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A PROBABILISTIC APPROACH TO PREVENTIVE MAINTENANCE

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INTRODUCTION

Sometimes it is impossible to know beforehand when particular machine parts may break down. If every break-down could bring about a grave damage, it would seem unwise to use such machine parts over the whole span of their natural life. Throwing away the long-lived parts before their lives expire and replace them by new ones will usually cut down production cost. Some typical models of preventive maintenance are showm in Fig. 1. In this paper, assuming the life of machine parts to be stochastic variables, some formulae for determining the optimal time of parts replacement in order to minimize the total cost are discussed from the stand-point of probability theory, and graphs for practical use are also prepared.

Notations used in this paper are as follows:

a =the price of a machine part.

a_1=loss of labour and time in replacing a broken part by a new one.

a_2=loss of labour and time in replacing a long-lived part by a

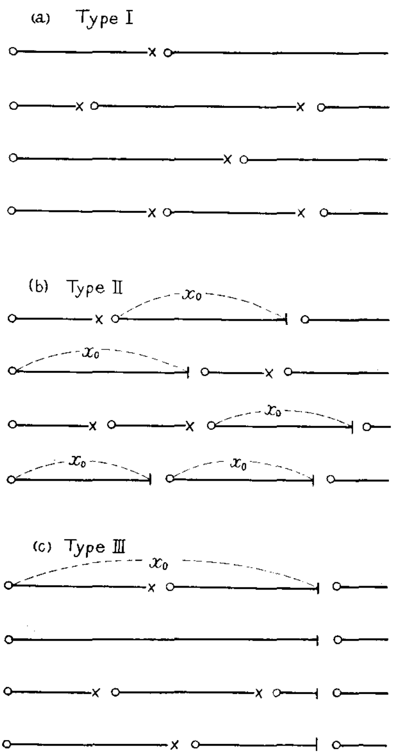

Fig. 1.

new one before its life expires.

b = the average loss due to a break-down of a machine part.

COST OF TYPE I

This is the case where every machine part is replaced by a new one after its natural life expires, as shown in Fig. 1, (*a*).

If $f(x)$ is the probability density function of the natural life of machine parts, then the mean m of the life is given by

$$m=\int_0^\infty xf(x)\,dx. \qquad (1)$$

The average number n of break-downs during a sufficiently long period of time T is given by T/m.

Therefore, the expected value C_1 of the total cost during the time period T is given as follows :

$$C_1=(a+a_1+b)\times\frac{T}{m}. \qquad (2)$$

COST OF TYPE II.

This is the case where a machine part which survives a time interval x_p is replaced by a new one without awaiting the natural expiration of its life. Fig. 1 (*b*) shows this type of maintenance policy. In this case, the expected value C_2 of the total cost during a long peroid T is given as follows:

$$C_2=\frac{(a+a_1+b)\cdot p+(a+a_2)\cdot q}{\int_0^{x_p} xf(x)\,dx+x_p\cdot q}\times T, \qquad (3)$$

where

$$p=\int_0^{x_p} f(x)\,dx, \quad q=\int_{x_p}^{\infty} f(x)\,dx.$$

If C_2 is to be minimized for $x_p=x_0$, such a value x_0 is obtained by putting the derivative of C_2 with respect to x equal to zero.

In this way, it is derived that the following equation should be satisfied by x_0 :

$$x_0 f(x_0)+\frac{1}{q}f(x_0)\int_0^{x_0} xf(x)\,dx-p=\frac{a+a_2}{a_1-a_2+b} \qquad (4)$$

Fig. 2 (a) shows the ratio of the optimal time length x_0 to the mean m of the life as plotted against $(a_1-a_2+b)/(a+a_2)$, when the

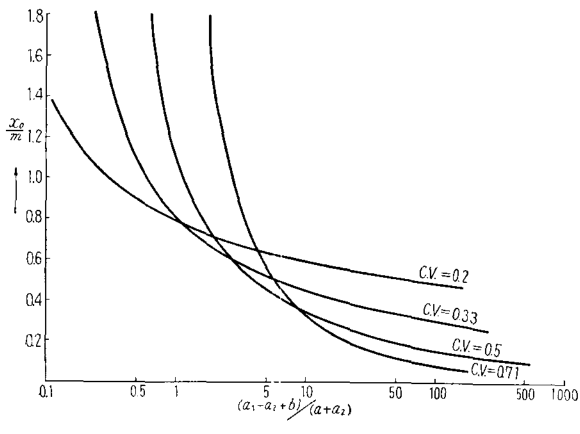

Fig. 2. (a)

probability density function $f(x)$ of the life time is a gamma distribution

$$\frac{1}{\Gamma(\nu)}\left(\frac{\nu}{m}\right)^{\nu} x^{\nu-1} e^{-\frac{\nu}{m}x} \qquad (\nu>0) \qquad (5)$$

or a normal distribution. The economy of adopting this maintenance policy is exhibited in Fig. 2(*b*), where the ratio C_2/C_1 of costs is plotted against $(a_1-a_2+b)/(a+a_2)$,

If we were using the optimal time length x_0 in this type of preventive maintenance, the actual frequency of the break-downs of machine parts would come close to the theoretical probability given by Fig. 2 (*c*).

COST OF TYPE III.

When a great many machine parts of the same type are in a

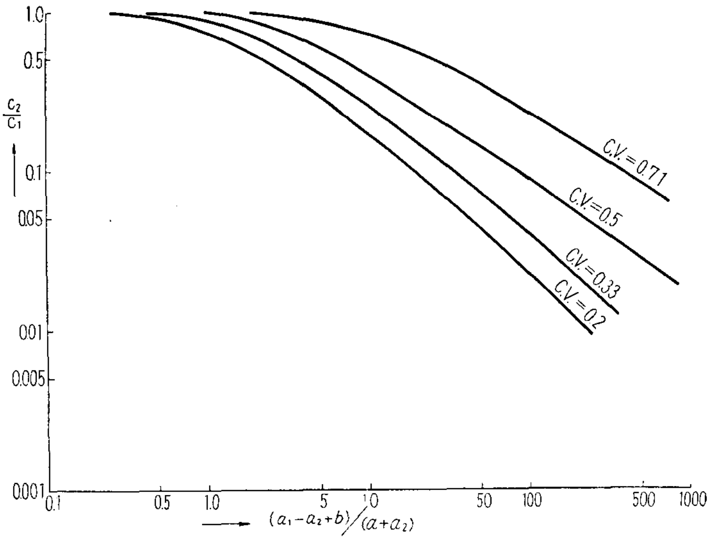

Fig. 2. (b)

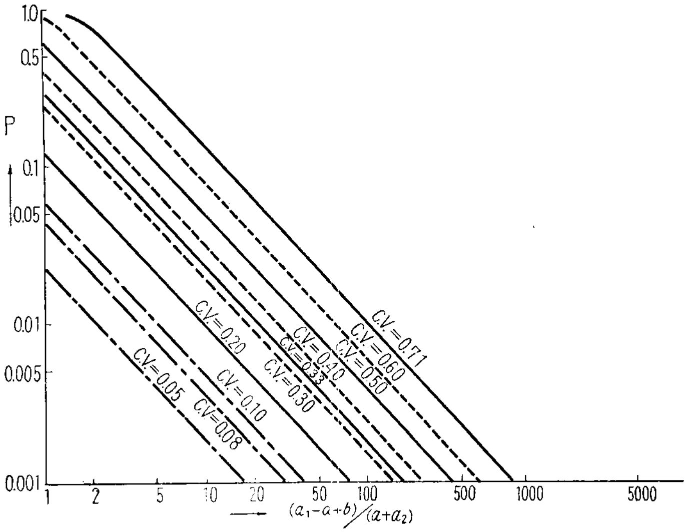

Fig. 2. (c)

factory, it is better to reglace at regular intervals all parts together regardless of their time duration since their last installation or renewal, as shown in Fig. 1 (*c*). Let $P(x_p < X_1 + \cdots + X_k)$ be the probability that the sum of lives of k machine parts

$$X = X_1 + X_2 + \cdots + X_k \tag{6}$$

(X_i is the life of the i-th part)
exceeds a particular time length x_p. Then

$$P(x_p < X_1 + \cdots + X_k) = \int_{x_p}^{\infty} f_k(x)\,dx = q_k, \tag{7}$$

where $f_k(x)$ is the probability density function of X expressed by the equation (6). Let q_k^0 be the probability that k machine parts are necessary and sufficient for the production in a time duration x_p, then

$$q_k^0 = q_k - q_{k-1}, \quad q_1^0 = q_1 \tag{8}$$

The average number $n(x_p)$ of machine parts necessary to cover a time length x_p is therefore

$$n(x_p)=\sum_{k=1}^{\infty} kq_k^0. \tag{9}$$

If we replace at regular interval x_p all parts together regardless of their history, then the expected value C_3 of the total cost during the period T is given as follows :

$$C_3=[(a+a_1+b)\{n(x_p)-1\}+(a+a_2)]\times\frac{T}{x_p}. \tag{10}$$

If C_3 is to be minimized for $x_p=x_0$, such a value x_0 is obtained by putting the derivative of C_3 with respect to x_p equal to zero.

Therefore, the optimal interval x_0 for such regular replacement should satisfy the following equation :

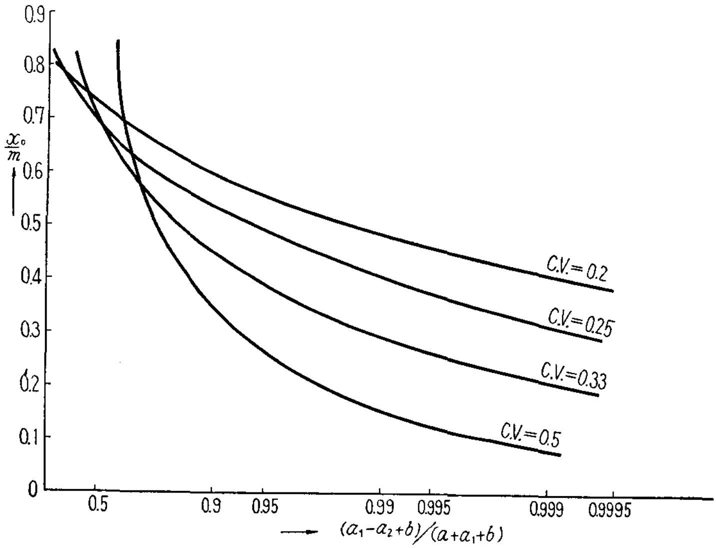

Fig. 3. (a)

$$n(x_0) - x_0 \cdot n'(x_0) = \frac{a_1 - a_2 + b}{a + a_1 + b} \tag{11}$$

where $n'(x_0)$ is the value of the derivative of $n(x_p)$ when $x_p = x_0$.

When the probability density function $f(x)$ of the life time is a gamma distribution, Fig. 3 (*a*), (*b*), (*c*) show the ratio of the optimal interval x_0 to the mean life m, the cost ratio C_3/C_1, and the probability of actual break-downs when the optimal solution x_0 is adopted in this type of maintenance.

If $f(x)$ is a gamma distribution given by equation (5), $f_k(x)$ is also of gamma type and given by

$$\frac{1}{\Gamma(k\nu)} \left(\frac{\nu}{m}\right)^{k\nu} x^{k\nu-1} e^{-\frac{\nu}{m}x} \qquad (\nu > 0).$$

Thus, useful tables [1] [2] are available for its integration from zero to any arbitrary upper limit.

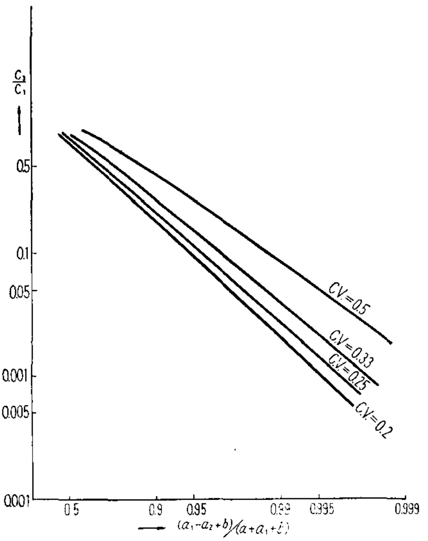

Fig. 3. (b)

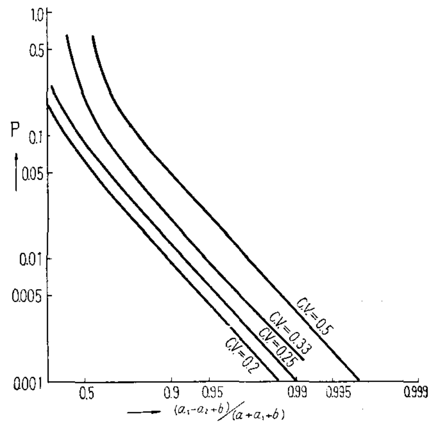

Fig. 3. (c)

USE OF HOLIDAYS

If we want to replace all parts together just like the type III, it is better to use holidays which come at regular intervals of x_p days, because the time loss necessary to replace parts is saved.

When this kind of maintenance is performed once in every r periodic holidays, the average C_4 of the total cost is expessed by the same equation as (10), thus :

$$C_4 = [(a+a_1+b)\{n(rx_p)-1\}+(a+a_2)]\times\frac{T}{rx_p} \tag{12}$$

where the value $n(rx_p)$ is given by Fig. 4.

Some integer r which minimizes C_3 given by (12) is the optimal period.

CASE OF MIXED CAUSES

If an accident should happen in a chemical plant, its whole function is generally stopped for a certain time period. In this case, preventive maintenance of other parts, which are still in their normal condition, may well be carried out in that repair period so as to avoid possible further accidents in the near future. The point to distinguish this case from the type III is that, in this case, there are many kinds of accidents, which have different cost parameters, and time intervals of which have their own distribution functions.

The same situation may prevail in the maintenance of railway vehicles, ships, automobiles, etc.

Dealing with this kind of maintenance, we use the following notations :

a_i, a_{i1}, a_{i2} are the cost parameters concerning the i-th kind of accidents.

b_i is the average loss brought about by the i-th kind of accident.

$n_i(x_p)$ is the average number of machine parts, belonging to the i-th kind, to cover a time length of x_p days.

The expected value C_5 of the total cost is given as follows :

$$C_5=\sum_i[(a_i+a_{i1}+b_i)\{n_i(x_p)-1\}+(a_i+a_{i2})]\times\frac{T}{x_p} \quad (13)$$

The optimal value of x_p which minimizes C_5 will be easily obtained referring to Fig. 4.

However, it is not always the best maintenance policy to repair completely all parts in every x_p days. Some parts may be best to be replaced in every $2x_p$ days or $3x_p$ days, and so on. Therefore, the expected value C_5 is

$$C_5=\sum_i[(a_i+a_{i1}+b_i)\{n_i(x_p)-1\}+(a_i+a_{i2})]\times\frac{T}{x_p}+\sum_j[(a_j+a_{j1}+b_j)\{n_j(2x_p)-1\}+(a_j+a_{j2})]\times\frac{T}{2x_p}+\cdots, \quad (14)$$

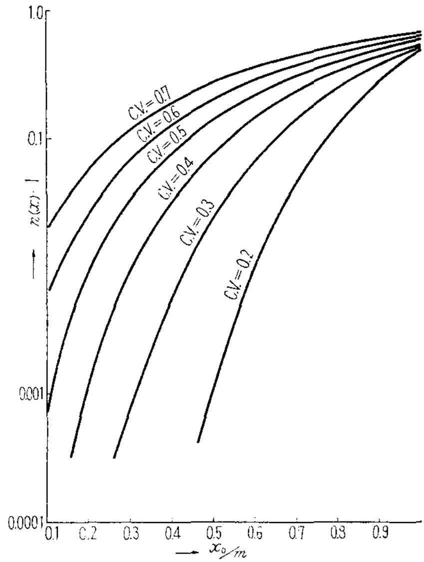

Fig. 4.

where Fig. 3 (*a*) is of use in classifying the types of accidents according to the time period rx_p in which it is desirable to take care of each type.

CONCLUSIONS

When this method of maintenance is applied appropriately, the total cost will be sharply decreased. However, the new problem will arise in finding out the distribution function of life time, because long-lived parts are replaced by new ones before their natural span of life come to its ends.

A literature [3] cited in the end of this paper is available in solving this problem.

Parts of this paper has been already published in Japanese.[4) 5)]

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