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A CERTAIN QUEUING PROBLEM OF TWO SERVICE STAGES AND THE EFFLUX DISTRIBUTION

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MOST OF THE PUBLISHED STUDIES of queues dealing with the type of problem in which one service operation is performed on each customer although one or more servers may be involved. Recently, a more general type of problem in which a sequence of service operations are served on each customer has discussed. [3] The problems considered in this paper are concerned with a two-stage queue, the first stage having general input, general service times and infinite number of servers, while the second stage has an arbitrary number of servers and general service times. In this case, it will be noted that the first service commences instantaneously when a customer arrives at the first stage. Also it is assumed that the second service operations perform on the customers transformed from the first stage to the second stage on the completion of their services and the customers are served in the order of arrival to the first stage. Thus the second stage has a queue. In the queuing system the waiting time of the i-th customer in the order of arrival at the first stage, i. e. the time which elapses between his arrival at the second stage and the begining of his second service is a chance variable whose distribution function depend on i. We shall consider a condition under which the distribution approaches a limit distribution function as $i \to \infty$. If the first stage has Poisson input and exponential service time, the efflux from the stage is also Poisson distributed with the same rate as the arrival rate. [1] Hence, this case can be simply discussed. In our case, since the first stage has general input and general service time, the distribution of efflux will be more complicated, but the condition for ergodicity of the above queuing system will be easily derived by using the similar methods as Lindley's [5] or Kiefer and Wolfowitz's. [4] Although the quoted statements are applied only to simple-server queue, the result will be generalized to multi-server queues.

Further, we shall treat the case where paying no attention to the order of arrival to the first stage the queues operate on a first-come and first-served basis only at the second stage.

Some notes of the distribution of the efflux from the queuing system will be given.

THE QUEUING SYSTEM OF TWO SERVICE STAGES

We shall now treat the case where the first stage has infinite number of servers and the second stage has a single server.

Let t_r be the time interval between the arrival of r-th and $(r+1)$ th customers to the first stage. Suppose that the t_r are independent random variables with identical probability distributions and the mean, $E(t_r)$, is finite. We put $E(t)=E(t_r)$. When the r-th customer arrives to the first stage, he receives immediately the first service. If his first service finishes when the second server is engaged with another customer, he has to wait in a queue, as his predecessors may be waiting. His second service time begins when his immediate predecessor's ((r-1)-th customer's) service time ends.

Let $s_r^{(1)}$ and $s_r^{(2)}$ be the first service time and second service time of the r-th customer, respectively. Suppose that $\{s_r^{(1)}\}$ and $\{s_r^{(2)}\}$ are sequences of random variables independently and identically distributed. Further suppose that $E(s_r^{(1)})$ and $E(s_r^{(2)})$ are finite, and three sets of random variables $\{s_r^{(1)}\}$, $\{s_r^{(2)}\}$, and $\{t_r\}$ $(r=1, 2, \cdots\cdots)$ are independent. We put $E(s^{(1)})=E(s_r^{(1)})$ and $E(s^{(2)})=E(s_r^{(2)})$.

Let w_r be the waiting time of the r-th customer at the second stage.

Then we have

$$\begin{aligned} w_{r+1} &= w_r+u_r, \qquad w_r+u_r>0, \\ &=0, \qquad w_r+u_r\leqq 0, \end{aligned} \tag{1}$$

where

$$u_r=s_r^{(2)}+s_r^{(1)}-s_{r+1}^{(1)}-t_r\ . \tag{2}$$

Let $F_r(x)$, $G_r(x)$ be the cumulative distribution function of w_r and u_r respectively. Then it will be shown that if $E(u_r)<0$ (we may write $E(u)<0$) w_r tends to a limit w as $r\rightarrow\infty$ by the same method as Lindley's.

When $P(u_r=0)<1$, the above condition $E(u)<0$ is the necessary and sufficient condition for the existence of a limit distribution of w_r.

If $E(u)\geqq 0$ and $P(u_r=0)<1$, the probability that the waiting time is not more than x tends to zero for any x.

Also it will be shown that, when it exists, this limit distribution is independent of the initial conditions.

From $E(u_r)=E(s_r^{(2)}-t_r)$, we note that the condition for the ergodic character of the waiting time in our system is independent of the first service time distribution.

Thus if $\rho \equiv \frac{E(s^{(2)})}{E(t)} < 1$, $F_r(x)$ tends to a limit distribution function $F(x)$. Then we have the following integral equation.

$$F(x)=\int_{x\geqq y} F(x-y)\,dG(y), \qquad (x<0) \ , \tag{3}$$

where $G(y)$ is the distribution function u.

This equation becomes the Wiener-Hopf equation which has been solved only in special cases.

Next, some special cases will be treated by the method which is found in Lindley's paper.

Let $G_1(y)$, $G_2(y)$ be the distribution function of $s_r^{(2)}+s_r^{(1)}-s_{r+1}^{(1)}$ and t_r respectively.

We shall consider first the case where the arrivals are random. Then

$$d\,G_2(y)=\lambda e^{-\lambda y}dy \ , \qquad (\lambda>0 \ , \ y>0). \tag{4}$$

Then the distribution function $G(y)$ has the following from:

$$G(y)=\lambda\int_0^\infty e^{-\lambda}G_1(y+z)\,dz. \tag{5}$$

Let $\phi(\tau)$, $\psi(\tau)$ be the characteristic functions of $F(x)$ and $G_1(y)$ respectively.

Then, it will be shown [5] that we have

$$\phi(\tau)=c\left[1+\lambda\cdot\frac{1-\psi(\tau)}{i\,\tau}\right]^{-1}, \tag{6}$$

where $$c=1-\frac{E(s_r^{(2)}+s_r^{(1)}-s_{r+1}^{(1)})}{E(t)}=1-\frac{E(s^{(2)})}{E(t)} \quad .$$

We shall note that c is the probability of not having to wait since

$$F(0) = \lim_{\tau\to\infty} \phi(\tau) = c.$$

In order to obtain the characteristic function $\phi(\tau)$, it is sufficient to find $\psi(\tau)$. Now, we shall consider the special case, where the distribution functions of $s^{(1)}$, $s^{(2)}$ are of the exponential functions type with $E(s^{(1)}) = \frac{1}{b_1}$ and with $E(s^{(2)}) = \frac{1}{b_2}$ respectively.

Then, by some simple calculations, we have the following form of the distribution function $G_1(x)$.

$$\begin{aligned} G_1(x) &= 1 - \frac{b_2}{2(b_2-b_1)} e^{-b_1 x} + \frac{b_1^2}{b_2^2-b_1^2} e^{-b_2 x}, \quad (x>0), \\ &= \frac{b_2}{2(b_2+b_1)} e^{b_1 x}, \qquad (x<0). \end{aligned} \tag{7}$$

Hence we have

$$\begin{aligned} \psi(\tau) &= \int_0^\infty e^{i\tau x}\left(\frac{b_2 b_1}{2(b_2-b_1)} e^{-b_1 x} - \frac{b_2 b_1^2}{b_2^2-b_1^2} e^{-b_2 x}\right) dx \\ &\quad + \int_{-\infty}^0 e^{i\tau x}\left(\frac{b_2 b_1}{2(b_1+b_2)} e^{b_1 x} dx\right. \\ &= \frac{b_2 b_1}{2(b_2-b_1)} \frac{1}{b_1 - i\tau} - \frac{b_2 b_1^2}{b_2^2-b_1^2} \frac{1}{b_2 - i\tau} + \frac{b_1 b_2}{2(b_1+b_2)} \frac{1}{b_1+i\tau} \\ &= \frac{b_1^2 b_2}{b_1^2+\tau^2} \cdot \frac{1}{b_2 - i\tau}. \end{aligned} \tag{8}$$

Thus, from (6) $\phi(\tau)$ can be obtained.

In the above case we shall find the mean $E(\tau)$ of the waiting time. By some calculations, we have

$$E(\tau) = \lambda\left(\frac{1}{b_1^2} + \frac{1}{b_2^2}\right) \Big/ \left(1 - \frac{\lambda}{b_2}\right) = \frac{b_2 \rho}{1-\rho}\left(\frac{1}{b_1^2} + \frac{1}{b_2^2}\right), \tag{9}$$

where

$$\rho = \frac{\lambda}{b_2}.$$

Next, we shall consider the case where the customers arrive at regular intervals, and for convenience we suppose the interval to the unit in the time scale.

Then we have

$$G(y) = \int G_1(y+z)\, d\, G_2(z) = G_1(y+1) \quad . \tag{10}$$

From this, the equation (3) becomes

$$F(x-1) = \int_{-\infty}^{x} F(x-y)\, d\, G_1(y) \ , \qquad (x \geqq 1) \tag{11}$$

where

$$G_1'(y) = \begin{cases} \dfrac{b_1 b_2}{2(b_2-b_1)} e^{-b_1 y} - \dfrac{b_1^2 b_2}{b_2^2-b_1^2}\, e^{-b_2 y} & y \geqq 0\ , \\[2ex] \dfrac{b_1 b_2}{2(b_1+b_2)} e^{b_1 y}\ , & y < 0\ . \end{cases} \tag{12}$$

Using the method in Lindley's paper, we shall solve the equation (11). Putting $F(z) = e^{zx}$ in the right-hand side of (11), we have

$$\int_{-\infty}^{0} e^{zx-zy} \frac{b_1 b_2}{2\,(b_2+b_1)}\, e^{b_1 y} + \int_0^x e^{zx-zy} \Big(\frac{b_1 b_2}{2\,(b_2-b_1)} e^{-b_1 y} - \frac{b_2 b_1^2}{b_2^2-b_1^2} e^{-b_2 y} \Big) dy$$

$$= \Big(\frac{b_2 b_1^2}{(b_1^2-z^2)\ (b_2+z)} e^{zx} - \frac{b_1 b_2}{2\,(b_2-b_1)}\, \frac{1}{b_1+z} e^{-b_1 x} + \frac{b_2 b_1^2}{b_2^2-b_1^2}\, \frac{1}{b_2+z} e^{-b_2 x},$$

The left-hand side of (11) becomes $e^{zx} e^{-z}$.

We shall, then, consider roots of the following equation

$$\frac{b_2\ b_1^2}{(b^2-z^2)\ (b_2+z)} = e^{-z}. \tag{13}$$

$z=0$ is certainly a root of (13). Further a simple calculation will show that there are two other negative real roots z_1, z_2.

Then it will be easily seen that the equation (11) has a solution $F(x)$:

$$F(x) = 1 + c_1 e^{z_1 x} + c_2 e^{z_2 x} \qquad x \geqq 0,$$
$$= 0, \qquad x < 0. \tag{14}$$

where c_1 and c_2 satisfy the following equations:

$$\frac{1}{b_1} + \frac{c_1}{b_1+z_1} + \frac{c_2}{b_1+z_2} = 0\,,$$
$$\frac{1}{b_2} + \frac{c_1}{b_2+z_1} + \frac{c_2}{b_2+z_2} = 0\,. \tag{15}$$

It is shown [5] that the solution $F(x)$ is a distribution function. From (15), we have

$$c_1 = \frac{z_2(b_1+z_1)(b_2+z_1)}{(z_1-z_2)\,b_1 b_2}\,,$$
$$c_2 = \frac{z_1(b_1+z_2)(b_2+z_2)}{(z_2-z_1)\,b_1 b_2}\,. \tag{16}$$

Hence, in this case the mean of the waiting time w is given by,

$$E(w) = -\frac{c_1}{z_1} - \frac{c_2}{z_2} = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{b_1} + \frac{1}{b_2}\,. \tag{17}$$

We shall remark that we can discuss the problem by the similar methods as Kiefer and Wolfowitz's when the second stage has many servers.

Further we shall consider the case where each customer at the second stage is served in the order in which he arrives to the second stage. If the first stage has Poisson input and exponential service time, our problem is reduced that of the one service stage, since the efflux from the stage is Poisson distributed with the same rate as the arrival rate.

Also, it is clear that when the first stage has a general input and constant service time, our problem becomes that of the one service stage. But it seems difficult to discuss the more general types of problems fully since the distribution of the efflux will be much complicated. Next, we shall give some notes on the efflux distribution.

THE EFFLUX DISTRIBUTION

It was previously considered by Morse that the efflux of a queuing system with Poisson input and exponential service times makes again Poisson process identical with the input, and this has been recently shown by Burke. Now we shall give another proof and consider the case where no queues is allowed.

Let the number of servers be n and the distribution of r, the number of customers arriving in any interval t be given by the Poisson law:

$$e^{-at}(at)^r/r! \qquad (a>0\ ,\ r=1,2\cdots\cdots). \tag{18}$$

Further let the probability density function of the service time be of the exponential type:

$$b\,e^{-bt}, \qquad (b>0). \tag{19}$$

Now let us write $W_{i,k}(t)$ for the conditional probability of the number k of efflux from the system during a period of time t when i customers are waiting or being served at the commencement of the period. Let $p_{i,m}(t)$ be the conditional probabpility that when i customersare waiting or being served at the commencement of a period of time t, exactly m units among them finish their service times during the period. Then we have the following result.

In case $i \geqq k$:

$$W_{i,k}(t)=\int_0^t \sum_{m=0}^{k} p_{i,m}(x)\,e^{-ax}\,W_{i-m+1,k-m}(t-x)\,adx$$
$$+p_{i,k}(t)\,e^{-at}\ . \tag{20}$$

In case $i<k$:

$$W_{i,k}(t)=\int_0^t \sum_{m=0}^{i} p_{i,m}(x)\,e^{-ax}\,W_{i-m+1,\,k-m}(t-x)\,adx. \tag{21}$$

Let $\varphi_i(z,t)$ be the moment generating function of $\{W_{i,k}(t)\}$. Then from (20), (21) we have

$$\varphi_i(z,t) \equiv \sum_{k=0}^{\infty} W_{i,k}(t) z^k = \sum_{k=0}^{i} \int_0^t \sum_{m=0}^{k} p_{i,m}(x) e^{-ax} z^m W_{i-m+1,k-m}(t-x) z^{k-m} a dx$$

$$+ \sum_{k=0}^{i} p_{i,k}(t) e^{-at} z^k$$

$$+ \sum_{k=i+1}^{\infty} \int_0^t \sum_{m=0}^{i} p_{i,m}(x) e^{-ax} z^m W_{i-m+1,\, k-m}(t-x) z^{k-m} a dx$$

$$= a \int_0^t \sum_{m=0}^{i} p_{i,m}(x) e^{-ax} z^m \sum_{k=m}^{i} W_{i-m+1,k-m}(t-x) z^{k-m} dx$$

$$+ \sum_{k=0}^{i} p_{i,k}(t) e^{-at} z^k$$

$$+ a \int_0^t \sum_{m=0}^{i} p_{im}(x) e^{-ax} z^m \sum_{k=i+1}^{\infty} W_{i-m+1,\, k-m}(t-x) z^{k-m} dx$$

$$= a \int_0^t \sum_{m=0}^{i} p_{im}(x) e^{-ax} z^m \varphi_{i-m+1}(z, t-x)\, dx$$

$$+ \sum_{k=0}^{i} p_{i,k}(t) e^{-at} z^k. \tag{22}$$

Now, we may consider the following Laplace transforms

$$\Phi_i(z,s) = \int_0^{\infty} \varphi_i(z,t) e^{-sbt} dt\ , \tag{23}$$

$$P_{im}(s) = \int_0^{\infty} p_{im}(x) e^{-ax} e^{-sbx} dx. \tag{24}$$

Then from (22) we have

$$\Phi_i(z,s) = a \sum_{m=0}^{i} P_{im}(s) z^m \Phi_{i-m+1}(z,s) + \sum_{m=0}^{i} P_{im}(s) z^m.$$

To obtain $P_{im}(s)$, we start with the following relations.
In the case $i \geqq n$:

$$p_{im}(x) = \int_0^x n b e^{-nbv} p_{i-1,\, m-1}(x-v)\, dv \tag{25}$$

In the case $i < n$:

$$p_{im}(x) = \int_0^x i\, b\, e^{-ibv}\, p_{i-1,\, m-1}(x-v)\, d v \, . \tag{26}$$

By a simple calculation, we have

$$P_{im}(s) = \frac{n}{n+\rho+s} P_{i-1,\, m-1}(s), \qquad (m \geqq 1,\ i \geqq n), \tag{27}$$

$$P_{im}(s) = \frac{i}{i+\rho+s} P_{i-1,\, m-1}(s), \qquad (m \geqq 1, 0 < i < n), \tag{28}$$

$$P_{io}(s) = \frac{1}{b} \cdot \frac{1}{n+\rho+s}, \qquad (i \geqq n), \tag{29}$$

$$P_{io}(s) = \frac{1}{b} \cdot \frac{1}{i+\rho+s}, \qquad (0 \leqq i \leqq n), \tag{30}$$

where $\rho = \dfrac{a}{b}$.

From (27), (28), (29) and (30), we can obtain all P_{im}, $(i \geqq m)$. In order to prove the quoted result, we examine the case $n=3$. (General cases being proved similarly). Thus taking $n = 3$ and putting $\Phi_i \equiv \Phi_i(s, z)$, we have

$$\Phi_0 = \frac{\rho}{\rho+s}\Phi_1 + \frac{1}{\rho+s} \cdot \frac{1}{b},$$

$$\Phi_1 = \frac{\rho}{1+\rho+s}\Phi_2 + \frac{z}{1+\rho+s}\Phi_0 + \frac{1}{1+\rho+s} \cdot \frac{1}{b},$$

$$\Phi_2 = \frac{\rho}{2+\rho+s}\Phi_3 + \frac{2z}{2+\rho+s}\Phi_1 + \frac{1}{2+\rho+s} \cdot \frac{1}{b}, \tag{31}$$

$$\Phi_3 = \frac{\rho}{3+\rho+s}\Phi_4 + \frac{3z}{3+\rho+s}\Phi_2 + \frac{1}{3+\rho+s} \cdot \frac{1}{b},$$

$$\Phi_4 = \frac{\rho}{3+\rho+s}\Phi_5 + \frac{4z}{3+\rho+s}\Phi_3 + \frac{1}{3+\rho+s} \cdot \frac{1}{b},$$

. . . .

Let p_k be the steady state (time independent) probability that there are k customers in the system waiting and in service.

It is known that p_k, are given by

$$p_k = p_0 \frac{\rho^k}{k!}, \qquad (0 \leqq k < 3),$$
$$p_k = p_0 \frac{3^3}{3!}\left(\frac{\rho}{3}\right)^k, \qquad (k \geqq 3), \tag{32}$$

where p_0 is determined by $\sum_{k=0}^{\infty} p_k = 1$.

Now, we may put such that

$$\Psi \equiv p_0\Phi_0 + p_0\rho\Phi_1 + p_0\frac{\rho^2}{2!}\Phi_2 + \sum_{r=3}^{\infty} p_0 \frac{3^3}{3!}\left(\frac{\rho}{3}\right)^r \Phi_r. \tag{33}$$

By some calculations, we have

$$\Psi = \frac{1}{b\ (\rho + s - \rho z)} \tag{34}$$

But the Laplace transform of Poisson distribution is given by

$$\int_0^{\infty} e^{-bst} \sum_{k=0}^{\infty} \frac{e^{-at}(at)^k}{k!} z^k\, dt = \frac{1}{b(\rho + s - \rho z)} . \tag{35}$$

Hence, from (34) and (35) we obtain the required result.

The above method is not simpler than Burke's. But by this method the efflux distribution in the case where no queue is allowed, will be discussed.

In this case, we shall treat the case $n=3$. Using the previous notations, we obatin the integral equation (22). From this, we have

$$\Phi_0 = \frac{\rho}{\rho + s}\Phi_1 + \frac{1}{\rho + s} \cdot \frac{1}{b} ,$$

$$\Phi_1 = \frac{\rho}{1 + \rho + s}\Phi_2 + \frac{z}{1 + \rho + s}\Phi_0 + \frac{1}{1 + \rho + s} \cdot \frac{1}{b} ,$$

$$\Phi_2=\frac{\rho}{2+\rho+s}\Phi_3+\frac{2z}{2+\rho+s}\Phi_1+\frac{1}{2+\rho+s}\cdot\frac{1}{b},$$

$$\Phi_3=\frac{\rho}{3+\rho+s}\Phi_3+\frac{3z}{3+\rho+s}\Phi_2+\frac{1}{3+\rho+s}\cdot\frac{1}{b}.$$

We can show that one and only one solution of the above equation exists. The existence of the solution of the integral equation (22) and its uniqueness will be shown. [2)]

In order to obtain explicit expressions we shall put

$$\triangle\equiv\begin{vmatrix} s+3 & -3z & 0 & 0 \\ -\rho & \rho+2+s & -2z & 0 \\ 0 & -\rho & \rho+1+s & -z \\ 0 & 0 & -\rho & \rho+s \end{vmatrix}.$$

If we denote the determinant obtained by replacing the $(i+1)$-th row of $\triangle$ by $\left(\frac{1}{b},\frac{1}{b},\frac{1}{b},\frac{1}{b}\right)$ by $\triangle_i$, then we have

$$\Phi_3=\frac{\triangle_0}{\triangle},\quad \Phi_2=\frac{\triangle_1}{\triangle},\quad \Phi_1=\frac{\triangle_2}{\triangle},\quad \Phi_0=\frac{\triangle_3}{\triangle}.$$

In this case, the equilibrium probabilities p_k's are given by

$$p_k=\rho^k\Big/\sum_{r=1}^{3}\frac{\rho^r}{r!},\qquad (0\leqq k\leqq 3).$$

Thus, the Laplace transform of the moment generating function of the efflux distribution in the case where no queue is allowed, is given by

$$\left(\frac{\rho^n}{n!}\Phi_n+\frac{\rho^{n-1}}{(n-1)!}\Phi_{n-1}+\cdots\cdots+\frac{\rho}{1!}\Phi_1+\Phi_0\right)\Big/\sum_{r=0}^{n}\frac{\rho^n}{r!}.$$

This result is different from the privious result. In the case $(n=\infty)$, discussed in the previous part it will be shown by the similar method that the efflux distribution is Poissonian.

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