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DECISION OF AN OPTIMUM TOLL RATE

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IN THIS REPORT we shall formulate a rule for decision of an optimum toll rate of a toll road in view of maximization of income therefrom.

Objections are ready against any such attempt to fix a system of charging toll on such public characters as toll roads, but we are confronted with the problem to make a schedule of toll rates in some way or other.

The toll road system in Japan was set very recently which is now governed by the Japan Highway Public Corporation.

The present study is only a model, a very simplified one, of a system of toll road. In application, the result should be carefully examined as there are many simplified assumptions involved, which we do not claim to be very near to reality.

Whether a passenger passes a toll road or he chooses other road if there is, or whether he wholly gives up the idea of making a travel depends upon the utility he expects to get by using the road (the benefit of the road) and the toll he has to pay in passing the toll road. If the utility exceeds the toll he has to pay, we simply assume, he takes the road, otherwise he does not. This assumption will sometimes be violated, for the utility or benefit of the road is hard to calculate. There will be other factors which are more decisive than this in the determ ination of one's passing a toll road. However, on a purely economical reasoning the idea is fairly natural,

Let the benefit of the toll road for a passenger be denoted by b. This b varies with the individual passenger and has a distribution function $F(b)$. Let n denote the number of times a passenger uses the toll road in a given interval of time (say one month or one year).

Let the mean value of n be denoted by λ, which also varies with the individual passenger and has a distribution function $H(\lambda)$.

Now we are in a position to formulate the rule of passenger as:

i) if a passenger's benefit b is larger than the toll x, *i. e.* if $b>x$, then the passenger takes the toll road,

ii) otherwise, he does not take the toll road.

Let $P(n, \lambda)$ denote the probability that a passenger whose mean frequency of using the toll road is λ actually uses it n times in the given interval of time. We have

$$\sum_{n=0}^{\infty} nP(n, \lambda) = \lambda.$$

If the toll rate is fixed at x we can calculate the expected income in the given interval of time as

$$E(I|x) = \sum_{n=0}^{\infty}\int_{b>x} nxP(n, \lambda)\, H(d\lambda) = x\int_{b>x} \lambda H(d\lambda). \qquad (1)$$

In order to perform the integral above, we must know the relation between λ and b,

$$\lambda = g(b).$$

It would be natural to assume that g is a monotone increasing function of b, so that we have

$$E(I|x) = x\int_{\lambda>g(x)} \lambda H(d\lambda). \qquad (2)$$

The relationship between H, F and g is

$$F(x) = H[g(x)] \qquad (3)$$

Our aim now is to find the x which maximizes the $E(I|x)$. Hence we have to solve the equation

$$\frac{\partial E}{\partial x} = 0$$

or

$$\int_{\lambda > g(x)} \lambda h(\lambda)\, d\lambda = x[g(x)\, h(g(x))]g'(x) = xg(x)f(x), \qquad (4)$$

where

$$h(x) = H'(x), \quad f(x) = F'(x).$$

In order to solve the above equation we must know the functions f, g and h. However we have no information about these functions. We do not even have statistical data of these functions, so that the following calculation is only for a trial.

We assume that the function $g(x) = Cx$, *i. e.* the individual benefit and the frequency of his using the toll road is proportional.

In this case without loss of generality we can take $C=1$.

The equation is then

$$\int_x^\infty \lambda h(\lambda)\, d\lambda = x^2 h(x). \qquad (5)$$

We shall solve the equation (5) for some typical distribution functions $h(\lambda)$.

1) Let $h(\lambda)$ be the uniform distribution over the interval (a, b)

$$h(\lambda) = \frac{1}{b-a} \quad \text{for} \quad a < \lambda < b,$$

$$= 0 \qquad \text{otherwise.}$$

In this case equation (5) becomes

$$b^2 - x^2 = 2x^2 \quad \text{for} \quad a < x < b, \qquad (6)$$

which yields

$$x = b/\sqrt{3}. \qquad (7)$$

Therefore, if

$$a < b/\sqrt{3}$$

(7) really gives the maximizing value x. However, if

$$b/\sqrt{3} \leqq a$$

(7) is not a solution of (6), the equation (5) has no solution. In this case the optimum value x should be $x=a$.

2) Let $h(\lambda)$ be the Γ-distribution given by

$$h(\lambda)=\begin{cases} \dfrac{1}{a^p\Gamma(p)}\lambda^{p-1}e^{-\lambda/a}, & 0<\lambda<\infty, \\ 0 & , \quad \lambda\leqq 0 \end{cases}$$

with a positive integral exponent p. In this case equation (5) becomes

$$\int_x^\infty \lambda^p e^{-\lambda/a}d\lambda = x^{p+1}e^{-x/a}$$

After a calculation this equation becomes

$$\left(\frac{x}{a}\right)^p\frac{1}{p!}+\left(\frac{x}{a}\right)^{p-1}\frac{1}{(p-1)!}+\cdots+\left(\frac{x}{a}\right)+1=\left(\frac{x}{a}\right)^{p+1}\frac{1}{p!} \qquad (8)$$

For $p=1$ (negative exponential distribution) (8)

is,

$$\left(\frac{x}{a}\right)^2-\left(\frac{x}{a}\right)-1=0.$$

Hence

$$x=\frac{1+\sqrt{5}}{2}a=1.6a.$$

3) Let $h(\lambda)$ be a so called logarithmic normal distribution, *i. e.* logarithm of mean frequency λ is normally distributed. We shall give some numerical solutions of the equation (5) for various parameters.

By definition we have

$$h(\lambda)=\frac{1}{\sqrt{2\pi}\,\sigma\lambda}e^{-(\log\lambda-m)^2/2\sigma^2}$$

where m and σ are constants and are related to the mean M and the variance V of $h(\lambda)$ by the following relations,

$$M=e^{m+\sigma^2/2},\qquad V=M^2(e^{c^2}-1)$$

Equation (5) becomes

$$\frac{\sigma}{\sqrt{2\pi}}\int_{\xi}^{\infty}e^{-\frac{z^2}{2}}dz=\frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}$$

where

$$\xi=\frac{\log x-m}{\sigma}-\sigma,$$

or

$$\frac{1}{\sqrt{\log\left(1+\frac{V}{M^2}\right)}}=\frac{1}{\sigma}=e^{\frac{\xi^2}{2}}\int_{\xi}^{\infty}e^{-\frac{z^2}{2}}dz \tag{9}$$

Equation (9) can be solved for ξ by using the table of normal probability, and the optimum x is given by

$$x=e^{\sigma\xi}\sqrt{M^2+V} \tag{10}$$

We give a table and a graph of x for $V=1$.

M	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
x	10.32	5.66	3.56	2.62	2.10	1.82	1.65	1.55	1.48	1.43	1.43

M	1.3	1.4	1.5	1.7	2.0	2.2	2.5	3.0
x	1.44	1.44	1.48	1.56	1.71	1.82	2.02	2.36

For example, if the distribution of frequency has mean $M=1.0$ and variance $V=1$, then the optimum $x=1.48$, *i. e.* if the benefit of the toll road for a passenger who uses the road once on an average in the given interval of time is calculated to be C yen, then if we decide the toll rate at 1.48 C yen, the income would be maximum.

We note that from the equations (9) and (10), x/M is a function of $\sqrt{V}/M$, *i. e.*

$$\frac{x}{M}=\phi\left(\frac{\sqrt{V}}{M}\right).$$

This shows that there is a similarity between graphs for various V' s. This similarity is useful in calculating the x's for various values of V. We have

$$x/M=\phi(1/M/\sqrt{V})=x'/M/\sqrt{V}. \qquad \text{so that} \quad x=x'\sqrt{V},$$

where x' denotes the value of x for variance 1 and mean value $M/\sqrt{V}$. In the figure the broken lines through the points 1.0 and $\sqrt{V}$ on the M axis are parallel, showing the similarity.

As is seen from the figure, x as a function of M has a minimum point at some M. This would seem paradoxical because larger the M, higher could the toll be charged. However in this case, taking a higher toll can yield more income than would be compensated by the reduction in number of passengers using the toll road.

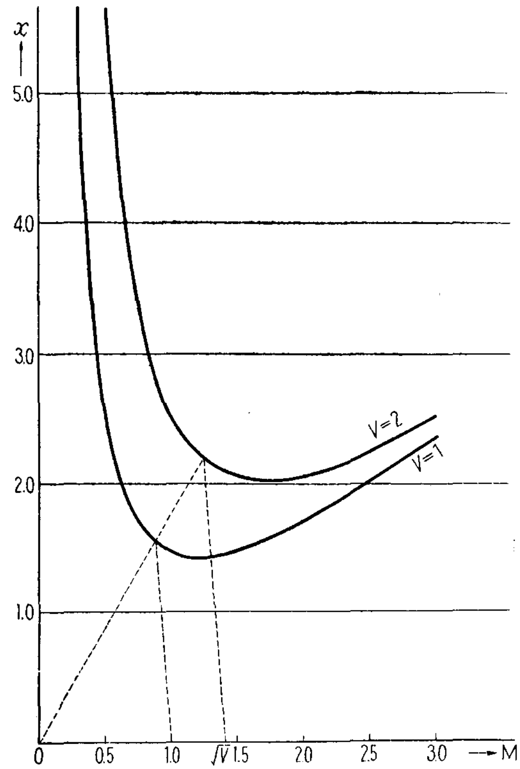

Fig. 1. Optimum toll rate x when the mean frequency λ has a logarithmic normal distribution with mean value M and variance V.