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ON A QUEUE IN WHICH JOINING CUSTOMERS GIVE UP THEIR SERVICES HALFWAY

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INTRODUCTION

It has been a generally accepted idea that the queue can be given if the following three assumptions are defined :

(1) input of customers at office ;
(2) service mechanism at office ;
(3) distribution of service time at office.

Many queues have so far been defined from various points of view, obtaining various results of much interest. It may, however, be said that almost all the theories of queue that have thus far been studied were established on the assumption that each customer arriving at the office would never leave there, unless he should be received his service. Professor T. Kawata [1] has now introduced and analyzed a new theory of queue, in which he assumed that the customer arriving at the office anew would not necessarily join the queue but join it with a certain probability p_n, of which n was to indicate the number of customers who had been forming the queue and waiting for the service. Taking much interest in this idea, present writers with other associate specialists, examined it from the viewpoint of the hidden demands, thereby introducing some results, and applied it to some practical cases. [2], [3]

Even in this theory, however, it is assumed that the customer once joined the queue woud not give up the service and leave the queue. Now, in making this report, an attempt was made to expand the theory as to consider that not only the customers arriving at the office would not necessarily join the queue, but also some of the customers already joined the queue might happen to give up the service halfway and leave the queue. Needless to say, there may be various ways of expanding the theory, but, taking into account its applicability to the practice and the relative difficulty of its theoretical analysis, the expansion of theory was made here on a specific assumption. The expanded theory, therefore, may in some cases be inapplicable to the practice. However, the assumption adopted here may, in a sense, have the reality of the same degree as of the assumption that a discrete additive process is to follow the Poisson process. (This will be verified later on). As for the other ways of expanding the theory, reports will be made in another article.

FORMULATION OF THE QUEUE

In order to formulate the queue, we set up some assumptions for each of the three assumptions referred to in the introduction. These assumptions are, with the exception of one for the customers who, giving up the service, leave the queue halfway, just the same as those set up for the queue introduced by Professor T. Kawata.

First, it is assumed, as for the input of customers at office, that :

(I) The input of customers at office is performed in accordance with the Poisson process of parameter λ, i.e., the probability of n customers' arriving at office within any time interval t is :

$$\frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

(II) The probability of the customer's joining the queue who arrives at office when n customers inclusive of one now being served stand in a queue is defined by p_n, provided, however, that he always receives the service when there is no customer but him at the office, i. e.,

$$p_0=1.$$

Next, it is assumed, as for the service mechanism, that :

(III) Services are furnished in the order of customers' standing in the queue.

Then, as for the distribution of service time, it is assumed that :

(IV) There is only one server where services are furnished, with the distribution of service time to be followed by the negative exponential law with the parameter b. Namely, the probability density with which a service is furnished within a time interval t is,

$$g(t) = be^{-bt}.$$

In addition to the above-described four assumptions, fifth assumption is made whereby the condition is defined, in which the customers standing in a queue give up their services halfway and leave the queue, i. e. :

(V) Supposing that n customers exclusive of one now being served at the office are standing in a queue and a server is occupied during a time interval t for a customer, the probability of one of standing customer's giving up the service halfway and leaving the queue is given by $q_n \Delta t$, provided, however, that Δt is a very little while and the probability of two or more customers' giving up the services halfway and leaving the queue in that time is $o(\Delta t)$.

It would be worth while to touch up a bit upon the five assumptions described above, which were set up to enable us to define our queue, except, the assumptions (I), (III) and (IV) that are customarily used in analyzing the queue as a matter of common sense.

Both of the assumptions (II) and (V) denote the disposition of the customers who come to the office with the object of being served, however, leave there without being served. Assumption (II) embodies the disposition of the customers who come to the office, however, leave there right away, and the possibility of such customers' leaving may, in the name of common sense, be considered to increase as the queue becomes longer. In fact, it has been made clear that this tendency approximately decreases exponentially. [3] In the theoretical analysis, however, only the definition, without regard to such tendency, of the fact that the possibility of the customer's leaving the office right away even when coming to it is affected by the number of customers already standing in a queue will suffice.

Assumption (V), on the other hand, embodies the disposition of the customers who have already stood in the queue give up their services and leave there halfway. In this case, the possibility of the customer's leaving will, as a matter of fact, be affected not only by the speed of services but also by the number of customers standing in a queue. In fact, if it takes so much time for furnishing service to a customer that the queue makes very slow progress, many customers will be tired of waiting and leave there. Therefore, it was assumed that the proportional constant q_n depends upon the number of customers standing in the queue, supposing that the probability of one customer's leaving while a customer is being served will be proportionate to the time so long as the time involved is very short. This assumption may, in a sense, have the reality of the same degree as of the assumption for the Poisson process, as mentioned in the introduction. For the Poisson process with parameter α, the probability of no occurrence coming out within the time Δt is $(1-e^{-\alpha\Delta t})$, which is $\alpha\Delta t$, thus being proportionate to Δt, if the infinitesimal of higher order of Δt are neglected. Our case, though, is rather similar to the Poisson process, in which parameter varies, since q_n varies with n.

DEATH PROCESS

The queue to be analyzed hereunder may be considered to have been defined in the preceding chapter, but it would be difficult to directly analyze how the number of customers standing in a queue will vary with the lapse of time. So, the consideration here will be limited to the time interval extending from the time when a customer arrives at the office to the time when the next customer comes to it. (It makes no difference whether the customer coming will join the queue or not.) Within this time interval, there should be no customer coming to the office and, accordingly, no customer joining the queue, so that the number of customers standing in a queue will go on falling away by the number of customers who, after having been served, will leave there or who will give up their services and leave there halfway. Then the numbers of customers standing in a queue will, in this case, create a death process. Supposing that, in this death process, n customers inclusive of the customer now being served at $t=0$ are standing in a queue at the office, the probability of $m(\leqq n)$ customers' leaving there until $t=T$ is defined as $P_m(T, n)$. The stochastic equation for $P_m(T, n)$ will then be obtainable. It would, however, be necessary to demonstrate, prior to this, some simple characters of the negative exponential distribution.

It was assumed in assumption (IV) that the distribution of service time, $g(t)$ was followed by the negative exponential distribution with the parameter b. Now, lets find out the probability of a customer's being fully served, in the case of such service time distribution within the following time t. Supposing that this customer has already been served for the time t_1, the probability of his being fully served within the following time t will be :

$$\frac{be^{-b(t_1+t)}}{\int_{t_1}^{\infty} be^{-bt}dt} = \frac{be^{-b(t_1+t)}}{e^{-bt_1}} = be^{-bt}.$$

This shows that, in the negative exponential distribution, the probability of a customer's being fully served within the following time t will always be equal to that in the initial distribution, regardless of the time for which he has already been served in advance. This quite distinguishing characteristic of the negative exponential distribution may be used for obtaining easily the stochastic equations for $P_m(t, n)$. Namely, considering the equilibrium conditions during a certain very short time Δt :

$$P_0(t+\Delta t, n) = P_0(t, n)e^{-b\Delta t}(1-q_{n-1}\Delta t) + o(\Delta t),$$

$$P_m(t+\Delta t, n) = P_m(t, n)e^{-b\Delta t}(1-q_{n-m-1}\Delta t)$$
$$+ P_{m-1}(t, n)\{(1-e^{-b\Delta t}) + e^{-b\Delta t}q_{n-m}\Delta t\} + o(\Delta t),$$
$$m \neq 0, \quad n > m,$$

$$P_n(t+\Delta t, n) = P_n(t, n) + P_{n-1}(t, n)(1-e^{-b\Delta t}) + o(\Delta t).$$

It is to be noted here that the probability of no customer's being fully served within a time interval Δt is $e^{-b\Delta t}$. Also the expression $q_0=0$ comes into existence. The three formulas given above are transformed as :

$$P_0(t+\Delta t,\ n)=\{1-(b+q_{n-1})\Delta t\}P_0(t,\ n)+o(\Delta t),$$

$$P_m(t+\Delta t,\ n)=\{1-(b+q_{n-m-1})\Delta t\}P_m(t,\ n)$$
$$+(b+q_{n-m})\Delta tP_{m-1}(t,\ n)+o(\Delta t),\quad m\neq 0,\ n>m,$$

$$P_n(t+\Delta t,\ n)=P_n(t,\ n)+b\Delta tP_{n-1}(t,\ n)+o(\Delta t),\ n\geqq 1.$$

Therefore, in the limit of $\Delta t\rightarrow 0$,

$$\frac{\partial}{\partial t}P_0(t,\ n)=-(b+q_{n-1})P_0(t,\ n),\quad n\geqq 1, \tag{3. 1}$$

$$\left.\begin{aligned}\frac{\partial}{\partial t}P_m(t,\ n)=-(b+q_{n-m-1})P_m(t,\ n)+(b+q_{n-m})P_{m-1}(t,\ n),\\ m\neq 0,\quad n>m.\end{aligned}\right\} \tag{3. 2}$$

$$\frac{\partial}{\partial t}P_n(t,\ n)=bP_{n-1}(t,\ n),\quad n\geqq 1. \tag{3. 3}$$

And the initial condition of these differential-difference equations is :

$$P_m(0,\ n)=\delta_{n0}, \tag{3. 4}$$

where δ denotes the Kronecker's delta and $\delta_{n0}=1$ or 0 for $n=0$ or $n\neq 0$ respectively. Since b and q in equations (3.1)$\sim$(3.3) always appear in the form of $b+q_n$, we define β_n as follows for the purpose of simplifying the symbols ;

$$\beta_n=b+q_n \tag{3. 5}$$

Then the solution of equation (3.1) is

$$P_0(t,\ n)=e^{-\beta_{n-1}t},\quad n\geqq 1 \tag{3. 6}$$

with the aid of (3.4).

And equations (3.2) and (3.3) are rewritten as :

$$\left.\begin{aligned}\frac{\partial}{\partial t}P_m(t,\ n)=-\beta_{n-m-1}P_m(t,\ n)+\beta_{n-m}P_{m-1}(t,\ n),\\ m\neq 0,\quad n\geqq m.\end{aligned}\right\} \tag{3. 2$'$}$$

Here, we use the definition of $\beta_{-1}=0$. Evidently the solution of equation (3.2)$'$ can be defined uniquely, if $m=1, 2, \cdots$ are substituted one after another with the aid of equations (3.4) and (3.6).

Now, in order to give the analytical representation to $P_m(t, n)$, some study will be made on q_n. If q_n is regarded as a function of n only, it is natural that, the more the customers standing in the queue, the greater will be the probability of a customer's leaving there. It is, in fact, reasonable to imagine that, if many customers are standing in the queue and the queue is advanced slowly, some of them may happen to give up their services and leave there. Supposing then that $q_n > q_{n-1}$ for all the positive numbers n, the similar relation, $\beta_n > \beta_{n-1}$, will also come into existence for every n. If such assumption as is quite probable is made for every n, the solution of equation (3.2)$'$ can be obtained easily in the case of $n > m$, as

$$P_m(t, n) = (-1)^m\left\{\prod_{i=1}^{m}\beta_{n-i}\right\}\sum_{j=1}^{n+1}\frac{e^{-\beta_{n-j}t}}{\prod_{k=1}^{m+1}{}'(\beta_{n-j}-\beta_{n-k})}. \tag{3. 7}$$

Here, Π' denotes the multiplication of all terms exclusive of $k=j$. To see that equation (3.7) is the solution, verifications of the fact that formulas (3.2)$'$ and (3.4) are thereby satisfied would suffice.

First, by substituting formula (3.7) in the left and the right sides of formula (3.2)$'$, we obtain

$$\text{left side} = (-1)^{m+1}\left\{\prod_{i=0}^{m}\beta_{n-i}\right\}\sum_{j=1}^{m+1}\frac{\beta_{n-j}e^{-\beta_{n-j}t}}{\prod_{k=0}^{m+1}{}'(\beta_{n-j}-\beta_{n-k})},$$

$$\text{right side} = (-1)^{m+1}\beta_{n-m-1}\left\{\prod_{i=1}^{m}\beta_{n-i}\right\}\sum_{j=1}^{m+1}\frac{e^{-\beta_{n-j}t}}{\prod_{k=1}^{m+1}{}'(\beta_{n-j}-\beta_{n-k})}$$

$$+(-1)^{m-1}\beta_{n-m}\left\{\prod_{i=1}^{m-1}\beta_{n-i}\right\}\sum_{j=1}^{m}\frac{e^{-\beta_{n-j}t}}{\prod_{k=1}^{m}{}'(\beta_{n-j}-\beta_{n-k})}$$

$$=(-1)^{m+1}\left\{\prod_{i=1}^{m}\beta_{n-i}\right\}\sum_{j=1}^{m+1}\frac{\beta_{n-m-1}e^{-\beta_{n-j}t}}{\prod_{k=1}^{m+1}{}'(\beta_{n-j}-\beta_{n-k})}$$

$$+(-1)^{m+1}\left\{\prod_{i=1}^{m}\beta_{n-i}\right\}\sum_{j=1}^{m}\frac{(\beta_{n-j}-\beta_{n-m-1})e^{-\beta_{n-j}t}}{\prod_{k=1}^{m+1}{}'(\beta_{n-j}-\beta_{n-k})}.$$

Just the comparison of the coefficients of $e^{-\beta_{n-j}t}$ of the left and right sides will make clear that both sides are equal. Next, in oder to see that the initial condition——formula (3.4)——are satisfied, $t=0$ will be put in formula (3.7), which will then be ;

$$P_m(0,\ n)=(-1)^m\left\{\prod_{i=1}^{m}\beta_{n-i}\right\}\sum_{j=1}^{m+1}\frac{1}{\prod_{k=1}^{m+1}{}'(\beta_{n-j}-\beta_{n-k})}.$$

So, the verification of :

$$D\equiv\sum_{j=1}^{m+1}\frac{1}{\prod_{k=1}^{m+1}{}'(\beta_{n-j}-\beta_{n-k})}=0$$

would suffice, since, in the case of $n>m$,

$$\left\{\prod_{i=1}^{m}\beta_{n-i}\right\}\neq 0.$$

For this purpose, we reduce to a common denominator of D, then we obtain

$$D=\sum_{j=1}^{m+1}\frac{(-1)^{m+1-j}\prod_{i>k}^{m+1}{}'(\beta_{n-i}-\beta_{n-k})}{\prod_{i>k}^{m+1}(\beta_{n-i}-\beta_{n-k})}.$$

As, in this formula, numerators and denominators are expressed, respectively, in the form of a differente, it may be expressed in a determinant by making use of Vandermonde's determinant. If jth numerator is expressed in this determinant, it will be :

$$\prod_{i>k}^{m+1}{}'(\beta_{n-i}-\beta_{n-k})=\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \beta_{n-1} & \beta_{n-2} & \cdots & \beta_{n-j+1} & \beta_{n-j-1} & \cdots & \beta_{n-m-1} \\ \beta_{n-1}^2 & \beta_{n-2}^2 & \cdots & \beta_{n-j+1}^2 & \beta_{n-j-1}^2 & \cdots & \beta_{n-m-1}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{n-1}^{m-1} & \beta_{n-2}^{m-1} & \cdots & \beta_{n-j+1}^{m-1} & \beta_{n-j-1}^{m-1} & \cdots & \beta_{n-m-1}^{m-1} \end{vmatrix}.$$

Accordingly, the numerator of D is identically zero, as a result of the lowest row of the following being expanded :

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \beta_{n-1} & \beta_{n-2} & \cdots & \beta_{n-m-1} \\ \beta_{n-1}^2 & \beta_{n-2}^2 & \cdots & \beta_{n-m-1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n-1}^{m-1} & \beta_{n-2}^{m-1} & \cdots & \beta_{n-m-1}^{m-1} \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

Since, however, the denominator can not be zero because of the assumption $(\beta_j > \beta_{j-1})$, the result will be $D=0$, hence it may be said that formula (3.4) has been verified.

Lastly, to find out the value of $P_m(t, n)$, $m=n-1$ is put in formula (3.7), which will then be :

$$P_{n-1}(t, n)=(-1)^{n-1}\left\{\prod_{i=1}^{n-1}{}'\beta_{n-i}\right\}\sum_{j=1}^{n}\frac{e^{-\beta_{n-j}t}}{\prod_{k=1}^{n}{}'(\beta_{n-j}-\beta_{n-k})}$$

Therefore, taking into account the initial condition of $P_m(t, n)$:

$$P_m(t, n)=(-1)^{n-1}\left\{\prod_{i=1}^{n}\beta_{n-i}\right\}\sum_{j=1}^{n}\int_0^t\frac{e^{-\beta_{n-j}t}}{\prod_{k=1}^{n}(\beta_{n-j}-\beta_{n-k})}dt. \qquad (3.\ 8)$$

MARKOV CHAIN

It has been described in chapter 3 that, in the lapse of time extending from the time when a customer arrives at the office to the time when the next customer comes to it, a death process was created by the numbers of customers standing in the queue at the office, and. at the same time, some study was made of the features of such death process. Now, in this chapter, explanations will be given, through application of the features of such death process, for Markov chain to be formed by the numbers of customers standing in the queue with respect only to the time points when customers come to the office. Also some study will be made of the features of the Markov chain.

Supposing that i customers are standing in the queue at the time point when a customer arrives at the office, the probability of j customers' standing in a queue at the time point when the next customer comes to it is defined as $p(i, j)$. (If the arriving customer joins the queue, he must be included in the number.) Since the time t between the two successive time points will, as may easily be introduced from assumption (I), show the distribution of $\lambda e^{-\lambda t}$, $p(i, j)$ is represented by

$$p(i, j)=\int_0^\infty \{p_{j-1}P_{i-j+1}(t, i)+(1-p_j)P_{i-j}(t, i)\}\lambda e^{-\lambda t}dt, \quad i\geqq j>1,$$

$$p(i, j)=\int_0^\infty p_i P_0(t, i)\lambda e^{-\lambda t}dt, \quad i=j-1>0,$$

$$p(i, j)=0, \ i<j-1,$$

where p_j is to represent the probability defined by the assumption (II), the probability of the customer's joining the queue, who newly arrives at the office when j customers are standing in the queue. The transition probability $p(i, j)$ will now be sought for one after another. First, $p(i, i+1)$ can be calculated with ease. Namely, by using formula (3.6),

$$p(i, i+1)=\lambda p_i\int_0^\infty e^{-(\lambda+\beta_{i-1})t}\,dt=\frac{\lambda p_i}{\lambda+\beta_{i-1}}, \qquad i\geqq 1. \qquad (4.\ 1)$$

Then, to calculate $p(i, j)$, $i\geqq j>0$, both sides of formula (3.2)' are multiplied by $\lambda e^{-\lambda t}$, and integrated by t from 0 to ∞. Thus we have

$$\int_0^\infty \left\{\frac{\partial}{\partial t} P_m(t, n)\right\} \lambda e^{-\lambda t} dt = -\beta_{n-m-1}\int_0^\infty P_m(t, n)\lambda e^{-\lambda t} dt$$

$$+\beta_{n-m}\int_0^\infty P_{m-1}(t, n)\lambda e^{-\lambda t} dt, \qquad n \geqq m > 0.$$

By integrating the left side by parts and taking into account the initial condition (3.4),

$$\lambda\int_0^\infty p_m(t, n)\lambda e^{-\lambda t} dt = -\beta_{n-m-1}\int_0^\infty p_m(t, n)\lambda e^{-\lambda t} dt$$

$$+\beta_{n-m}\int_0^\infty P_{m-1}(t, n)\lambda e^{-\lambda t} dt, \quad n \geqq m > 0.$$

Accordingly,

$$\int_0^\infty P_m(t, n)\lambda e^{-\lambda t} dt = \frac{\beta_{n-m}}{\lambda+\beta_{n-m-1}}\int_0^\infty P_{m-1}(t, n)\lambda e^{-\lambda t} dt, \quad n \geqq m > 0. \qquad (4.\ 2)$$

By using formula (4.2), for the case of $i \geqq j > 0$,

$$p(i, j) = p_{j-1}\int_0^\infty P_{i-j+1}(t, i)\lambda e^{-\lambda t} dt + (1-p_j)\int_0^\infty P_{i-j}(t, i)\lambda e^{-\lambda t} dt$$

$$= p_{j-1}\frac{\beta_{j-1}}{\lambda+\beta_{i-2}}\int_0^\infty P_{i-j}(t, i)\lambda e^{-\lambda t} dt + (1-p_j)\int_0^\infty P_{i-j}(t, i)\lambda e^{-\lambda t} dt$$

$$= \left\{\frac{p_{j-1}\beta_{j-1}}{\lambda+\beta_{j-2}} + (1-p_j)\right\}\int_0^\infty P_{i-j}(t, i)\lambda e^{-\lambda t} dt. \qquad (4.\ 3)$$

By putting $i=j>0$ in this formula

$$p(i, i) = \left\{\frac{p_{i-1}\beta_{i-1}}{\lambda+\beta_{i-2}} + (1-p_i)\right\}\int_0^\infty P_0(t, i)\lambda e^{-\lambda t} dt$$

$$= \left\{\frac{p_{i-1}\beta_{i-1}}{\lambda+\beta_{i-2}} + (1-p_i)\right\}\int_0^\infty \lambda e^{-(\lambda+\beta_{i-1})t} dt$$

$$=\left\{\frac{p_{i-1}\beta_{i-1}}{\lambda+\beta_{i-2}}+(1-p_i)\right\}\frac{\lambda}{\lambda+\beta_{i-1}}. \qquad (4.\ 4)$$

Supposing that $i>j>0$, it may be obtained from formula (4.3),

$$p(i,\ j+1)=\left\{\frac{p_j\beta_j}{\lambda+\beta_{j-1}}+(1-p_{j+1})\right\}\int_0^\infty P_{i-j-1}(t,\ i)\lambda e^{-\lambda t}dt.$$

Also,

$$p(i,\ j)=\left\{\frac{p_{j-1}\beta_{j-1}}{\lambda+\beta_{j-2}}+(1-p_j)\right\}\int_0^\infty P_{i-j}(t,\ i)\lambda e^{-\lambda t}dt$$

$$=\left\{\frac{p_{j-1}\beta_{j-1}}{\lambda+\beta_{j-2}}+(1-p_j)\right\}\frac{\beta_j}{\lambda+\beta_{j-1}}\int_0^\infty P_{i-j-1}(t,\ i)\lambda e^{-\lambda t}dt.$$

Therefore,

$$\frac{p(i,\ j)}{p(i,\ j+1)}=\frac{\left\{\frac{p_{i-1}\beta_{j-1}}{\lambda+\beta_{j-2}}+(1-p_j)\right\}\frac{\beta_j}{\lambda+\beta_{j-1}}}{\left\{\frac{p_j\beta_j}{\lambda+\beta_{j-1}}+(1-p_{j+1})\right\}}$$

$$=\frac{\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}/\{(\lambda+\beta_{j-2})(\lambda+\beta_{j-1})\}}{\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})/(\lambda+\beta_{j-1})}$$

$$=\frac{\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}}{(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}}, \qquad i>j>0.$$

Consequently, $p(i,\ j)$ can be expressed by $p(i,\ j+1)$ as :

$$p(i,\ j)=\frac{\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}}{(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}}p(i,\ j+1), \qquad i>j>0.$$

$$(4.\ 5)$$

This is a recurrence equation of $p(i,\ j)$. Since $p(i,\ i)$ has been calculated in formula (4.4), $p(i,\ j)$ for the case of $i>j>0$ can be defined uniquely by formula (4.4).

Now, since Markov chain to be defined according to the transition probability $p(i,\ j)$ is, as is easily supposed, irreducible and aperiodic, it is either in null state or in ergodic one. It is known, anyway, that, if only the time points where a customer arrives at the office are to be taken into account and the probability of j customers standing in a queue after n time points where i customers are standing at a certain time point is supposed to be $p^{(n)}(i, j)$, the relation of

$$u_j = \lim_{n\to\infty} p^{(n)}(i, j) \geqq 0 \tag{4. 6}$$

will always exist with respect to j, regardless of i. If this Markov chain is in null state, the relation of $u_j=0$ will exist with respect to all j, which means that the number of customers standing in a queue will increase indenfinitely with the lapse of time. If, however, Markov chain is in ergodic state, the relation of

$$\left.\begin{aligned} & u_j > 0, \\ & \sum_{i=1}^{\infty} u_i p(i, j) = u_j, \\ & \sum_{j=1}^{\infty} u_j = 1 \end{aligned}\right\} \tag{4. 7}$$

exist with respect to all j. As, in our Markov chain, $p(i, j)=0$ for the case of $i<j-1$, 2nd formula of formula (4.7) may be transformed as:

$$\sum_{i=j-1}^{\infty} u_i\, p(i, j) = u_j. \tag{4. 8}$$

Next, some consideration will be given to the limiting probability, u_j in ergodic state of Markov chain, for which the transition probability, $p(i, j)$ is given by formulas (4.1), (4.4) and (4.5). First, it will be obtained from formula (4.5) :

$$(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}p(i, j)$$
$$=\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}p(i, j+1), \quad i>j>0. \tag{4. 9}$$

By multiplying both sides by u_i and adding them for i,

$$(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}\sum_{i=j+1}^{\infty}u_i p(i,\ j)$$

$$=\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}\sum_{i=j+1}^{\infty}u_i p(i,\ j+1),\quad j\geqq 1.$$

Here, by making use of the relation of formula (4.8),

$$(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}[\{1-p(j,\ j)\}u_j-p(j-1,\ j)u_{j-1}]$$

$$=\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}\{u_{j+1}-p(j,\ j+1)u_j\},\quad j\geqq 2,$$

This represents a linear difference equation with variable coefficients for u_{j-1}, u_j and u_{j+1}.

If, in this formula, the relation of

$$x_j u_j=y_j u_{j-1}+z_j u_{j+1}$$

is introduced, it will make:

$$x_j=(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}\{1-p(j,\ j)\}$$
$$+\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}p(j,\ j+1).$$

Then, using formulas (4.1) and (4.4),

$$x_j=(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}$$
$$\times\left[1-\left\{\frac{p_{j-1}\beta_{j-1}}{\lambda+\beta_{j-2}}+(1-p_j)\right\}\frac{\lambda}{\lambda+\beta_{j-1}}\right]$$
$$+\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\frac{\lambda p_j}{\lambda+\beta_{j-1}}$$
$$=\frac{1}{\lambda+\beta_{j-1}}[\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}\{(\lambda+\beta_{j-2})(\lambda+\beta_{j-1})$$

$$-\lambda p_{j-1}\beta_{j-1}-\lambda(1-p_j)(\lambda+\beta_{j-2})\}$$

$$+\lambda p_j\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})]$$

$$=\frac{1}{\lambda+\beta_{j-1}}[(1-p_{j+1})(\lambda+\beta_{j-1})\{(\lambda+\beta_{j-2})(\lambda+\beta_{j-1})$$

$$-\lambda p_{j-1}\beta_{j-1}-\lambda(1-p_j)(\lambda+\beta_{j-2})\}$$

$$+p_j\beta_j(\lambda+\beta_{j-2})(\lambda+\beta_{j-1})]$$

$$=(1-p_{j+1})\{(\lambda+\beta_{j-2})(\lambda+\beta_{j-1})-\lambda p_{j-1}\beta_{j-1}$$

$$-\lambda(1-p_j)(\lambda+\beta_{j-2})\}+p_j\beta_j(\lambda+\beta_{j-2})$$

$$=(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}$$

$$-\lambda(1-p_{j+1})\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\},$$

$$y_j=(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}\frac{\lambda p_{j-1}}{\lambda+\beta_{j-2}}$$

$$=\lambda p_{j-1}\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\},$$

$$z_j=\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}.$$

Accordingly, formula (4.10) becomes

$$[(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}$$

$$-\lambda(1-p_{j+1})\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2}\}]u_j$$

$$=\lambda p_{j-1}\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}u_{j-1}$$

$$+\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}u_{j+1},\quad j\geqq 2. \tag{4.11}$$

As, in the case of $j=1$, $p(i-1, j)=0$, hence $y_j=0$, the following formula will be obtained ;

$$[(\lambda+\beta_{j-2})\{p_j\beta_j+(1-p_{j+1})(\lambda+\beta_{j-1})\}$$

$$-\lambda(1-p_{j+1})\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}]u_j$$

$$=\beta_j\{p_{j-1}\beta_{j-1}+(1-p_j)(\lambda+\beta_{j-2})\}u_{j+1},\quad j=1. \tag{4.12}$$

The co-relations between $u_j (j=1, 2, \cdots)$ will be perfectly determined by formulas (4.11) and (4.12), since, in fact, u_2 can be expressed in u_1 by means of formula (4.12) and u_3 can be expressed in both u_1 and u_2, if $j=2$ is put in formula (4.13). By substituting $j=3, 4, \cdots$ one after another in formula (4.12), $u_4, u_5, \cdots$ will be determined successively. Threfore, by using

$$\sum_{j=1}^{\infty} u_j = 1,$$

each u_j can be determined but the analytical representation of u_j may be troublesome.

EXAMPLE IN A SPECIAL CASE

While, in chapter 2, various studies have been made of formulated queues, actual calculation will be made here in this chapter, such as derivation of ergodic condition etc.. The queue to be handled in this chapter is contemplated as follows. Namely, any customer who arrives at the office is supposed to join the queue without fail, while subsequently, some of the customers standing in a queue is expected to leave there, giving up service halfway, according to the proceeding conditions of the queue. In this case, the relation of

$$p_n = 1 \tag{5. 1}$$

exists for every n. No peculiarity, therefore, can be seen here with respect to the death process that has been examined in chapter 3.

By substituting formula (5.1) into formulas (4.11) and (4.12) which will come into existence in ergodic conditions, following formulas may be obtained :

$$(\lambda + \beta_{j-2})\, \beta_j u_j = \lambda \beta_j u_{j-1} + \beta_j \beta_{j-1} u_{j+1}, \quad j \geqq 2, \tag{5. 2}$$

$$(\lambda + \beta_{j-2})\, \beta_j u_j = \beta_j \beta_{j-1} u_{j+1}, \quad j=1. \tag{5. 3}$$

Taking into account $\beta_{-1}=0$, it may be obtained from formula (5.3) ;

$$u_2 = \frac{\lambda}{\beta_0} u_1.$$

By substituting above in formula (5.2),

$$u_3 = \frac{\lambda}{\beta_1}.$$

Calculation one after another will readily show that

$$u_{j+1} = \frac{\lambda}{\beta_{j-1}} u_j. \tag{5. 4}$$

Accordingly,

$$\sum_{j=1}^{\infty} u_j = \left(1 + \frac{\lambda}{\beta_0} + \frac{\lambda^2}{\beta_0\beta_1} + \frac{\lambda^2}{\beta_0\beta_1\beta_2} + \cdots\right) u_1 = 1,$$

where $$\rho = \frac{\lambda}{\beta_{j-1}}, \quad j \geqq 1. \tag{5. 5}$$

By putting in

$$p_j = \frac{\lambda}{\beta_{j-1}}, \quad j \geqq 1, \tag{5. 6}$$

following formulas will be obtained :

$$\left.\begin{aligned} u_1 &= \frac{1}{1 + \sum_{j=1}^{\infty} \rho_1\rho_2\rho_3 \cdots \rho_j}, \\ u_j &= \frac{\rho_1\rho_2\rho_3 \cdots \rho_j}{1 + \sum_{j=1}^{\infty} \rho_1\rho_2\rho_3 \cdots \rho_j}, \quad j \geqq 2. \end{aligned}\right\} . \tag{5. 7}$$

Hence it is quite clear that ergodic condition is :

$$\sum_{j=1}^{\infty} \rho_1\rho_2\rho_3 \cdots \rho_j < \infty.$$

If no customer is supposed to leave the queue, giving up service halfway, i. e., on the supposition of $q_n=0$ for every n, the ergodic condition will be $\rho=\lambda/b<1$, which accords with an ordinary condition.

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