heavier loss on himself than upon his opponents.

To illustrate the situation we shall first show the following theorem which is a restatement of a result of Raiffa [3] and Ville [4].

Theorem 1. *Let A and B be finite matrices with the same size. Consider the non-zero-sum two-person cooperative game with payoff matrices A and B to each players respectively. Then a "reasonable" distribution of payments fulfilling the above principles exists uniquely yielding to each player*

$$x_1 = \frac{1}{2}(M+\Delta), \qquad x_2 = \frac{1}{2}(M-\Delta), \tag{1}$$

where $M \equiv \max\ (A+B)$ *and* $\Delta \equiv \text{val.}\ (A-B)$.

Proof. Let $\| x_1, x_2 \|$ with $x_1+x_2=M$ be the final payment to be sought. The principle (ii) requires the existence of threat strategies $\xi^* \varepsilon S_m$ and $\eta^* \varepsilon S_n$ (S_m being the set of all m-dimensional probability vectors) such that

$$x_1 - \xi^* \cdot A\eta \leqq x_2 - \xi^* \cdot B\eta, \qquad \text{for all } \eta \varepsilon S_n$$

$$x_2 - \xi \cdot B\eta^* \leqq x_1 - \xi \cdot A\eta^*, \qquad \text{for all } \xi \varepsilon S_m,$$

that is,

$$\xi \cdot (A-B)\eta^* \leqq x_1 - x_2 \leqq \xi^* \cdot (A-B)\eta$$

for all $\xi \varepsilon S_m$ and $\eta \varepsilon S_n$. The min-max theorem of matrix games asserts the existence of such ξ^* and η^* and unique existence of such value $x_1 - x_2$.

Solving the equations $x_1 + x_2 = M$ and $x_1 - x_2 = \text{val.}\ (A-B) \equiv \Delta$ we have (1). We next show that this pair of numbers satisfies our principle (i). It is easily shown that the characteristic functions of our non-zero-sum game in the von Neumann-Morgenstern's sense are

$$v(O)=0, \quad v(\{1\}) = \text{val. } A, \quad v(\{2\}) = \text{val. } B^T, \quad v(\{1, 2\}) = M.$$

Due to the simple inequality val. $H \leqq$ val. K + max$(H-K)$ and the equality val. $(-A^T) = -$val. A for matrix games we have

$$\text{val. } A \leqq \text{val.} \frac{A-B}{2} + \max \frac{A+B}{2} = \frac{1}{2}(M+\Delta)$$

$$\text{val. } B^T \leqq \text{val.} \frac{B^T - A^T}{2} + \max \frac{A^T + B^T}{2} = \frac{1}{2}(M-\Delta).$$

Thus we get $x_i \geqq v(\{i\})$ $(i=1, 2)$. This completes the proof.

EQUILIBRIUM-POINTS OF ORDER 2 IN THREE-PERSON GAMES.

This section treats some non-cooperative theory.

Farquharson [1] defined an equilibrium-point (eq. pt.) of order r in an n-person game to be a point in strategy space (product space of each player's ones) at which no set of r players can improve their individual positions by any changes in their strategies. An eq. pt. of order 1 is clearly a Nash eq. pt. [2]. Two strategy points which are indifferent to all players are said to be equivalent. A strategy point s is said to be admissible if there exist no strategy point s' such that

$$p_i(s') \geqq p_i(s), \qquad i=1,\cdots,n,$$

where at least one strict inequality holds.

If there exist two or more admissible strategy points, the set, **A**, of all of them has the property that for any points u and v in **A** whenever player k prefers u to v some player j prefers v to u. We may, of course, consider only the strategy points in **A**, and so the game which has the above property.

Theorem 2. *In three-person games any two equilibrium points of order* 2 *in* **A** *are equivalent.*

Proof. Let the two eq. pt. of order 2 be s^* and t^*. We have by definition of s^*

$$p_i(s^*) \geqq p_i(s_1,\ s_2,\ s_3^*), \qquad i=1,\ 2$$
$$p_i(s^*) \geqq p_i(s_1^*,\ s_2,\ s_3), \qquad i=2,\ 3$$
$$p_i(s^*) \geqq p_i(s_1,\ s_2^*,\ s_3), \qquad i=3,\ 1$$

for any strategies s_1, s_2 and s_3. By admissibility we have

$$p_3(s^*) \leqq p_3(s_1,\ s_2,\ s_3^*)$$
$$p_1(s^*) \leqq p_1(s_1^*,\ s_2,\ s_3)$$
$$p_2(s^*) \leqq p_2(s_1,\ s_2^*,\ s_3)$$

Thus we obtain

$$p_1(s^*) = p_1(s_1^*,\ s_2,\ s_3^*) = p_1(s_1^*,\ s_2^*,\ s_3)$$
$$p_2(s^*) = p_2(s_1,\ s_2^*,\ s_3^*) = p_2(s_1^*,\ s_2^*,\ s_3)$$
$$p_3(s^*) = p_3(s_1^*,\ s_2,\ s_3^*) = p_3(s_1,\ s_2^*,\ s_3^*)$$

for all s_1, s_2 and s_3. From the first set of inequalities and the third set of equalities and those in which s and s^* are replaced by t and t^* respectively, we can easily derive the relations

$$p_i(u_1^*,\ u_2^*,\ u_3^*) = p_i(s^*) = p_i(t^*), \qquad i=1,\ 2,\ 3$$

where $\qquad u_j^* \equiv s_j^*$ or t_j^* $(j=1,\ 2,\ 3)$.

For any eq. pt. s^* of order 2, if exist, let $p_i(s^*) = v_i (i=1,\ 2,\ 3)$. The above theorem asserts the existence of the unique value v_i for each player. Since these values represent in some sense economic positions of the players, we may take

$$x_i = \frac{v_i M}{v_1+v_2+v_3}, \quad i=1,\ 2,\ 3 \tag{2}$$

provided $v_1+v_2+v_3 \neq 0$. The payment $\| x_1,\ x_2,\ x_3 \|$ satisfies our principle (i) if all v_i are positive but this, of course, does not fullfill the principle (ii) in any satisfactory interpretations.

ECONOMIC POSITIONS IN THE COOPERATIVE TEAM

Let the final payment to the players be $\| x_1,\ x_2,\ x_3 \|$ and we shall derive in the following the conditions by which the x_i 's reflect in some reasonable sense the economic positions of the players. If the player 1 refuses his payment x_1, then the players 2 and 3 may cooperate opposing the player 1. The corporation by the both players may admit the joint strategies over the product space of their pure-strategy spaces, and may even behave just like a single player. Thus the composite player (2, 3) uses a threat strategy (sometimes joint) s_{23}^* inflicting a heavier loss on the opponent than upon the team of them, *i. e.*,

$$x_1 - p_1(s_1,\ s_{23}^*) \geqq x_2 + x_3 - p_2(s_1,\ s_{23}^*) - p_3(s_1,\ s_{23}^*)$$

for all strategies s_1 of the player 1. Conversely if the composite player (2, 3) refuses their payment and leads the play non-cooperatively, the player 1 uses his threat strategy s_1^* yielding the result that

$$x_2 + x_3 - p_2(s_1^*,\ s_{23}) - p_3(s_1^*,\ s_{23}) \geqq x_1 - p_1(s_1^*,\ s_{23})$$

for all joint strategies s_{23} of his opponent. From the two inequalities above we have

$$x_1 - (x_2 + x_2) = v_{1\cdot 23} \equiv \min_{s_{23}} \max_{s_1} (p_1 - p_2 - p_3)\ (s_1,\ s_{23})$$

in which the existence of $v_{1\cdot 23}$ is assured by the well known min-max theorem in matrix games.

Similarly we have

$$x_2-x_3-x_1=v_{2\cdot 31}\equiv \min_{s_{31}} \max_{s_2} (p_2-p_3-p_1)\,(s_2,\ s_{31})$$

$$x_3-x_1-x_2=v_{3\cdot 12}\equiv \min_{s_{12}} \max_{s_3} (p_3-p_1-p_2)\,(s_{12},\ s_3).$$

Hence we get

$$\left.\begin{aligned} x_1&=-\frac{1}{2}(v_{2\cdot 31}+v_{3\cdot 12})=\frac{1}{2}(v_{1\cdot 23}+K)\\ x_2&=-\frac{1}{2}(v_{3\cdot 12}+v_{1\cdot 23})=\frac{1}{2}(v_{2\cdot 31}+K)\\ x_3&=-\frac{1}{2}(v_{1\cdot 23}+v_{2\cdot 31})=\frac{1}{2}(v_{3\cdot 12}+K)\end{aligned}\right\} \tag{3}$$

where we have set

$$K\equiv -(v_{1\cdot 23}+v_{2\cdot 31}+v_{3\cdot 12})=x_1+x_2+x_3.$$

This payment $\|x_1,\ x_2,\ x_3\|$ to the players satisfies our principle (ii) but whether the principle (i) is satisfied is not clear. And it is, moreover, doubtful whether this payment can be realizable, that is, whether the players are able to choice jointly their strategies to get the amount K in total, unless we can prove the relation

$$\min\ (p_1+p_2+p_3)\leqq K\leqq M\equiv \max\ (p_1+p_2+p_3).$$

Consider the special case where the third player can use only one pure strategy and the payoff to him is always the constant amount c. In this case we have essentially a two-person game and the payoffs to the players 1 and 2 are represented by the matrices A and B with the same size. We can easily show that

$$v_{1\cdot 23}=\Delta-c,\qquad v_{2\cdot 31}=-\Delta-c,\qquad v_{3\cdot 12}=-M+c$$

where

$$\Delta\equiv \mathrm{val}\,(A-B),\qquad M\equiv \max\,(A+B).$$

Hence we have

$$x_1=\frac{1}{2}(M+\Delta),\qquad x_2=\frac{1}{2}(M-\Delta),\qquad x_3=c$$

and

$$K=M+c=\max(p_1+p_2+p_3).$$

AN AMALGAMATION

We shall imagine the random formation of a coalition of all of the players, starting with a single member and adding one player at a time. All coalition formations are considered as equally likely. For the first player we have three cases, that is, (i) he is added to a coalition formed by the players 2 and 3, (ii) he first forms a coalition with the player 2 and then the team admits adherence of the player 3, and (iii) he first forms a coalition with the player 3 and then the team admits adherence of the player 2. By the Theorem 1 the player 1 gets in the first case

$$x_{1\cdot 23}=\frac{1}{2}(M+v_{1\cdot 23})$$

where $M=\max(p_1+p_2+p_3)$ and $v_{1\cdot 23}=\min_{s_{23}}\max_{s_1}(p_1-p_2-p_3)\ (s_1,\ s_{23})$, and in the latter two cases the teams $\{1,\ 2\}$ and $\{3,\ 1\}$ get

$$x_{12\cdot 3}=\frac{1}{2}(M-v_{3\cdot 12})\quad\text{and}\quad x_{31\cdot 2}=\frac{1}{2}(M-v_{2\cdot 31})$$

respectively.

Thus we may take as the amount promised to the player 1 the expected value

$$\begin{aligned} x_1&=\frac{1}{3}\left(x_{1\cdot 23}+\frac{v_1}{v_1+v_2}x_{12\cdot 3}+\frac{v_1}{v_1+v_3}x_{31\cdot 2}\right)\\ &=\frac{M}{6}\left(1+\frac{v_1}{v_1+v_2}+\frac{v_1}{v_1+v_3}\right)+\frac{1}{6}\left(v_{1\cdot 23}-\frac{v_1v_{3\cdot 12}}{v_1+v_2}-\frac{v_1v_{2\cdot 31}}{v_1+v_3}\right). \qquad (4)\end{aligned}$$

Similarly we obtain

$$x_2=\frac{M}{6}\left(1+\frac{v_2}{v_1+v_2}+\frac{v_2}{v_2+v_3}\right)+\frac{1}{6}\left(v_{2\cdot 31}-\frac{v_2v_{3\cdot 12}}{v_1+v_2}-\frac{v_2v_{1\cdot 23}}{v_2+v_3}\right),$$

$$x_3=\frac{M}{6}\left(1+\frac{v_3}{v_2+v_3}+\frac{v_3}{v_3+v_1}\right)+\frac{1}{6}\left(v_{3\cdot12}-\frac{v_3v_{1\cdot23}}{v_2+v_3}-\frac{v_3v_{2\cdot31}}{v_1+v_3}\right).$$

We have at once $x_1+x_2+x_3=M$. Hence this payment to the players is realizable and most profitable for the aggregation and satisfies our principle (ii) described in § 1, still remaining the question whether the principle (i) there is satisfied or not.

SIMPLE EXAMPLES

We shall show some simple examples which illustrate the concepts in the paper.

Ex. 1. We consider a three-person game in which each player shows one face of his coin. One who shows a different face to the faces of the other two players gets one dollar. If all players show the same face of their coins, nothing is paid. The mixed strategies of each players are represented by the numbers u, v and w in [0, 1]. The payoff functions are

$$p_1(u, v, w)=u(1-v)(1-w)+(1-u)vw=u(1-v-w)+vw,$$
$$p_2(u, v, w)=v(1-w-u)+wu,$$
$$p_3(u, v, w)=w(1-u-v)+uv.$$

Thus we easily find that the strategy point where $u=v=w=1/2$ is the unique Nash eq. pt., and that eq. points of order 2 do not exist. But we may in this case take v_i in (2) the Nash eq. value $1/4$ and we have $x_1=x_2=x_3=1/3$. And we have $v_{1\cdot23}=v_{2\cdot31}=v_{3\cdot12}=-1$ and $K\equiv-(v_{1\cdot23}+v_{2\cdot31}+v_{3\cdot12})=3\neq M=1$. Hence the payment (3) cannot be realizable, but our amalgamation concept leads again to the payment $x_1=x_2=x_3=1/3$ by (4).

Ex. 2. The first player has the roman letter strategies and the payoff in the first column, *etc.*

a	α	$\mathfrak{a}$	1	1	1
a	β	$\mathfrak{a}$	0	1	0

b	α	$\mathfrak{a}$	1	0	0
b	β	$\mathfrak{a}$	0	0	−1
a	α	$\mathfrak{b}$	1	1	1
a	β	$\mathfrak{b}$	1	0	0
b	α	$\mathfrak{b}$	0	1	0
b	β	$\mathfrak{b}$	0	0	−1

The payoff functions are as in the previous example,

$$p_1(u,\ v,\ w)=vw+u(1-w)$$

$$p_2(u,\ v,\ w)=uw+v(1-w)$$

$$p_3(u,\ v,\ w)=uv-(1-u)(1-v).$$

Thus we easily find that all strategy points where $u=v=1$ and w is arbitrary is Nash eq. points and, at the same time, eq. points of order 2 with the common value $v_1=v_2=v_3=1$. Since $M=3$ the payment (2) results $x_1=x_2=x_3=1$. Moreover we have $v_{1\cdot 23}=v_{2\cdot 31}=v_{3\cdot 12}=-1$ and $K=3=M$, and so the payment (3) is realizable yielding again $x_1=x_2=x_3=1$.

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