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RENEWAL THEOREMS IN THE PROBLEMS OF QUEUES*

TATSUO KAWATA
Tokyo Institute of Technology

INTRODUCTION

THE BEHAVIOR OF QUEUES is strongly affected by the distribution of successive arrival times of customers as well as customer servicing times. In classical theory of queues, it used to be assumed that each distribution depends on exponential law, in which case the mathematical treatment is the easiest. For instance we can find the probability of queue lengths at counters in an ergodic case (or an equilibrium), the mean waiting time of a newly arrived customer, the autocorrelation of the process of queues and so on. In recent years many authors have attempted to generalize the theory to the case where either or both of the distributions of servicing and arrival times obey the general distribution or other type of the distribution.

STANDING TIME PROBLEM

Dealing with some actual problems, we have sometimes met to consider the case where certain distributions of arrivals rather complicated took place. For example, in the problem of the standing time of freight cars in a marshalling yard, we had to consider the following situation. [1),2)]

The formation of a train composed of some fixed number h of cars in a classification track is considered as an arrival of a customer in the theory of queues. A freight train for some direction is composed by successive cars for the destination in arrival trains. Hence let

$$X_1,\ X_2,\ \cdots \tag{1}$$

be the number of cars for the destination in successive arrival trains. Then we ask the distribution of n which is the largest number such that

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$$X_1+X_2+\cdots+X_n>h. \tag{2}$$

If freight trains arrive at the yard regularly every a minutes, then the arrival interval is

$$a\cdot n \tag{3}$$

If successive arrival intervals of trains are

$$a_1,\ a_2,\ a_3,\ \cdots \tag{4}$$

then we ask the distribution of

$$a_1+a_2+\cdots+a_n. \tag{5}$$

Having been composed, a train undergoes some service (for example to be led to and set up at the departure line).

SOME RENEWAL THEOREMS

The problem concerning n in (2) appears as is well known in the problem in connection with life times of individuals and the renewal theory deals with it.

In the renewal theory, we shall further consider the distribution of

$$h-\sum_1^\infty X_k,$$

in other words, the age distribution at time h if we consider X_i as life time of an individual.

Here we denote n as $n(h)$ which is a random variable. We shall state some known results.

Let X_1, X_2, $\cdots\cdots$ be a sequence of random variables identically and independently distributed. Let the mean of X_i be $EX_i=m$. Then

Theorem 1. $$\lim_{h\to\infty}\frac{n(h)}{h}=\frac{1}{m}$$

with probability 1.

Theorem 2. $$\lim_{h\to\infty}\frac{En(h)}{h}=\frac{1}{m}.$$

These were proved first by W. Feller[3] using Laplace transform methods. Täcklind has obtained accurate asymptotic expressions for $En(h)$.

We shall consider the number N of n such that

$$x\leqq X_1+X_2+\cdots+X_n<x+h \tag{6}$$

the expectation of N being expressed as

$$EN=\sum_{n=1}^{\infty}P(x\leqq X_1+X_2+\cdots+X_n<x+h). \tag{7}$$

J. L. Doob[4] proved that

Theorem 3. $$\lim_{x\to\infty}EN=\frac{h}{m},\quad m>0 \tag{3}$$

for fixed h.

Many authors haves discussed the situation. When X_1, X_2, $\cdots$ are not necessarily distributed identically, D. R. Cox and W. L. Smith[5] treated the problem and under certain cmditions, requiring that the distributions of X_i are not so much different each other, they proved (8) with

$$m=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}EX_i.$$

But in general it would be more natural to replace the mean limit in the left side of (8). We then have

Theorem 4. *If* $EX_i=m_i>0$ *and*

$$\lim_{n\to\infty}\frac{m_1+\cdots+m_n}{n}=m,\quad m>0, \tag{9}$$

then we have

$$\lim_{X\to\infty}\frac{1}{X}\int_0^X EN\cdot dx=\frac{h}{m}. \tag{10}$$

This is due to the author.[6]

If X_1, X_2, $\cdots$ will take times

$$a_1,\ a_2,\ \cdots \tag{11}$$

respectively to take their values, for instance as in the example in 2. Then we must consider

$$\sum_{n=1}^{\infty} a_n P(0\leqq X_1+\cdots+X_n<h)$$

which means the expectation of $a_1+\cdots+a_N$, instead of $En(h)$ in 1, and we consider

$$U(x,\ h)=\sum_{n=1}^{\infty} a_n P(x\leqq X_1+\cdots X_n<x+h) \tag{12}$$

Then we have the theorem*

Theorem 5. *If* (9) *is assumed and*

$$\lim_{n\to\infty}\frac{\alpha_1+\cdots+\alpha_n}{n}=\alpha \tag{13}$$

then

$$\lim_{X\to\infty}\frac{1}{X}\int_0^X U(x,\ h)\,dx=\frac{h\alpha}{m} \tag{14}$$

To evaluate the exact value of the expectation $a_1+\cdots+a_N$ seems not to be easy and we can use (14) as an approximate value of it.

THEOREMS CONCERNING AGE DISTRIBUTION

Concerning the age distribution, that is, the distribution of

$$R(h)=h-\sum_{1}^{n(h)} X_k \tag{15}$$

* The proof is done dy completely similar arguments as in Theorem IV.

where $n(h)$ is the largest integer n such that

$$0<X_1+\cdots+X_n<h,$$

the following fact is own:

Theorem 6. *If X_1, X_2. $\cdots\cdots$ are independently and indentically distributed with the finite second moment μ_2', then*

$$\lim_{h\to\infty}\frac{1}{H}\int_0^H R(h)\,dh=\frac{\mu_2'}{2m}. \tag{16}$$

This is due to J. L. Doob.[4]

To apply the theorem, it is necessary to know the second moment which seems to be difficult to find. Hence we shall estimate the second moment by another value. Connecting with this we shall have the following theorem.

Theorem 7.

$$\lim_{X\to\infty}\frac{1}{X}\int_0^X\sum_{n=1}^{\infty}\left(n-\frac{x}{M_n}\right)P(x\leqq X_1+\cdots+X<x_n+h)\,dx=\frac{h}{m^2}\left(\frac{v}{m}+\frac{h}{2}\right), \tag{17}$$

where $M_n=\frac{1}{n}\sum_1^n m_i\to m$ and v is the variance of X_i.

The left side is rather artificial but the one corresponding to $\left(\mu_2'+\frac{h}{m}\right)$ /2, which we have applied to the research of the problem of the classification of freight cars. This is due to H. Morimura.[7]

ANOTHER REMARK

To evaluate the distribution function of N or $n(h)$ seems to be difficult. We have concerning this, the following limit theorem.*

Theorem 8. *If $\{X_i\}$ is a sequence of random variables identically and independently distributed with the mean $m>0$ and the variance v, and $\{a_i\}$ is also a sequence of random variables obeying the exponential law with the mean a, then the distribution of*

* Not yet published.

$$\frac{\sqrt{2}\,(\pi mv)^{\frac{1}{4}}}{hx^{\frac{1}{4}}}\left\{\sum_{n=1}^{\infty} a_n P(x \leqq X_1+\cdots+X_n < x+h) - \frac{ah}{m}\right\}$$

tends to the normal distribution $N(o,\ a^2)$ *as* $x \to \infty$.
This is due to H. Morimura.

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