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A NOTE ON THE DETERMINATION OF ALL OPTIMAL SOLUTIONS IN LINEAR PROGRAMMING*

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There may be more than one optimal solution to a linear programming problem. A general means for locating all optimal solutions is known. [1] The author will provide an improved method.

DETERMINATION OF ALL OPTIMAL BASIC SOLUTIONS TO A GENERALIZED MATRIX PROBLEM

Let $P=[P_0, P_1, \cdots, P_n]$ be a given matrix whose j-th column, P_j, is a vector of $(m+1)$-components. Let M be a fixed matrix of rank $m+1$ consisting of $m+1$ l-components row vectors. The generalized matrix problem [2] is concerned with finding a matrix X satisfying

$$PX=P_0\bar{x}_0+\sum_1^n P_j\bar{x}_j=M, \qquad (1)$$

where $\bar{x}_j$ (the $(j+1)$-th row of X) is a row vector of l-components

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satisfying the conditions, in the lexicographic sense,

$$\bar{x}_j \geqq 0 \quad (j=1, 2, \cdots, n), \tag{2}$$

$$\bar{x}_0 = \max. \tag{3}$$

A basic solution [2] is one in which only $m+1$ variables (including $\bar{x}_0$) are considered in (1), the remainder being set equal to zero; that is, it is of the form

$$BV = P_0\bar{v}_0 + \sum_{i=1}^{m} P_{j_i}\bar{v}_i = M \qquad (\bar{v}_i > 0, \quad j_i \neq 0), \tag{4}$$

where $B=[P_0, P_{j_1}, \cdots, P_{j_m}]$ is a $(m+1)$-rowed square matrix of rank $m+1$ and V is a matrix of $m+1$ rows and l columns whose $(i+1)$-th row is denoted by $\bar{v}_i$ $(i=0, 1, \cdots, m)$.

Let us augment M and P so that

$$\begin{bmatrix} P_0 \\ 0 \end{bmatrix}\bar{y}_0 + \sum_1^n \begin{bmatrix} P_j \\ \delta_j \end{bmatrix}\bar{y}_j + \begin{bmatrix} \cdot \\ 1 \end{bmatrix}\bar{y}_{n+1} = \begin{bmatrix} & M & & & \cdot \\ K & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{5}$$

where $\bar{y}$ has one more component than $\bar{x}$ and $\cdot$ represents the null vector, and δ_j, K are arbitrary constants. $\bar{y}_j$ is required to be non-negative in the lexicographic sense, but neither $\bar{y}_0$ nor $\bar{y}_{n+1}$ is restricted to be non-negative. It is also required that a basic solution to (5) consists of $m+2$ variables including $\bar{y}_0$ and $\bar{y}_{n+1}$.

There is a one-to-one correspondence between the set of basic solutions to (1) and the set of basic solutions to (5). This correspondence associates with a basic solution (4) a basic solution to (5)

$$\left.\begin{array}{l} \bar{y}_0 = [\bar{x}_0, 0], \qquad \bar{y}_{j_i} = [\bar{v}_i, 0] \qquad (i=1, 2, \cdots, m), \\ \bar{y}_{n+1} = [K, 0, \cdots, 0, 1] - \sum_{i=1}^{m} \delta_{j_i}[\bar{v}_i, 0], \quad \bar{y}_j = [\cdot, 0] \quad (j \neq j_i). \end{array}\right\} \tag{6}$$

Conversely, the correspondence associates with a basic solution to (5) a basic solution to (1) whose variable vectors $\bar{x}$ consist of the first l-components of $\bar{y}$.

The generalized simplex method [2] gives a means of transition from a basic solution to any adjacent basic solution. If we consider basic solutions as nodes and we connect every pair of two nodes representing two adjacent basic solutions by an edge, we will obtain a finite linear graph. [3] It is clear that two basic solutions to (5), which correspond to two adjacent basic solutions to (1), are also adjacent, and conversely. Therefore, the graphs so obtained from (1) and (5) are identical from the point of view of abstract graph theory.

Let us assume that a basic solution to (1),

$$P_0\bar{w}_0+\sum_{i=1}^{m}P_{k_i}\bar{w}_i=M \tag{7}$$

is given. Then, let us set $\delta_{k_i}=1$ $(i=1, 2, \cdots, m)$, $\delta_j=1$ $(j\neq k_i)$ in (5), and maximize $\bar{y}_{n+1}$. Starting from a basic solution (6), after some iterative calculations we will obtain a maximizing basic solution to (5), which corresponds to (7) since max $\bar{y}_{n+1}=[K, 0, \cdots, 0, 1]$ occurs only when $\bar{y}_j=0$ $(j\neq k_i)$. This shows that the graph of (5) has a path connecting any two nodes.

According to the above, the graph is finite and connected; that is, it is a labyrinth. [3] Thus, applying the known solution to labyrinth problems, such as Tarry's one,[3] we can determine all basic solutions to (1) by the simplex technique whenever one basic solution is found.

Let us assume that an optimal basic solution (4) is found. Let β_i denote the $(i+1)$-th row of B^{-1}:

$$B^{-1}=[P_0,\ P_{j_1},\ \cdots,\ P_{j_m}]^{-1}=[\beta_0',\ \beta_1',\ \cdots,\ \beta_m']', \tag{8}$$

where primed letters stand for transpose. It is known [2] that

$$\beta_0P_j\geq 0 \qquad (j=1,\ 2,\ \cdots,\ n) \tag{9}$$

and a solution is optimal if and only if it has the property that $\bar{x}_j=0$ whenever $(\beta_0P_j)>0$. Therefore, the set of optimal basic solutions to (1) coincides with the set of basic solutions to the reduced problem which is given from the problem (1) by eliminating all $\bar{x}_j$ such that $(\beta_0P_j)>0$. Hence, whenever one optimal basic solution is found, we can determine all optimal basic solutions by applying the abovementioned wandering procedure to the reduced problem which is given from the original one by eliminating all $\bar{x}_j$ such that $(\beta_0P_j)>0$.

DETERMINATION OF ALL OPTIMAL SOLUTIONS TO A LINEAR PROGRAMMING PROBLEM

The fundamental problem is to find a set $(x_0, x_1, \cdots, x_{n'})$ satisfying the equations

$$x_0+\sum_1^{n'}a_{0j}x_j=0, \quad \sum_1^{n'}a_{kj}x_j=b_k \quad (b_k\geq 0\,;\ k=2,\ 3,\ \cdots,\ m) \tag{10}$$

such that

$$x_j\geq 0 \qquad (j=1,\ 2,\ \cdots,\ n'), \tag{11}$$

$$x_0=\max. \tag{12}$$

It is convenient to augment a redundant equation

$$\sum_1^{n'}a_{1j}x_j=b_1 \qquad (a_{1j}=-\sum_{k=2}^{m}a_{kj},\ b_1=-\sum_2^{m}b_k). \tag{13}$$

Let us consider the following generalized problem : To maximize $\bar{x}_0$ under the constraints

$$\left.\begin{aligned}&\bar{x}_0+\sum_1^{n'}a_{0j}\bar{x}_j=[0,\ 1,\ 0,\ \cdots,\ 0],\\ &\bar{x}_{n'+k}+\sum_1^{n'}a_{kj}\bar{x}_j=[b_k,\ 0,\ 0,\ \cdots,\ 1,\ \cdots,\ 0] \quad (k=1,\ 2,\ \cdots,\ m),\\ &\bar{x}_j\geq 0 \qquad (j=1,\ 2,\ \cdots,\ n'+m).\end{aligned}\right\} \tag{14}$$

Rewriting the equations we have (1) such that $M=[b,\ I]$, where I stands for $(m+1)$-rowed identity matrix and $\bar{x}$ is a $(m+2)$-components row vector, and $b=[0,\ b_1,\ \cdots,\ b_m]'$, $n=n'+m$.

It is clear that the first components of every solution to (14) constitute a solution to (10)$\sim$(13). Conversely, with a solution to (10) $\sim$(13) we can associate a solution to (14)

$$\left.\begin{aligned}&\bar{x}_0=[x_0,\ 1,\ 0,\ \cdots,\ 0], \quad \bar{x}_j=[x_j,\ 0,\ 0,\ \cdots,\ 0] \quad (j=1,\ 2,\ \cdots,\ n'),\\ &\bar{x}_{n'+k}=[0,\ 0,\ \cdots,\ 1,\ \cdots,\ 0] \qquad (k=1,\ 2,\ \cdots,\ m).\end{aligned}\right\} \tag{15}$$

An extreme point solution to (10) ~ (13) is one in which only $(q+1)$ $(q \leq m)$ variables (including x_0) are considered, the remainder being set equal to zero ; that is, it is of the form

$$P_0 v_0 + \sum_{i=1}^{q} P_{j_i} v_i = b \qquad (v_i > 0,\ 0 < j_i \leq n'), \tag{16}$$

where P_0, $P_{j_1} \cdots$, P_{j_q} are linearly independent.[4)]

It is easy to see that the first components of a basic solution to (1) form an extreme point solution to (10) ~ (13). Conversely, to an extreme point solution there corresponds at least one basic solution to (1). The existence of a basic solution to (1) which corresponds to an extreme point solution (16) is demonstrated as follows: Set $\delta_{j_i}=0$ $(i=1, \cdots, q)$, $\delta_j=1$ $(j \neq j_i,\ 0<j\leq n)$ in (5) and maximize $\bar{y}_{n+1}$ starting from a basic solution which necessarily exists.[2)] From (15) and the assumption $v_j=0$ $(j \neq j_i,\ 0<j\leq n)$ it is clear that max $\bar{y}_{n+1}$

$$= \max \{[K, 0, \cdots, 0, 1] - \sum_{j \neq j_i} \bar{y}_j\} > [K, 0, -1, \cdots, -1, 1]$$

occurs only when the first components of $\bar{y}_j$ $(j \neq j_i,\ j \neq 0)$ are zero. Hence, the maximizing basic solution so obtained has the property that the first components of $\bar{y}_j$ $(j \neq j_i,\ j \neq 0)$ are zero, and consequently the associated basic solution to (1) is one which corresponds to the extreme point solution (16).

According to the general theory,[2)] it is known that every optimal basic solution to (1) provides an optimal (extreme point) solution. Conversely, it will be shown that with an optimal extreme point solution there is at least one corresponding optimal basic solution to (1). Let us assume that the basic solution (4) is associated with an optimal extreme point solution, then the first component of $\bar{v}_0$ is equal to max x_0.

Let us assume that the first components of $\bar{v}_i$ are all positive. If there exists a P_s such that $(\beta_0 P_s) < 0$, then by introducing P_s into the basis we will obtain a new solution which gives a strictly larger value of x_0 than max x_0. This is a contradiction and $(\beta_0 P_j) \geq 0$ for all j. Therefore, the basic solution under consideration is an optimal one.

On the other hand, let us assume that some of the first components of $\bar{v}_i$ are zero; that is, degeneracy occurs. If there exists a P_s such that $(\beta_0 P_s)<0$, then by introducing P_s into the basis and dropping a P_{j_r} such that the first component of $\bar{x}_{j_r}$ is zero we can obtain a new basic solution which gives the value of $\bar{x}_0$ strictly larger than before. The new basic solution is also associated with the same extreme point solution as before. Such an iterative procedure necessarily terminates in a basic solution such that $\beta_0 P_j \geq 0$ for all j. This is an optimal basic solution associated with the optimal extreme point solution under consideration.

If the set of optimal solutions to (10) ~ (13) is bounded, any optimal solution can be represented as a convex combination of all optimal extreme point solutions. [4] It is known from the discussion above that, in order to find all optimal extreme point solutions to (10) ~ (13), it is sufficient to find all optimal basic solutions to (1). The procedure for latter purpose is described in the preceding section.

If the set of optimal solutions to (10) ~ (13) is not bounded, any optimal solution is, in general, a convex combination of all optimal extreme point solutions to the new problem which has one more equation than (10) ~ (13),

$$x_{n+1}+\sum_{1}^{n'} x_j = K, \tag{17}$$

where K is sufficiently large and x_{n+1} is restricted to be non-negative. The generalized problem associated with this new augmented problem is given by setting $\delta_j=1$ $(j=1, 2, \cdots, n')$, $\delta_{n'+k}=0$ $(k=1, 2, \cdots, m)$ in (5). There is a one-to-one correspondence between the set of optimal basic solutions to (1) and the set of optimal basic solutions to (5), which have $\bar{y}_{n+1}(>0)$ in the basis. This correspondence is characterized by (4), (6).

It is easy to see that given an optimal basic solution to (5) in which $\bar{y}_{n+1}=0$ we can obtain a new optimal basic solution in which $\bar{y}_{n+1}>0$, by introducing $\bar{y}_{n+1}$ into the basis and dropping an unique $\bar{y}_r$. Then, if we introduce $\bar{x}_r$ into the basis of the optimal basic solution to (1) which corresponds to the basic solution with $\bar{y}_{n+1}>0$ mentioned above, $\bar{x}_r$ can be made arbitrarily large without violating the feasibility and optimality of the solution. Consequently, we know that a class of optimal solutions in which the value of $\bar{x}_r$ can be made arbitrarily large corresponds to the optimal basic solution (to (5)) with $\bar{y}_{n+1}=0$ mentioned above.

To the graph of optimal basic solutions to (1), we will add every class of optimal solutions, which arise from an optimal basic solution when we introduce some $\bar{x}_j$ into the basis and we can make $\bar{x}_j$ arbitrarily large without violating the feasibility, as a node. And we will connect two nodes, which correspond to a class and an optimal basic solution which produce the class, by an edge (end-edge). This graph is identical with a connected subgraph of the graph of optimal basic solutions to (5) and this graph has same nodes as the graph of (5). Therefore, it is not necessary to write down explicitly the additional constraint (17). If we wander this labyrinth, we will obtain a set of optimal solutions to (10) ~ (13)

$$\left.\begin{aligned}
&E_1=(\max x_0,\ x_{11},\ \cdots,\ x_{n'1}),\\
&E_2=(\max x_0,\ x_{12},\ \cdots,\ x_{n'2}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&F_1=(\max x_0,\ x_{1i_1}+\lambda_1 z_{11},\ \cdots,\ x_{n'i_1}+\lambda_1 z_{n'1}),\\
&F_2=(\max x_0,\ x_{1i_2}+\lambda_2 z_{12},\ \cdots,\ x_{n'i_2}+\lambda_2 z_{n'2}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}\right\} \tag{18}$$

where E_1, E_2, $\cdots$, are optimal extreme point solutions to (10) ~ (13), F_1, F_2, $\cdots$ are the parametric representations of all classes mentioned above and λ_1, λ_2, $\cdots$ are arbitrary positive constants (parameters). To a $E_j(F_j)$ there may be more than one corresponding basic solution (class of solutions). Then every optimal solution to (10) ~ (13) is represented as follows:

$$\left.\begin{aligned}
&(x_0,\ x_1,\ \cdots,\ x_{n'})\\
&=\mu_1 E_1+\mu_2 E_2+\cdots+\rho_1 F_1+\rho_2 F_2+\cdots.
\end{aligned}\right\} \tag{19}$$

where $\mu_1\geq 0$, $\mu_2\geq 0$, $\cdots$; $\rho_1\geq 0$, $\rho_2\geq 0$, $\cdots$, and $\sum_i \mu_i+\sum_j \rho_j=1$.

CONCLUSION

A general means for locating all optimal solutions to a linear programming problem is as follows : find an optimal basic solution by the generalized simplex method by which cycling can be avoided. Then, eliminate all variables x_j such that $(\beta_0 P_j) > 0$. Starting from an optimal basic solution obtained above, wander the labyrinth, for example, by Tarry's rule.[3] In the labyrinth, every node is a basic solution or a class of solutions which arise from a basic solution when we introduce some variable into the basis and the variable can be made arbitrarily large. Every edge connects two adjacent basic solutions, or otherwise it is an end-edge connecting two nodes which correspond to a basic solution and a class arisen from the basic solution. The resulting records (18) give a general solution (19).

In Charnes' wandering, x_j such that $(\beta_0 P_j) > 0$ is not rejected in principle, and his graph is more complicated than the graph presented here. However, when degeneracy does not occur, Charnes' labyrinth and the author's one is identical. If degeneracy occurs, the present method is simpler than Charnes'. It must be noted, however, that when degeneracy occurs Charnes' method will give, perhaps, all the extreme point solutions to the dual problem though the author's method gives only one solution to the dual.

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