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A BAYESIAN SEQUENTIAL SINGLE MACHINE BATCHING AND SCHEDULING PROBLEM WITH RANDOM SETUP TIME

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Abstract We consider single machine batching and scheduling problem in which the processing time of each job is known, but the setup time is a random variable from the distribution with an unknown parameter. Batch sizes are determined sequentially, that is, the size and jobs of the first batch are determined by using the prior knowledge and observed the value of a setup time, then the size and jobs of the second batch are determined by using the value of the first setup time, and so on. Since this problem is a sequential decision problem, it is formulated by dynamic programming and several properties are obtained.

Keywords: Dynamic programming, Bayesian learning, batching, scheduling

1. Introduction

Batching and scheduling problem considered in this paper is the problem to decide both the batch size and the sequence of jobs to be processed sequentially so as to minimize the sum of the total completion times of all the jobs. In this kind of problem, the length of setup time is the important factor to influence the optimal solution. The setup time of this paper is a random variable with an unknown parameter and the Bayesian learning mechanism is included.

There are several cases that the setup time is a random variable, for example, the time of the material handling to bring the raw materials from the warehouse to the workstation is not constant but a random variable if they were brought by a worker. In this case, the distribution function usually has several parameters whose values are known if the material handling is always done by the same worker. If a new worker begins to work in this workstation and he brings the raw material from the warehouse to the workstation, the values of several parameters are unknown. In this case, we can update the values of the unknown parameters by gathering the information about the time of each material handling and using the Bayesian learning method.

Batching and scheduling problem has been studied in these twenty years mainly in the deterministic case where both the setup time and the processing times are known and the problems are static in the sense that sizes of all the batches are determined at once. This kind of problem is discussed in Santos and Magazine [9], where the jobs are said to be in batch availability. Dobson, Karmarker and Rummel [4] called this as batch flow problem. Naddef and Santos [7] considered one-pass batching algorithm to construct the optimal solution. Coffman, Yannakakis, Magazine and Santos [2] found the property that sequencing of jobs in increasing order of processing time is optimal.

Dynamic programming is useful in several kinds of batching and scheduling problem, for example, Baker [1], Coffman, Yannakakis, Magazine, and Santos [2], Wagelmans and

Gerodimos [11], Ng, Cheng and Kovalyov [8]. In this paper, we formulate the model by using Bayesian dynamic programming.

In the case that the setup time is a constant, the optimal batch sizes and sequence of jobs are predetermined. If the setup time is short or long, the optimal batch size is small or large, respectively. In this paper, we assume that the setup time is not constant but the random variable whose distribution has at least one unknown parameter with some prior information. After observing the setup time of the first batch, we will determine the second batch size by using the expected setup time updated by the Bayesian learning mechanism.

Some number of jobs are selected as a batch and processed. After observing the value of a setup time, we will obtain the posterior information by using the value and the prior information. Now, this posterior information is used as a new prior information and a new batch size at the next stage is decided. This process is repeated until all the jobs have been processed. In this process, a Bayesian learning mechanism is included and the problem is now the sequential decision problem with learning. The special case, where all the processing times are the same and the setup time is distributed in the gamma distribution with one unknown parameter, was discussed in Hamada [5].

In Section 3, the problem is formulated by dynamic programming and after deriving some properties, more simple recursive equations are obtained. Some properties for this recursive equations and the optimal strategy are derived in Section 4. The optimal strategy for the special case of sequence of processing times is also derived in Section 5. The properties of that case of sequence of processing times are discussed in Section 6.

2. Model

Consider the case that several size of boards have to be cut by a cutting machine. For example, there are n boards, $1, 2, \cdots, n$, of the same length $2l$ with width $w_1, w_2, \cdots, w_n$, respectively, in a warehouse and we bring some of them from the warehouse to the workstation where we cut them to get the board with length l. If the several width of board had to be cut by the cutting machine with the same cutting speed, the time to cut each board would be in proportion to its width. Let p_i for $i = 1, 2, \cdots, n$ be the time to cut the board i. If the boards i and j are cut without any interval time, the total time to cut them is $p_i + p_j$. Let X be the time to bring boards from warehouse to the workstation. Even if we bring two or more boards simultaneously, the time to bring them is X. The total time to bring boards i and j from the warehouse to the workstation and cut them continuously is $X + p_i + p_j$. This is the motivation of our model discussed in this paper.

The Bayesian sequential single machine batching and scheduling problem considered in this paper is described as follows: There are n jobs, $1, 2, \cdots, n$, which is processed by a single machine. Let $\Omega = \{1, 2, \cdots, n\}$, the set of all the jobs to be considered and to be available at time 0. The processing time of job i ($i = 1, 2, \cdots, n$) is p_i, which is a known value and satisfies $p_1 \geq p_2 \geq \cdots \geq p_n$. Let $\Lambda_n = \{(p_1, p_2, \cdots, p_n) | 1 \geq p_1 \geq p_2 \geq \cdots \geq p_n > 0\}$. Several number of jobs are selected as a batch and these selected jobs are processed one by one sequentially. Let B_1 be the set of all the jobs in the first batch. After completing all the jobs in the batch, some jobs are selected from $\Omega \setminus B_1$ as the jobs of the second batch B_2 and processed, where $\Omega \setminus B_1$ is the set of jobs which are included in Ω, but not in B_1. Jobs of the third batch B_3 is selected from $(\Omega \setminus B_1) \setminus B_2$ and processed. We continue to make the batch until the set of remaining jobs is empty. The completion time of each job in a batch is the same as that of the last job in the batch. A setup time is necessary before processing the first job in a batch. Therefore, the completion time of the k-th batch is the sum of the completion time of the last job of $(k-1)$-th batch, the setup time of k-th batch, and sum of all the processing times of jobs in the k-th batch. The setup time is not a constant but a random variable whose distribution function is $F(x|\theta)$, but the true value of the parameter θ is unknown and there is a conjugate prior distribution $G(\theta|u,v)$, where (u,v) is the parameter vector with $u>0$ and $v>0$. Let $(\varphi(u,v;X),\psi(u,v;X))$ be the parameter vector of the posterior distribution of θ after observing a setup time X when the parameter vector of θ is $(u,v)\in W$, where W be the set of all the possible vector values of (u,v). When the current set of the remaining jobs is $S(\subset\Omega)$, several jobs are selected from S as a batch and after observing the setup time X, all the jobs in the batch are processed one by one, and after completing the last job in the batch, several jobs of the next batch is selected by using the posterior distribution $G(\theta|\varphi(u,v;X),\psi(u,v;X))$ which is derived from both $G(\theta|u,v)$ and X by using the Bayes' Theorem. The objective is to minimize the expected sum of completion times of all the jobs of Ω.

Let

$$h(u,v)=\mathrm{E}_\Theta[\mathrm{E}_X[X|\Theta]|u,v],$$

where $\mathrm{E}_X[\cdot|\Theta]$ is the expectation operation with the distribution $F(x|\theta)$ and $\mathrm{E}_\Theta[\cdot|u,v]$ is that with the distribution $G(\theta|u,v)$. Several distribution with an unknown parameter has its own conjugate prior distribution (See, for example, DeGroot [3]), that is, the posterior distribution is the same kind as the prior one except that values of the parameters are revised.

Now, we make five assumptions, whose second, third, and forth are the assumptions A_3, A_4, and A_1, of Hamada and Ross [6], respectively.

Assumption 1. For any continuous and strictly increasing function $f(x)$ of x, $\mathrm{E}_\Theta[\mathrm{E}_X[f(X)|\Theta]|u,v]$ is continuous and strictly increasing in u and continuous and strictly decreasing in v.

Assumption 2. $h(u,v)>0$.

Assumption 3. For any $c>0$ and $v>0$, the equation $h(u,v)=c$ of u has a unique root in the interval $(0,\infty)$.

Assumption 4. $\varphi(u,v;x)$ is continuous in u, nondecreasing in u and x, and nonincreasing in v, and $\psi(u,v;x)$ is continuous in v, nonincreasing in u and x, and nondecreasing in v.

Assumption 5. $\mathrm{E}[h(\varphi(u,v;x),\psi(u,v;x))|u,v]=h(u,v)$.

As an immediate consequence of Assumption 1, we have some properties of $h(u,v)$ as follows:

Lemma 2.1 *$h(u,v)$ is continuous and strictly increasing in u and continuous and strictly decreasing in v.*

For example, consider the gamma distribution with an unknown parameter θ whose density function is

$$f(x|\theta,\alpha)=\begin{cases}\Gamma(\alpha)^{-1}\theta^\alpha x^{\alpha-1}e^{-\theta x}, & \text{if } 0<x,\\ 0, & \text{otherwise,}\end{cases}$$

where α is a known parameter. The unknown parameter θ has the gamma distribution with the density function

$$g(\theta|u,v)=\begin{cases}\Gamma(v)^{-1}u^v\theta^{v-1}e^{-u\theta}, & \text{if } 0<u,\\ 0, & \text{otherwise,}\end{cases}$$

as the conjugate prior distribution. In this case, Assumption 1 is satisfied with Lemma 3.1 of Hamada [5], and

$$h(u,v)=\alpha u(v-1)^{-1},$$

which satisfies Assumptions 2 and 3. After observing $X = x$, the posterior distribution is

$$g(\theta|\varphi(u,v;x),\psi(u,v;x)) = \begin{cases} \Gamma(v+\alpha)^{-1}(u+x)^{v+\alpha}\theta^{v+\alpha-1}e^{-(u+x)\theta}, & \text{if } 0 < u, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$\varphi(u,v;x) = u + x$$

and

$$\psi(u,v;x) = v + \alpha,$$

which satisfy Assumption 4. Assumption 5 is satisfied with

$$h(\varphi(u,v;x),\psi(u,v;x)) = \alpha(u+x)(v+\alpha-1)^{-1}.$$

3. Formulation

Let $\boldsymbol{p}_S$ be the vector of p_i for $i \in S$, whose elements are sequenced in decreasing order. Let $(S; u, v; \boldsymbol{p}_S)$ be the state which denotes that the current set of jobs is S, the current parameter vector of the prior distribution is (u, v) and the vector of the processing times is $\boldsymbol{p}_S$. Let B denote any subset of S that is called a batch and $|B|$ the number of elements in B. Also, let $V(S; u, v; \boldsymbol{p}_S)$ be the minimum expected sum of completion times for the set S of remaining jobs when the current state is $(S; u, v; \boldsymbol{p}_S)$ and the optimal strategy is followed thereafter. Let $V^B(S; u, v; \boldsymbol{p}_S)$ be the minimum expected sum of completion times for the set S of remaining jobs when the current state is $(S; u, v; \boldsymbol{p}_S)$, all the jobs in B are processed one by one sequentially, and the optimal strategy is followed thereafter. Then, the recursive equations are derived as follows:

$$V(S; u, v; \boldsymbol{p}_S) = \min_{B \subset S} V^B(S; u, v; \boldsymbol{p}_S), \tag{3.1}$$

$$V(\phi; u, v; \boldsymbol{p}_\phi) = 0 \tag{3.2}$$

where

$$\begin{aligned} V^B(S; u, v; \boldsymbol{p}_S) = {} & |S|\{\mathrm{E}_\Theta[\mathrm{E}_X[X|\Theta]|u,v] + \sum_{i \in B} p_i\} \\ & + \mathrm{E}_\Theta[\mathrm{E}_X[V(S \setminus B; \varphi(u,v;X), \psi(u,v;X); \boldsymbol{p}_{S\setminus B})|\Theta]|u,v]. \end{aligned} \tag{3.3}$$

The first term on the right hand side of (3) means that all the jobs in the first batch have the same release time, $\mathrm{E}_\Theta[\mathrm{E}_X[X|\Theta]|u,v] + \sum_{i\in B} p_i$, as the one machine batching problem considered in Shallcross [10]

Let

$$\mathrm{E}[f(X)|u,v] = \mathrm{E}_\Theta[\mathrm{E}_X[f(X)|\Theta]|u,v]$$

for any function $f(x)$. Then,

$$\begin{aligned} V^B(S; u, v; \boldsymbol{p}_S) = {} & |S|\{h(u,v) + \sum_{i \in B} p_i\} \\ & + \mathrm{E}\left[V(S \setminus B; \varphi(u,v;X), \psi(u,v;X); \boldsymbol{p}_{S\setminus B})|u,v\right]. \end{aligned}$$

Lemma 3.1 *Let S and S' be two sets of n jobs which satisfy $S \setminus \{i\} = S' \setminus \{j\}$ with $i \neq j$. Then, for any parameter vector (u, v), if $p_i > p_j$,*

$$p_i - p_j \leq V(S; u, v; \boldsymbol{p}_S) - V(S'; u, v; \boldsymbol{p}_{S'}) \leq n(p_i - p_j) \tag{3.4}$$

and if $p_i < p_j$,

$$p_j - p_i \leq V(S'; u, v; \boldsymbol{p}_{S'}) - V(S; u, v; \boldsymbol{p}_S) \leq n(p_j - p_i), \tag{3.5}$$

and therefore

$$V(S; u, v; \boldsymbol{p}_S) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} V(S'; u, v; \boldsymbol{p}_{S'}) \quad \text{if } p_i \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} p_j.$$

Proof. Let $\pi(S)$ and $\pi(S')$ be the optimal policies for $(S; u, v; \boldsymbol{p}_S)$ and $(S'; u, v; \boldsymbol{p}_{S'})$, respectively. Then, $V(S; u, v; \boldsymbol{p}_S) = V^{\pi(S)}(S; u, v; \boldsymbol{p}_S)$, $V(S'; u, v; \boldsymbol{p}_{S'}) = V^{\pi(S')}(S'; u, v; \boldsymbol{p}_{S'})$, $V(S; u, v; \boldsymbol{p}_S) \leq V^{\pi(S')}(S; u, v; \boldsymbol{p}_S)$, and $V(S'; u, v; \boldsymbol{p}_{S'}) \leq V^{\pi(S)}(S'; u, v; \boldsymbol{p}_{S'})$, where $V^{\pi(S')}(S; u, v; \boldsymbol{p}_S)$ is the expected total completion time for the state $(S; u, v; \boldsymbol{p}_S)$ when the policy $\pi(S')$ is applied by regarding p_i as p_j and also $V^{\pi(S)}(S'; u, v; \boldsymbol{p}_{S'})$ is the expected total completion time for the state $(S'; u, v; \boldsymbol{p}_{S'})$ when the policy $\pi(S)$ is applied by regarding p_j as p_i. Then, if $p_i > p_j$,

$$V(S; u, v; \boldsymbol{p}_S) - V(S'; u, v; \boldsymbol{p}_{S'}) \geq V^{\pi(S)}(S; u, v; \boldsymbol{p}_S) - V^{\pi(S)}(S'; u, v; \boldsymbol{p}_{S'}).$$

Let job i is in the kth batch in S. Also, let $|S|$, $|B_1|$, $|B_2|$,$\cdots$,$|B_{k-1}|$ be the numbers of jobs in $S, B_1, B_2, \cdots, B_{k-1}$, respectively. As job j is in the kth batch in S' if job i is in the kth batch in S. Since we use the same policy to the state $(S; u, v; \boldsymbol{p}_S)$ and $(S'; u, v; \boldsymbol{p}_{S'})$,

$$\begin{aligned} V^{\pi(S)}(S; u, v; \boldsymbol{p}_S) - V^{\pi(S)}(S'; u, v; \boldsymbol{p}_{S'}) &= (|S| - |B_1| - |B_2| - \cdots - |B_{k-1}|)(p_i - p_j) \\ &\geq p_i - p_j, \end{aligned}$$

which means $V(S; u, v; \boldsymbol{p}_S) > V(S'; u, v; \boldsymbol{p}_{S'})$ if $p_i > p_j$. Also, we have

$$\begin{aligned} V(S; u, v; \boldsymbol{p}_S) - V(S'; u, v; \boldsymbol{p}_{S'}) &\leq V^{\pi(S')}(S; u, v; \boldsymbol{p}_S) - V^{\pi(S')}(S'; u, v; \boldsymbol{p}_{S'}) \\ &\leq n(p_i - p_j). \end{aligned}$$

By the same way, if $p_i < p_j$, (3.5) is derived by interchanging S and S' and also i and j. In the case of $p_i = p_j$, we have $S = S'$ and $V(S; u, v; \boldsymbol{p}_S) = V(S'; u, v; \boldsymbol{p}_{S'})$. This completes the proof. □

In Lemma 1 of Coffman, Yannakakis, Magazine and Santos [2] gave that the job sequence in increasing order is an optimal sequence for the batching and scheduling problem. The following lemma gives the properties of the jobs for a batch.

Lemma 3.2 *For the state $(S; u, v; \boldsymbol{p}_S)$, if the optimal batch is B with batch size $|B| = k$, jobs in B are the smallest k jobs in S.*

Proof. Let $S = \{1, 2, \cdots, n\}$ and $p_1 \geq p_2 \geq \cdots \geq p_n$. If there is a job $i \in B$ such that $p_i > p_j$ for $n - k + 1 \leq j \leq n$, then there is the job l in $S \setminus B$ such that $p_i > p_l$ and $n - k + 1 \leq l \leq n$. Let $B' = B \cup \{l\} \setminus \{i\}$, then $i \in B$, $l \notin B$, $i \notin B'$, $l \in B'$, and $|B'| = |B|$. Now, we have both

$$\begin{aligned} V^B(S; u, v; \boldsymbol{p}_S) = |S| &\left\{ \mathrm{E}[X|u, v] + \sum_{j \in B} p_j \} \right\} \\ &+ \mathrm{E}[V(S \setminus B; \varphi(u, v; X), \psi(u, v; X)); \boldsymbol{p}_{S \setminus B})|u, v] \end{aligned}$$

and

$$V^{B'}(S;u,v;\boldsymbol{p}_S) = |S|\left\{\mathrm{E}[X|u,v] + \sum_{i\in B'} p_i\right\} + \mathrm{E}[V(S\setminus B';\varphi(u,v;X),\psi(u,v;X));\boldsymbol{p}_{S\setminus B'})|u,v],$$

from which

$$\begin{aligned}&V^{B}(S;u,v;\boldsymbol{p}_S) - V^{B'}(S;u,v;\boldsymbol{p}_S)\\ &= |S|(p_i - p_l)\\ &+ \mathrm{E}[V(S\setminus B;\varphi(u,v;X),\psi(u,v;X));\boldsymbol{p}_{S\setminus B}) - V(S\setminus B';\varphi(u,v;X),\psi(u,v;X));\boldsymbol{p}_{S\setminus B'})|u,v].\end{aligned}$$

As $(S\setminus B)\setminus\{l\} = (S\setminus B')\setminus\{i\}$ and $p_i > p_l$, Lemma 2 means

$$\begin{aligned}&V(S\setminus B';\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{S\setminus B'}) - V(S\setminus B;\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{S\setminus B})\\ &\le (|S|-|B|)(p_i - p_l)\end{aligned}$$

that is

$$\begin{aligned}&V(S\setminus B;\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{S\setminus B}) - V(S\setminus B';\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{S\setminus B'})\\ &\ge (|S|-|B|)(p_l - p_i),\end{aligned}$$

from which $V^{B}(S;u,v;\boldsymbol{p}_S) > V^{B'}(S;u,v;\boldsymbol{p}_S)$. This contradicts the assertion that B is the optimal batch for $(S;u,v;\boldsymbol{p}_S)$. □

From this lemma, if the size of the optimal batch B_1 for the state $(S;u,v;\boldsymbol{p}_S)$ with $S = \{1,2,\cdots,n\}$ is m, the set of jobs in B_1 are composed of jobs $n-m+1, n-m+2,\cdots,n$, and the second batch is decided for the state $(\{1,2,\cdots,n-m\};\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{\{1,2,\cdots,n-m\}})$. From this property, it is sufficient to consider the state $(n;u,v;\boldsymbol{p}_n)$ in place of $(S;u,v;\boldsymbol{p}_S)$ for $S=\{1,2,\cdots,n\}$ and $\boldsymbol{p}_n = (p_1,p_2,\cdots,p_n)$. Let $V_n(u,v;\boldsymbol{p}_n)$ be the minimum expected sum of completion times for the state $(n;u,v;\boldsymbol{p}_n)$. Also, let $V_n^k(u,v;\boldsymbol{p}_n)$ be the minimum expected sum of completion times for the state $(n;u,v;\boldsymbol{p}_n)$ when the smallest k jobs are processed as a batch and are followed by the optimal batching and scheduling of the remaining $n-k$ jobs. Then, the recursive equations (3.1), (3.2) and (3.3) are rewritten as follows:

$$V_n(u,v;\boldsymbol{p}_n) = \min_{1\le k\le n} V_n^k(u,v;\boldsymbol{p}_n), \tag{3.6}$$

$$V_0(u,v;\boldsymbol{p}_0) = 0, \tag{3.7}$$

and

$$V_n^k(u,v;\boldsymbol{p}_n) = nh(u,v) + n\sum_{j=n-k+1}^{n} p_j + \mathrm{E}[V_{n-k}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k})|u,v]. \tag{3.8}$$

Therefore,

$$V_1(u,v;\boldsymbol{p}_1) = h(u,v) + p_1, \tag{3.9}$$

and

$$V_2(u,v;\boldsymbol{p}_2) = \begin{cases} V_2^1(u,v;\boldsymbol{p}_2), & \text{if} \quad 0 < u < r_{2,1}(v;\boldsymbol{p}_2),\\ V_2^2(u,v;\boldsymbol{p}_2), & \text{if} \quad r_{2,1}(v;\boldsymbol{p}_2) \le u,\end{cases} \tag{3.10}$$

where

$$V_2^1(u,v;\boldsymbol{p}_2) = 3h(u,v) + 2p_2 + p_1, \tag{3.11}$$

$$V_2^2(u,v;\boldsymbol{p}_2) = 2h(u,v) + 2p_2 + 2p_1, \tag{3.12}$$

and $r_{2,1}(v;\boldsymbol{p}_2)$ is the unique root of the equation of u, $h(u,v) = p_1$. Also,

$$V_3(u,v;\boldsymbol{p}_3) = \begin{cases} V_3^1(u,v;\boldsymbol{p}_3), & \text{if} \quad 0 < u < r_{3,1}(v;\boldsymbol{p}_3), \\ V_3^2(u,v;\boldsymbol{p}_3), & \text{if} \quad r_{3,1}(v;\boldsymbol{p}_3) \le u < r_{3,2}(v;\boldsymbol{p}_3), \\ V_3^3(u,v;\boldsymbol{p}_3), & \text{if} \quad r_{3,2}(v;\boldsymbol{p}_3) \le u, \end{cases} \tag{3.13}$$

where

$$V_3^1(u,v;\boldsymbol{p}_3) = 3h(u,v) + 3p_3 + \mathrm{E}[V_2(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_2)|u,v], \tag{3.14}$$

$$V_3^2(u,v;\boldsymbol{p}_3) = 4h(u,v) + 3p_3 + 3p_2 + p_1, \tag{3.15}$$

and

$$V_3^3(u,v;\boldsymbol{p}_3) = 3h(u,v) + 3p_3 + 3p_2 + 3p_1. \tag{3.16}$$

Also, $r_{3,1}(v;\boldsymbol{p}_3)$ is the unique root of the equation of u,

$$\mathrm{E}[V_2(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_2)|u,v] = h(u,v) + 3p_2 + p_1,$$

and $r_{3,2}(v;\boldsymbol{p}_3)$ is the unique root of the equation of u, $h(u,v) = 2p_1$.

4. Several Properties of $\mathbf{V}_n^k(u,v;\boldsymbol{p}_n)$

$V_n^k(u,v;\boldsymbol{p}_n)$ satisfies several properties. Now we have the following Lemma.

Lemma 4.1 *(i) For $n \ge 2$ and $1 \le k \le n$, $V_n^k(u,v;\boldsymbol{p}_n)$ is continuous and strictly monotone increasing in u and also continuous and strictly monotone decreasing in v and (ii) for $n \ge 1$, $V_n(u,v;\boldsymbol{p}_n)$ is continuous and strictly monotone increasing in u and also continuous and strictly monotone decreasing in v.*

The proof of this lemma is done by using (3.8), Assumption 1, and induction on n.

From (3.8) and

$$V_{n-m}^{k-m}(u,v;\boldsymbol{p}_{n-m}) = (n-m)h(u,v) + (n-m)\sum_{j=n-k+1}^{n-m} p_j + \mathrm{E}[V_{n-k}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k})|u,v],$$

which is derived from (3.8) by replacing n and k by $n-m$ and $k-m$, respectively, we have

$$\mathrm{B}_{n,k,m}(\boldsymbol{p}_n): \quad V_n^k(u,v;\boldsymbol{p}_n) - V_{n-m}^{k-m}(u,v;\boldsymbol{p}_{n-m}) = mh(u,v) + m\sum_{j=n-k+1}^{n-m} p_j + n\sum_{j=n-m+1}^{n} p_j.$$

for $p_1 \ge p_2 \ge \cdots \ge p_n > 0$, $1 \le m < k \le n$ and $(u,v) \in W$. From $\mathrm{B}_{n,k,1}(\boldsymbol{p}_n)$, we have

$$V_n^k(u,v;\boldsymbol{p}_n) = V_{n-1}^{k-1}(u,v;\boldsymbol{p}_{n-1}) + h(u,v) + \sum_{j=n-k+1}^{n-1} p_j + np_n \tag{4.1}$$

for $2 \le k \le n$.

Lemma 4.2 *For $n \geq 2$, $p_1 \geq p_2 \geq \cdots \geq p_n > 0$ and $(u,v) \in W$,*

$$C_{n,k,l}(\boldsymbol{p}_n)\text{:} \quad V_n^k(u,v;\boldsymbol{p}_n) - V_n^{k+1}(u,v;\boldsymbol{p}_n) = V_{n-l}^{k-l}(u,v;\boldsymbol{p}_{n-1}) - V_{n-l}^{k+1-l}(u,v;\boldsymbol{p}_{n-1}) - lp_{n-k},$$

hold for $1 \leq l < k \leq n-1$ and

$$D_{n,k}(\boldsymbol{p}_n)\text{:} \quad V_n^k(u,v;\boldsymbol{p}_n) - V_n^{k+1}(u,v;\boldsymbol{p}_n) = -np_{n-k}$$
$$+E[V_{n-k}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k}) - V_{n-k-1}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-1})|u,v]$$

holds for $1 \leq k \leq n-1$.

Proof. Replacing k in (4.1) by $k+1$,

$$V_n^{k+1}(u,v;\boldsymbol{p}_n) = V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) + h(u,v) + \sum_{j=n-k}^{n-1} p_j + np_n.$$

Also, subtracting this equation from (4.1), we have

$$V_n^k(u,v;\boldsymbol{p}_n) - V_n^{k+1}(u,v;\boldsymbol{p}_n) = V_{n-1}^{k-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - p_{n-k},$$

which means that $\mathrm{C}_{n,k,1}(\boldsymbol{p}_n)$ holds. Also,

$$V_{n-1}^{k-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) = V_{n-2}^{k-2}(u,v;\boldsymbol{p}_{n-2}) - V_{n-2}^{k-1}(u,v;\boldsymbol{p}_{n-2}) - p_{n-k},$$

$$\vdots$$

$$V_{n-k+2}^2(u,v;\boldsymbol{p}_{n-k+2}) - V_{n-k+2}^3(u,v;\boldsymbol{p}_{n-k+2}) = V_{n-k+1}^1(u,v;\boldsymbol{p}_{n-k+1}) - V_{n-k+1}^2(u,v;\boldsymbol{p}_{n-k+1}) - p_{n-k}.$$

Adding first l of these $k-1$ equations side by side, we have

$$V_n^k(u,v;\boldsymbol{p}_n) - V_n^{k+1}(u,v;\boldsymbol{p}_n) = V_{n-l}^{k-l}(u,v;\boldsymbol{p}_{n-l}) - V_{n-l}^{k+1-l}(u,v;\boldsymbol{p}_{n-l}) - lp_{n-k}.$$

and $\mathrm{C}_{n,k,l}(\boldsymbol{p}_n)$ holds for $2 \leq l < k \leq n-1$. Also, for $1 \leq k \leq n-1$, we have from (3.8)

$$\begin{aligned}
&V_n^k(u,v;\boldsymbol{p}_n) - V_n^{k+1}(u,v;\boldsymbol{p}_n) \\
&= n\left\{h(u,v) + \sum_{j=n-k+1}^{n} p_j\right\} + \mathrm{E}[V_{n-k}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k})|u,v] \\
&- n\left\{h(u,v) + \sum_{j=n-k}^{n} p_j\right\} - \mathrm{E}[V_{n-k-1}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-1})|u,v] \\
&= -np_{n-k} \\
&+ \mathrm{E}[V_{n-k}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k}) - V_{n-k-1}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-1})|u,v],
\end{aligned}$$

and $\mathrm{D}_{n,k}(\boldsymbol{p}_n)$ is derived. □

From this lemma, we have

$$V_2^1(u,v;\boldsymbol{p}_2) - V_2^2(u,v;\boldsymbol{p}_2) = h(u,v) - p_1, \tag{4.2}$$

$$V_3^1(u,v;\boldsymbol{p}_3) - V_3^2(u,v;\boldsymbol{p}_3)$$
$$= \mathrm{E}[V_2(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_2) - V_1(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_1)|u,v] - 3p_2, \tag{4.3}$$
$$V_3^2(u,v;\boldsymbol{p}_3) - V_3^3(u,v;\boldsymbol{p}_3) = h(u,v) - 2p_1, \tag{4.4}$$
$$V_4^1(u,v;\boldsymbol{p}_4) - V_4^2(u,v;\boldsymbol{p}_4)$$
$$= \mathrm{E}[V_3(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_3) - V_2(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_2)|u,v] - 4p_3, \tag{4.5}$$
$$V_4^2(u,v;\boldsymbol{p}_4) - V_4^3(u,v;\boldsymbol{p}_4) = V_3^1(u,v;\boldsymbol{p}_3) - V_3^2(u,v;\boldsymbol{p}_3) - p_2$$
$$= \mathrm{E}[V_2(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_2) - V_1(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_1)|u,v] - 4p_2, \tag{4.6}$$

and

$$\begin{aligned} V_4^3(u,v;\boldsymbol{p}_4) - V_4^4(u,v;\boldsymbol{p}_4) &= V_3^2(u,v;\boldsymbol{p}_3) - V_3^3(u,v;\boldsymbol{p}_3) - p_1 \\ &= h(u,v) - 3p_1. \end{aligned} \tag{4.7}$$

Lemma 4.3 *For* $p_1 \geq p_2 \geq \cdots \geq p_n > 0$, $1 \leq k \leq n-1$ *and* $(u,v) \in W$,

$$E_{n,k}(\boldsymbol{p}_n)\text{:} \qquad V_n^k(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) = h(u,v) + \sum_{j=n-k+1}^{n-1} p_j + np_n - (n-1)p_{n-k}$$
$$+E\left[V_{n-k}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k}) - V_{n-k-1}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-1})|u,v\right].$$

Proof. $\mathrm{E}_{n,k}(\boldsymbol{p}_n)$ is obtained from (3.8) and

$$\begin{aligned} V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) = (n-1)h(u,v) + (n-1)\sum_{j=n-k}^{n-1} p_j \\ + \mathrm{E}\left[V_{n-k-1}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-1}) \mid u,v\right] \end{aligned}$$

which is the immediate consequence of replacing n in (3.8) by $n-1$. □

Lemma 4.4 *For* $n \geq 2$, $p_1 \geq p_2 \geq \cdots \geq p_n > 0$, $1 \leq k \leq n-1$ *and any* $(u,v) \in W$,

$$h(u,v) + \sum_{j=1}^{n} p_j \leq V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1}) \leq nh(u,v) + \sum_{j=1}^{n} p_j. \tag{4.8}$$

Proof. The proof is in almost the same way as that of Lemma 3.6 of Hamada [5]. □

Lemma 4.5 *For* $n \geq 2$, $p_1 \geq p_2 \geq \cdots \geq p_n > 0$, $1 \leq k \leq n-1$ *and any* $(u,v) \in W$,

$$-np_{n-k} + h(u,v) + \sum_{j=1}^{n-k} p_j \leq V_n^k(u,v;\boldsymbol{p}_n) - V_n^{k+1}(u,v;\boldsymbol{p}_n) \leq -np_{n-k} + (n-k)h(u,v) + \sum_{j=1}^{n-k} p_j.$$

Proof. These inequalities are the immediate consequence of $\mathrm{D}_{n,k}(\boldsymbol{p}_n)$ and Lemma 7. □

Lemma 4.6 *For $n \geq 2$, $p_1 \geq p_2 \geq \cdots \geq p_n > 0$, $1 \leq k \leq n-1$ and any $(u,v) \in W$,*

$$(i)\quad -np_1+h(u,v)+\sum_{j=1}^{1} p_j \leq -np_2+h(u,v)+\sum_{j=1}^{2} p_j \leq \cdots \leq -np_{n-1}+h(u,v)+\sum_{j=1}^{n-1} p_j.$$

$$(ii)\quad -np_1+h(u,v)+\sum_{j=1}^{1} p_j \leq -np_2+2h(u,v)+\sum_{j=1}^{2} p_j \leq \cdots \leq -np_{n-1}+(n-1)h(u,v)+\sum_{j=1}^{n-1} p_j.$$

Proof. These inequalities are derived from $h(u,v) > 0$, $p_1 \geq p_2 \geq \cdots \geq p_{n-1} > 0$, and the trivial inequalities $\sum_{j=1}^{1} p_j < \sum_{j=1}^{2} p_j < \cdots < \sum_{j=1}^{n-1} p_j$. □

Theorem 4.1 *For $n \geq 2$, $p_1 \geq p_2 \geq \cdots \geq p_n > 0$, $1 \leq k \leq n-1$ and any $(u,v) \in W$,*
(i) if $(n-1)p_1 < h(u,v)$, then $V_n(u,v;\boldsymbol{p}_n) = V_n^n(u,v;\boldsymbol{p}_n)$.
(ii) if $h(u,v) < p_{n-1} - (n-1)^{-1}\sum_{j=1}^{n-2} p_j$, then $V_n(u,v;\boldsymbol{p}_n) = V_n^1(u,v;\boldsymbol{p}_n)$.

Proof. If $(n-1)p_1 < h(u,v)$, then $0 < -np_1 + h(u,v) + \sum_{j=1}^{1} p_j$, and we have from Lemmas 8 and 9

$$V_n^n(u,v;\boldsymbol{p}_n) \leq V_n^{n-1}(u,v;\boldsymbol{p}_n) \leq \cdots \leq V_n^2(u,v;\boldsymbol{p}_n) \leq V_n^1(u,v;\boldsymbol{p}_n),$$

that is,

$$V_n(u,v;\boldsymbol{p}_n) = V_n^n(u,v;\boldsymbol{p}_n).$$

Also, if $h(u,v) < p_{n-1} - (n-1)^{-1}\sum_{j=1}^{n-2} p_j$, then $-np_{n-1} + (n-1)h(u,v) + \sum_{j=1}^{n-1} p_j < 0$, and we have from (ii) of Lemma 9 that

$$V_n^1(u,v;\boldsymbol{p}_n) \leq V_n^2(u,v;\boldsymbol{p}_n) \leq \cdots \leq V_n^{n-1}(u,v;\boldsymbol{p}_n) \leq V_n^n(u,v;\boldsymbol{p}_n),$$

that is

$$V_n(u,v;\boldsymbol{p}_n) = V_n^1(u,v;\boldsymbol{p}_n),$$

which completes the proof. □

From this theorem, if $h(u,v) < p_{n-1} - (n-1)^{-1}\sum_{j=1}^{n-2} p_j$, the optimal batch size is 1. Also, if $(n-1)p_1 < h(u,v)$, then the optimal batch size is n. Since $h(u,v) > 0$, we consider the case that the inequality $(n-1)p_{n-1} \geq \sum_{j=1}^{n-2} p_j$ holds.

For the state $(n;u,v;\boldsymbol{p}_n)$, the optimal batch size is l which satisfies

$$V_n^l(u,v;\boldsymbol{p}_n) = \min_{1 \leq k \leq n} V_n^k(u,v;\boldsymbol{p}_n).$$

The optimal strategy for the state $(n;u,v;\boldsymbol{p}_n)$ is given as follows:
Algorithm.

Step 1. Let $k = 1$, $m = n$ and $C = 0$, where C be the total completion time for the all the jobs completed up to now.

Step 2. If $V_m(u,v;\boldsymbol{p}_m) = V_m^l(u,v;\boldsymbol{p}_m)$, then let $B_k = \{m, m-1, \cdots, m-l+1\}$ and $C \longleftarrow C + m(x + p_m + p_{m-1} + \cdots + p_{m-l+1})$, where x is the observed value of the random setup time X.

Step 3. If $l = m$, stop. Otherwise, let $m \longleftarrow m-l$, $u \longleftarrow \varphi(u,v;x)$, and $v \longleftarrow \psi(u,v;x)$.

Step 4. $k \longleftarrow k+1$ and go to Step 2.

In Step 2, the jobs, $m, m-1, \cdots, m-l+1$, are processed as a batch and their contribution to the total completion time is $m(x + p_m + p_{m-1} + \cdots + p_{m-l+1})$. By using this algorithm, we obtain the optimal batches, B_1, B_2, $\cdots$, B_k, and the total completion time C for the optimal strategy.

5. Special class of sequence of processing times

In this section, we consider the case that $\sum_{j=1}^{n-k-1} p_j \le (n-k)p_{n-k}$ holds for every k with $1 \le k \le n-2$. Furthermore, we restrict our attention to the special class of vector $\boldsymbol{p}_n = (p_1, p_2, \cdots, p_n)$ of processing times which satisfies

$$-kp_{n-k} + (k+1)p_{n-k-1} - p_1 \ge 0$$

for $1 \le k \le n-2$. Let

$$A_n = \left\{ (p_1, p_2, \cdots, p_n) \,\middle|\, p_1 \ge p_2 \ge \cdots \ge p_n > 0, \ -kp_{n-k} + (k+1)p_{n-k-1} - p_1 \ge 0 \right.$$
$$\left. \text{for } 1 \le k \le n-2 \text{ and } \sum_{j=1}^{n-l-1} p_j \le (n-l)p_{n-l} \text{ for } 1 \le l \le n-2 \right\}$$

for $n \ge 3$.

For example, if $n = 10$, let $p_1 = 1$, $p_2 = 0.990$, $p_3 = 0.988$, $p_4 = 0.986$, $p_5 = 0.983$, $p_6 = 0.978$, $p_7 = 0.970$, $p_8 = 0.955$, and $p_9 = 0.910$. Then, $(p_1, p_2, \cdots, p_{10})$ satisfies $p_8 \ge (p_9 + p_1)/2$, $p_7 \ge (2p_8 + p_1)/3$, $p_6 \ge (3p_7 + p_1)/4$, $p_5 \ge (4p_6 + p_1)/5$, $p_4 \ge (5p_5 + p_1)/6$, $p_3 \ge (6p_4 + p_1)/7$, $p_2 \ge (7p_3 + p_1)/8$, $p_1 \ge p_2$, $p_2 > p_1/2$, $p_3 > (p_1 + p_2)/3$, $p_4 > (p_1 + p_2 + p_3)/4$, $p_5 > (p_1 + p_2 + \cdots + p_4)/5$, $p_6 > (p_1 + p_2 + \cdots + p_5)/6$, $p_7 > (p_1 + p_2 + \cdots + p_6)/7$, $p_8 > (p_1 + p_2 + \cdots + p_7)/8$, $p_9 > (p_1 + p_2 + \cdots + p_8)/9$, and $p_1 \ge p_2 \ge \cdots \ge p_{10} > 0$. Therefore, $(p_1, p_2, \cdots, p_{10}) \in A_{10}$.

Now, let

(T$_2$): $V_2(u, v; \boldsymbol{p}_2) = \begin{cases} V_2^1(u, v; \boldsymbol{p}_2), & \text{if } 0 < u < r_{2,1}(v; \boldsymbol{p}_1), \\ V_2^2(u, v; \boldsymbol{p}_2), & \text{if } r_{2,1}(v; \boldsymbol{p}_1) \le u, \end{cases}$

(U$_2$): $V_2(u, v; \boldsymbol{p}_2) - V_1(u, v; \boldsymbol{p}_1) = \begin{cases} V_2^1(u, v; \boldsymbol{p}_2) - V_1(u, v; \boldsymbol{p}_1), & \text{if } 0 < u < r_{2,1}(v, \boldsymbol{p}_1), \\ V_2^2(u, v; \boldsymbol{p}_2) - V_1(u, v; \boldsymbol{p}_1), & \text{if } r_{2,1}(v, \boldsymbol{p}_1) \le u, \end{cases}$

and

(U$_3$): $V_3(u, v; \boldsymbol{p}_3) - V_2(u, v; \boldsymbol{p}_2)$

$$= \begin{cases} V_3^1(u, v; \boldsymbol{p}_3) - V_2^1(u, v; \boldsymbol{p}_2), & \text{if } 0 < u < r_{3,1}(v; \boldsymbol{p}_2), \\ V_3^2(u, v; \boldsymbol{p}_3) - V_2^1(u, v; \boldsymbol{p}_2), & \text{if } r_{3,1}(v; \boldsymbol{p}_2) \le u < r_{2,1}(v; \boldsymbol{p}_1), \\ V_3^2(u, v; \boldsymbol{p}_3) - V_2^2(u, v; \boldsymbol{p}_2), & \text{if } r_{2,1}(v; \boldsymbol{p}_1) \le u < r_{3,2}(v; \boldsymbol{p}_2), \\ V_3^3(u, v; \boldsymbol{p}_3) - V_2^2(u, v; \boldsymbol{p}_2), & \text{if } r_{3,2}(v; \boldsymbol{p}_2) \le u, \end{cases}$$

and furthermore we define (P$_n$), (Q$_n$), and (W$_n$) for $n \ge 2$, (R$_n$), (S$_n$), and (T$_n$) for $n \ge 3$, (U$_n$) for $n \ge 4$, and (V$_n$) for $n \ge 1$, as follows:

(P$_n$): For $1 \le k \le n-1$, $V_n^k(u, v; \boldsymbol{p}_n) - V_n^{k+1}(u, v; \boldsymbol{p}_n)$ is continuous and strictly monotone increasing in u and continuous and strictly monotone decreasing in v.

(Q$_n$): For $1 \le k \le n-1$, the equation $V_n^k(u, v; \boldsymbol{p}_n) - V_n^{k+1}(u, v; \boldsymbol{p}_n) = 0$ of u has a unique root $r_{n,k}(v; \boldsymbol{p}_{n-1})$ such that $V_n^k(u, v; \boldsymbol{p}_n) - V_n^{k+1}(u, v; \boldsymbol{p}_n) < 0$ if $0 < u < r_{n,k}(v; \boldsymbol{p}_{n-1})$ and $V_n^k(u, v; \boldsymbol{p}_n) - V_n^{k+1}(u, v; \boldsymbol{p}_n) > 0$ if $r_{n,k}(v; \boldsymbol{p}_{n-1}) < u$. Also, $r_{n,k}(v; \boldsymbol{p}_{n-1})$ is strictly increasing in v.

(R$_n$): For $2 \le k \le n-1$, $r_{n-1,k-1}(v; \boldsymbol{p}_{n-2}) < r_{n,k}(v; \boldsymbol{p}_{n-1})$.

(S$_n$): For $1 \le k \le n-2$, $r_{n,k}(v; \boldsymbol{p}_{n-1}) \le r_{n-1,k}(v; \boldsymbol{p}_{n-2})$.

$$(\mathrm{T}_n):\ V_n(u,v;\boldsymbol{p}_n)=\begin{cases} V_n^1(u,v;\boldsymbol{p}_n), & \text{if}\ \ 0<u<r_{n,1}(v;\boldsymbol{p}_{n-1}),\\ V_n^k(u,v;\boldsymbol{p}_n), & \text{if}\ \ r_{n,k-1}(v;\boldsymbol{p}_{n-1})\le u<r_{n,k}(v;\boldsymbol{p}_{n-1})\\ & \qquad\qquad (2\le k\le n-1),\\ V_n^n(u,v;\boldsymbol{p}_n), & \text{if}\ \ r_{n,n-1}(v;\boldsymbol{p}_{n-1})\le u. \end{cases}$$

(U_n): $V_n(u,v;\boldsymbol{p}_n)-V_{n-1}(u,v;\boldsymbol{p}_{n-1})$

$$=\begin{cases} V_n^1(u,v;\boldsymbol{p}_n)-V_{n-1}^1(u,v;\boldsymbol{p}_{n-1}), & \text{if}\ \ 0<u<r_{n,1}(v;\boldsymbol{p}_{n-1}),\\ V_n^2(u,v;\boldsymbol{p}_n)-V_{n-1}^1(u,v;\boldsymbol{p}_{n-1}), & \text{if}\ \ r_{n,1}(v;\boldsymbol{p}_{n-1})\le u<r_{n-1,1}(v;\boldsymbol{p}_{n-2}),\\ V_n^k(u,v;\boldsymbol{p}_n)-V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}), & \text{if}\ \ r_{n-1,k-1}(v;\boldsymbol{p}_{n-2})\le u<r_{n,k}(v;\boldsymbol{p}_{n-1}),\\ V_n^{k+1}(u,v;\boldsymbol{p}_n)-V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}), & \text{if}\ \ r_{n,k}(v;\boldsymbol{p}_{n-1})\le u<r_{n-1,k}(v;\boldsymbol{p}_{n-2}),\\ & \quad (2\le k\le n-2)\\ V_n^{n-1}(u,v;\boldsymbol{p}_n)-V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}), & \text{if}\ \ r_{n-1,n-2}(v;\boldsymbol{p}_{n-2})\le u<r_{n,n-1}(v;\boldsymbol{p}_{n-1}),\\ V_n^n(u,v;\boldsymbol{p}_n)-V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}), & \text{if}\ \ r_{n,n-1}(v;\boldsymbol{p}_{n-1})\le u. \end{cases}$$

(V_n): $V_n(u,v;\boldsymbol{p}_n)-V_{n-1}(u,v;\boldsymbol{p}_{n-1})$ is continuous and strictly increasing in u and continuous and strictly decreasing in v.

$$(\mathrm{W}_n):\ V_n(u,v;\boldsymbol{p}_n)-V_{n-1}(u,v;\boldsymbol{p}_{n-1})\ge V_{n-1}(u,v;\boldsymbol{p}_{n-1})-V_{n-2}(u,v;\boldsymbol{p}_{n-2})$$
$$+np_n-(n-2)p_{n-1}-p_1.$$

Now we have

Lemma 5.1 *(V_1), (P_2), (Q_2), (U_2), (V_2), and (W_2) hold.*

Proof. (V_1), (P_2), (Q_2), (U_2), (V_2), and (W_2) are derived from (7), (9), (10), (11) and Assumption 1. □

Theorem 5.1 *For $n\ge 3$, if $\boldsymbol{p}_n\in A_n$, then (P_n), (Q_n), (R_n), (S_n), (T_n), (U_n), (V_n), and (W_n) hold.*

Proof. For $n=3$,

$$\begin{aligned} V_3^1(u,v;\boldsymbol{p}_3)-V_3^2(u,v;\boldsymbol{p}_3) &= -h(u,v)-3p_2-p_1+\mathrm{E}[V_2(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_2)|u,v]\\ &= \mathrm{E}[V_2(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_2)-h(\varphi(u,v;X),\psi(u,v;X))|u,v]\\ &\quad -3p_2-p_1. \end{aligned}$$

Since $V_2^1(u,v;\boldsymbol{p}_2)-h(u,v)=2h(u,v)+2p_2+p_1$ and $V_2^2(u,v;\boldsymbol{p}_2)-h(u,v)=h(u,v)+2p_2+2p_1$, $V_2(u,v;\boldsymbol{p}_2)-h(u,v)$ is strictly increasing in u and strictly decreasing in v. This means from Assumption 1 that $V_3^1(u,v;\boldsymbol{p}_3)-V_3^2(u,v;\boldsymbol{p}_3)$ is strictly monotone increasing in u and strictly monotone decreasing in v. Also, from (4.4) and Lemma 1, $V_3^2(u,v;\boldsymbol{p}_3)-V_3^3(u,v;\boldsymbol{p}_3)$ is strictly monotone increasing in u and strictly monotone decreasing in v. This completes the proof of (P_3). Since $2h(u,v)+2p_2+p_1\le V_2(u,v;\boldsymbol{p}_2)\le 3h(u,v)+2p_2+p_1$, we have $h(u,v)-p_2\le V_3^1(u,v;\boldsymbol{p}_3)-V_3^2(u,v;\boldsymbol{p}_3)\le 2h(u,v)-p_2$. For $1\le k\le 2$, from (P_3), Assumption 3, and Lemma 1, the equation $V_3^k(u,v;\boldsymbol{p}_n)-V_3^{k+1}(u,v;\boldsymbol{p}_n)=0$ of u has a unique root $r_{3,k}(v;\boldsymbol{p}_2)$ such that $V_3^k(u,v;\boldsymbol{p}_3)-V_3^{k+1}(u,v;\boldsymbol{p}_3)<0$ if $0<u<r_{3,k}(v;\boldsymbol{p}_2)$ and $V_3^k(u,v;\boldsymbol{p}_3)-V_3^{k+1}(u,v;\boldsymbol{p}_3)>0$ if $r_{3,k}(v;\boldsymbol{p}_2)<u$. Also, $r_{3,k}(v;\boldsymbol{p}_2)$ is strictly increasing in v, which means (Q_3) holds. From (4.2) and (4.4), we have $r_{2,1}(v;\boldsymbol{p}_1)<r_{3,2}(v;\boldsymbol{p}_1)$ and (R_3) holds. From $h(u,v)-p_2\le V_3^1(u,v;\boldsymbol{p}_3)-V_3^2(u,v;\boldsymbol{p}_3)$, $p_1\ge p_2$, and (4.2), (S_3) holds. From (R_3) and (S_3), we have $r_{3,1}(v;\boldsymbol{p}_2)<r_{3,2}(v;\boldsymbol{p}_2)$, that is, (T_3) holds. (U_3) is derived from (T_2), (T_3), (R_3) and (S_3). (V_3) is derived from (3.14), (3.15), (3.16), (3.11), (3.12), Lemma 4 and Assumptions 1 and 4. Since $V_3^1(u,v;\boldsymbol{p}_3)-V_2^1(u,v;\boldsymbol{p}_2)\ge 2h(u,v)+3p_3$, $V_3^2(u,v;\boldsymbol{p}_3)-$

$V_2^1(u,v;\boldsymbol{p}_2) = h(u,v) + 3p_3 + p_2$, $V_3^2(u,v;\boldsymbol{p}_3) - V_2^2(u,v;\boldsymbol{p}_2) = 2h(u,v) + 3p_3 + p_2 - p_1$, and $V_3^3(u,v;\boldsymbol{p}_n) - V_2^2(u,v;\boldsymbol{p}_2) = h(u,v) + 3p_3 + p_2 + p_1$, $V_2^1(u,v;\boldsymbol{p}_2) - V_1(u,v;\boldsymbol{p}_1) = 2h(u,v) + 2p_2$ and $V_2^2(u,v;\boldsymbol{p}_2) - V_1(u,v;\boldsymbol{p}_1) = h(u,v) + 2p_2 - p_1$, ($\mathrm{W}_3$) is derived from ($\mathrm{U}_3$).

For $n \geq 4$, (P_n) is derived from $\mathrm{D}_{n,k}(\boldsymbol{p}_n)$ and (V_{n-k}) for $1 \leq k \leq n-1$ and Assumptions 1 and 4. (Q_n) is derived from (P_n), Lemma 8 and $\boldsymbol{p}_n = (p_1, p_2, \cdots, p_n) \in A_n$. ($\mathrm{R}_n$) is derived from $\mathrm{C}_{n,k,1}(\boldsymbol{p}_n)$ and the definitions of $r_{n,k}(v;\boldsymbol{p}_{n-1})$ and $r_{n-1,k-1}(v;\boldsymbol{p}_{n-2})$. Since ($\mathrm{W}_{n-k}$) holds for $1 \leq k \leq n-2$, we have

$$\begin{aligned}&V_{n-k}(u,v;\boldsymbol{p}_{n-k}) - V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1})\\ &\geq V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1}) - V_{n-k-2}(u,v;\boldsymbol{p}_{n-k-2}) + (n-k)p_{n-k} - (n-k-2)p_{n-k-1} - p_1\end{aligned}$$

for $1 \leq k \leq n-2$, that is,

$$\begin{aligned}&E[V_{n-k}(u,v;\boldsymbol{p}_{n-k}) - V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1})|u,v]\\ &\geq E[V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1}) - V_{n-k-2}(u,v;\boldsymbol{p}_{n-k-2})|u,v]\\ &+ (n-k)p_{n-k} - (n-k-2)p_{n-k-1} - p_1\end{aligned}$$

for $1 \leq k \leq n-2$, which are equivalent to

$$\begin{aligned}&-np_{n-k} + E[V_{n-k}(u,v;\boldsymbol{p}_{n-k}) - V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1})|u,v]\\ &\geq -(n-1)p_{n-k-1} + E[V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1}) - V_{n-k-2}(u,v;\boldsymbol{p}_{n-k-2})|u,v]\\ &+ \{-kp_{n-k} + (k+1)p_{n-k-1} - p_1\}\end{aligned}$$

for $1 \leq k \leq n-2$, respectively. From these inequalities and $\boldsymbol{p}_n \in A_n$, we have

$$\begin{aligned}&-np_{n-k} + E[V_{n-k}(u,v;\boldsymbol{p}_{n-k}) - V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1})|u,v]\\ &\geq -(n-1)p_{n-k-1} + E[V_{n-k-1}(u,v;\boldsymbol{p}_{n-k-1}) - V_{n-k-2}(u,v;\boldsymbol{p}_{n-k-2})|u,v]\end{aligned}$$

for $1 \leq k \leq n-2$, and from $\mathrm{D}_{n,k}(\boldsymbol{p}_n)$ and $\mathrm{D}_{n-1,k}(\boldsymbol{p}_{n-1})$, these are equivalent to

$$V_n^k(u,v;\boldsymbol{p}_n) - V_n^{k+1}(u,v;\boldsymbol{p}_n) \geq V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-1}^{k+1}(u,v;\boldsymbol{p}_{n-1})$$

for $1 \leq k \leq n-2$, respectively. From these inequalities and the definitions of $r_{n,k}(v;\boldsymbol{p}_{n-1})$ and $r_{n-1,k}(v;\boldsymbol{p}_{n-2})$, ($\mathrm{S}_n$) is derived. ($\mathrm{T}_n$) is the immediate consequence of (R_n), (S_n), (Q_n) and also (U_n) is that of (R_n), (S_n), (T_n) and (T_{n-1}). (V_n) is derived from $\mathrm{E}_{n,k}(\boldsymbol{p}_n)$ for $1 \leq k \leq n-1$, Lemma 4 and Assumptions 1 and 4. To prove (W_n), we have from (U_n) and (U_{n-1})

$$\begin{aligned}&V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\\ &= \begin{cases} V_n^1(u,v;\boldsymbol{p}_n) - V_{n-1}^1(u,v;\boldsymbol{p}_{n-1}), & \text{if } 0 < u < r_{n,1}(v;\boldsymbol{p}_{n-1}),\\ V_n^2(u,v;\boldsymbol{p}_n) - V_{n-1}^1(u,v;\boldsymbol{p}_{n-1}), & \text{if } r_{n,1}(v;\boldsymbol{p}_{n-1}) \leq u < r_{n-1,1}(v;\boldsymbol{p}_{n-2}),\\ V_n^k(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}), & \text{if } r_{n-1,k-1}(v;\boldsymbol{p}_{n-2}) \leq u < r_{n,k}(v;\boldsymbol{p}_{n-1}),\\ V_n^{k+1}(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}), & \text{if } r_{n,k}(v;\boldsymbol{p}_{n-1}) \leq u < r_{n-1,k}(v;\boldsymbol{p}_{n-2}),\\ & \quad (2 \leq k \leq n-2)\\ V_n^{n-1}(u,v;\boldsymbol{p}_n) - V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}), & \text{if } r_{n-1,n-2}(v;\boldsymbol{p}_{n-2}) \leq u < r_{n,n-1}(v;\boldsymbol{p}_{n-1}),\\ V_n^n(u,v;\boldsymbol{p}_n) - V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}), & \text{if } r_{n,n-1}(v;\boldsymbol{p}_{n-1}) \leq u, \end{cases}\end{aligned}$$

and

$$V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})$$

$$= \begin{cases} V_{n-1}^1(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^1(u,v;\boldsymbol{p}_{n-2}), & \text{if } 0 < u < r_{n-1,1}(v;\boldsymbol{p}_{n-2}), \\ V_{n-1}^2(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^1(u,v;\boldsymbol{p}_{n-2}), & \text{if } r_{n-1,1}(v;\boldsymbol{p}_{n-2}) \le u < r_{n-2,1}(v;\boldsymbol{p}_{n-3}), \\ V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^k(u,v;\boldsymbol{p}_{n-2}), & \text{if } r_{n-2,k-1}(v;\boldsymbol{p}_{n-3}) \le u < r_{n-1,k}(v;\boldsymbol{p}_{n-2}), \\ V_{n-1}^{k+1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^k(u,v;\boldsymbol{p}_{n-2}), & \text{if } r_{n-1,k}(v;\boldsymbol{p}_{n-2}) \le u < r_{n-2,k}(v;\boldsymbol{p}_{n-3}), \\ & \quad (2 \le k \le n-3) \\ V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{n-2}(u,v;\boldsymbol{p}_{n-2}), & \text{if } r_{n-2,n-2}(v;\boldsymbol{p}_{n-3}) \le u < r_{n-1,n-2}(v;\boldsymbol{p}_{n-2}), \\ V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{n-2}(u,v;\boldsymbol{p}_{n-2}), & \text{if } r_{n-1,n-2}(v;\boldsymbol{p}_{n-2}) \le u. \end{cases}$$

(i) If $0 < u < r_{n,1}(v;\boldsymbol{p}_{n-1})$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^1(u,v;\boldsymbol{p}_n) - V_{n-1}^1(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^1(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^1(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

(ii) If $r_{n,1}(v;\boldsymbol{p}_{n-1}) \le u < r_{n-1,1}(v;\boldsymbol{p}_{n-2})$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^2(u,v;\boldsymbol{p}_n) - V_{n-1}^1(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^1(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^1(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

(iii) If $r_{n-1,k-1}(v;\boldsymbol{p}_{n-2}) \le u < \min\{r_{n,k}(v;\boldsymbol{p}_{n-1}), r_{n-2,k-1}(v;\boldsymbol{p}_{n-3})\}$ for $2 \le k \le n-2$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^k(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{k-1}(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

If $\min\{r_{n,k}(v;\boldsymbol{p}_{n-1}), r_{n-2,k-1}(v;\boldsymbol{p}_{n-3})\} \le u < \max\{r_{n,k}(v;\boldsymbol{p}_{n-1}), r_{n-2,k-1}(v;\boldsymbol{p}_{n-3})\}$ for $2 \le k \le n-2$, two cases (iv) $r_{n,k}(v;\boldsymbol{p}_{n-1}) < r_{n-2,k-1}(v;\boldsymbol{p}_{n-3})$ and (v) $r_{n-2,k-1}(v;\boldsymbol{p}_{n-3}) < r_{n,k}(v;\boldsymbol{p}_{n-1})$ have to be considered and the case $r_{n,k}(v;\boldsymbol{p}_{n-1}) = r_{n-2,k-1}(v;\boldsymbol{p}_{n-3})$ need not be considered.

(iv) If $r_{n,k}(v;\boldsymbol{p}_{n-1}) \le u < r_{n-2,k-1}(v;\boldsymbol{p}_{n-3})$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^{k+1}(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{k-1}(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

(v) If $r_{n-2,k-1}(v;\boldsymbol{p}_{n-3}) \le u < r_{n,k}(v;\boldsymbol{p}_{n-1})$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^k(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^k(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

(vi) If $\max\{r_{n,k}(v;\boldsymbol{p}_{n-1}), r_{n-2,k-1}(v;\boldsymbol{p}_{n-3})\} \le u < r_{n-1,k}(v;\boldsymbol{p}_{n-2})$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^{k+1}(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^k(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

(vii) If $r_{n-1,n-2}(v;\boldsymbol{p}_{n-2}) \le u < r_{n,n-1}(v;\boldsymbol{p}_{n-1})$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^{n-1}(u,v;\boldsymbol{p}_n) - V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{n-2}(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

(viii) If $r_{n,n-1}(v;\boldsymbol{p}_{n-1}) \le u$,

$$\begin{aligned} &\left\{V_n(u,v;\boldsymbol{p}_n) - V_{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}(u,v;\boldsymbol{p}_{n-2})\right\} \\ &= \left\{V_n^n(u,v;\boldsymbol{p}_n) - V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^{n-1}(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{n-2}(u,v;\boldsymbol{p}_{n-2})\right\}. \end{aligned}$$

In the cases of (i) and (v), by using $\mathrm{E}_{n,k}(\boldsymbol{p}_n)$, $\mathrm{E}_{n-1,k}(\boldsymbol{p}_{n-1})$ and (W_{n-k}), we have

$$\begin{aligned}
&\left\{V_n^k(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^k(u,v;\boldsymbol{p}_{n-2})\right\} \\
&= np_n - (n-2)p_{n-1} - np_{n-k} + (n-2)p_{n-k-1} \\
&+ E\left[\left\{V_{n-k}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k}) - V_{n-k-1}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-1}\right\}\right. \\
&\left. - \left\{V_{n-k-1}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-1})\right\} - \left\{V_{n-k-2}(\varphi(u,v;X),\psi(u,v;X);\boldsymbol{p}_{n-k-2}\right\} |u,v\right] \\
&\geq np_n - (n-2)p_{n-1} - np_{n-k} + (n-2)p_{n-k-1} + (n-k)p_{n-k} - (n-k-2)p_{n-k-1} - p_1 \\
&= np_n - (n-2)p_{n-1} - p_1 + k(p_{n-k-1} - p_{n-k}) \\
&\geq np_n - (n-2)p_{n-1} - p_1.
\end{aligned}$$

In the cases of (ii) and (vi), by using $\mathrm{E}_{n,k+1}(\boldsymbol{p}_n)$, $\mathrm{E}_{n-1,k}(\boldsymbol{p}_{n-1})$ and $p_1 \geq p_{n-k-1}$, we have

$$\begin{aligned}
&\left\{V_n^{k+1}(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^k(u,v;\boldsymbol{p}_{n-2})\right\} \\
&\geq \left\{V_n^{k+1}(u,v;\boldsymbol{p}_n) - V_{n-1}^{k+1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^k(u,v;\boldsymbol{p}_{n-2})\right\} \\
&= np_n - (n-2)p_{n-1} - p_{n-k-1} \\
&\geq np_n - (n-2)p_{n-1} - p_1.
\end{aligned}$$

(15) is useful for (iii), (iv), (vii) and (viii). In the cases of (iii) and (vii),

$$\begin{aligned}
&\left\{V_n^k(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{k-1}(u,v;\boldsymbol{p}_{n-2})\right\} \\
&\geq \left\{V_n^k(u,v;\boldsymbol{p}_n) - V_{n-1}^{k-1}(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{k-1}(u,v;\boldsymbol{p}_{n-2})\right\} \\
&= np_n - (n-2)p_{n-1} \\
&\geq np_n - (n-2)p_{n-1} - p_1.
\end{aligned}$$

In the cases of (iv) and (viii),

$$\begin{aligned}
&\left\{V_n^{k+1}(u,v;\boldsymbol{p}_n) - V_{n-1}^k(u,v;\boldsymbol{p}_{n-1})\right\} - \left\{V_{n-1}^k(u,v;\boldsymbol{p}_{n-1}) - V_{n-2}^{k-1}(u,v;\boldsymbol{p}_{n-2})\right\} \\
&\geq np_n - (n-2)p_{n-1} \\
&> np_n - (n-2)p_{n-1} - p_1.
\end{aligned}$$

Therefore, (W_n) is derived. □

Theorem 5.2 *For $n \geq 3$, if $\boldsymbol{p}_n \in A_n$, then there exist $r_{n,k}(v;\boldsymbol{p}_{n-1})$ $(k = 1, 2, \cdots, n-1)$ which satisfy $r_{n,1}(v;\boldsymbol{p}_{n-1}) < r_{n,2}(v;\boldsymbol{p}_{n-1}) < \cdots < r_{n,n-1}(v;\boldsymbol{p}_{n-1})$ and the optimal strategy is to perform job n as a batch if $0 < u < r_{n,1}(v;\boldsymbol{p}_{n-1})$, to perform jobs $n, n-1, \cdots, n-k+1$ as a batch if $r_{n,k-1}(v;\boldsymbol{p}_{n-1}) \leq u < r_{n,k}(v;\boldsymbol{p}_{n-1})$ for $2 \leq k \leq n-1$, and to perform jobs $n, n-1, \cdots, 1$ as a batch if $r_{n,n-1}(v;\boldsymbol{p}_{n-1}) \leq u$.*

Proof. This theorem is the immediate consequence of Theorem 2. □

Example 1. Let X be the random variable from the exponential distribution with an unknown mean θ^{-1} and the conjugate prior distribution of θ be the gamma distribution whose density function is given by

$$g(\theta|u,v) = \begin{cases} \Gamma(v)^{-1}u^v\theta^{v-1}e^{-u\theta}, & \text{if } \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$h(u,v) = u(v-1)^{-1},$$

and

$$V_n^k(u,v,\boldsymbol{p}_n) = nu(v-1)^{-1} + n\sum_{j=n-k+1}^{n} p_j + \mathrm{E}\left[V_{n-k}(u+X,v+1,\boldsymbol{p}_{n-k})|u,v\right]$$

for $n = 1,2,3,\cdots$ and $k = 1,2,\cdots,n$ with $V_0(u,v,\boldsymbol{p}_0) = 0$. In this case,

$$V_1(u,v,\boldsymbol{p}_1) = u(v-1)^{-1} + p_1,$$

$$V_2^1(u,v,\boldsymbol{p}_2) = 3u(v-1)^{-1} + 2p_2 + p_1,$$

and

$$V_2^2(u,v,\boldsymbol{p}_2) = 2u(v-1)^{-1} + 2p_2 + 2p_1,$$

that is,

$$V_2(u,v,\boldsymbol{p}_2) = \begin{cases} 3u(v-1)^{-1} + 2p_2 + p_1, & \text{if } 0 < u < r_{2,1}(v;\boldsymbol{p}_1) \\ 2u(v-1)^{-1} + 2p_2 + 2p_1, & \text{if } r_{2,1}(v;\boldsymbol{p}_1) \le u, \end{cases}$$

where

$$r_{2,1}(v;\boldsymbol{p}_1) = p_1(v-1).$$

For $n = 3$, we have

$$V_3^1(u,v,\boldsymbol{p}_3) = \begin{cases} 6u(v-1)^{-1} + 3p_3 + 2p_2 + p_1 + H_{3,1}(v;\boldsymbol{p}_1)u^v, & \text{if } 0 < u < r_{2,1}(v+1;\boldsymbol{p}_1), \\ 5u(v-1)^{-1} + 3p_3 + 2p_2 + 2p_1, & \text{if } r_{2,1}(v+1;\boldsymbol{p}_1) \le u, \end{cases}$$

$$V_3^2(u,v,\boldsymbol{p}_3) = 4u(v-1)^{-1} + 3p_3 + 3p_2 + p_1,$$

and

$$V_3^3(u,v,\boldsymbol{p}_3) = 3u(v-1)^{-1} + 3p_3 + 3p_2 + 3p_1,$$

where

$$H_{3,1}(v;\boldsymbol{p}_1) = -(v-1)^{-1}\left\{r_{2,1}(v+1;\boldsymbol{p}_1)\right\}^{-v+1} + p_1\left\{r_{2,1}(v+1;\boldsymbol{p}_1)\right\}^{-v}.$$

As the inequalities $r_{3,1}(v;\boldsymbol{p}_2) \le r_{2,1}(v+1;\boldsymbol{p}_1) < r_{3,2}(v;\boldsymbol{p}_2)$ are obtained from (S_3) and (Q_3),

$$V_3(u,v,\boldsymbol{p}_3) = \begin{cases} 6u(v-1)^{-1} + 3p_3 + 2p_2 + p_1 + H_{3,1}(v;\boldsymbol{p}_1)u^v, & \text{if } 0 < u < r_{3,1}(v;\boldsymbol{p}_2), \\ 4u(v-1)^{-1} + 3p_3 + 3p_2 + p_1, & \text{if } r_{3,1}(v;\boldsymbol{p}_2) \le u < r_{3,2}(v;\boldsymbol{p}_2), \\ 3u(v-1)^{-1} + 3p_3 + 3p_2 + 3p_1 & \text{if } r_{3,2}(v;\boldsymbol{p}_2) \le u, \end{cases}$$

where

$$r_{3,2}(v;\boldsymbol{p}_2) = 2p_1(v-1)$$

and $r_{3,1}(v;\boldsymbol{p}_2)$ is the unique root of the equation of u;

$$6u(v-1)^{-1} + 3p_3 + 2p_2 + p_1 + H_{3,1}(v;\boldsymbol{p}_1)u^v = 4u(v-1)^{-1} + 3p_3 + 3p_2 + p_1$$

that is,

$$-p_2 + 2u(v-1)^{-1} + \left[-(v-1)^{-1}\left\{r_{2,1}(v+1;\boldsymbol{p}_1)\right\}^{-v+1} + p_1\left\{r_{2,1}(v+1;\boldsymbol{p}_1)\right\}^{-v}\right]u^v = 0.$$

By using $r_{2,1}(v+1;\boldsymbol{p}_1) = p_1 v$ with $p_1 = 1$ in this equation of u, we have

$$-p_2 + 2u(v-1)^{-1} + \left[-(v-1)^{-1}v^{-v+1} + v^{-v}\right]u^v = 0$$

that is

$$(u/v)^v - 2v(u/v) + p_2(v-1) = 0.$$

This equation of u can be solved for $p_2 = 0.99, 0.98, \cdots, 0.80$ and $v = 2, 3, \cdots, 10$ and the values of $r_{3,1}(v; \boldsymbol{p}_2)$ are tabulated in Table 1. Furthermore, for $n = 4$,

$$V_4^1(u,v,\boldsymbol{p}_4) = \begin{cases} 10u(v-1)^{-1} + 4p_4 + 3p_3 + 2p_2 + p_1 + \{H_{4,1}(v;\boldsymbol{p}_2) + H_{4,2}(v;\boldsymbol{p}_2)\}u^v \\ \qquad - H_{3,1}(v+1;\boldsymbol{p}_1)vu^{v+1}, \qquad \text{if } \ 0 < u < r_{3,1}(v+1;\boldsymbol{p}_2), \\ 8u(v-1)^{-1} + 4p_4 + 3p_3 + 3p_2 + p_1 + H_{4,2}(v;\boldsymbol{p}_2)u^v, \\ \qquad\qquad\qquad \text{if } \ r_{3,1}(v+1;\boldsymbol{p}_2) \le u < r_{3,2}(v+1;\boldsymbol{p}_2), \\ 7u(v-1)^{-1} + 4p_4 + 3p_3 + 3p_2 + 3p_1 \qquad \text{if } \ r_{3,2}(v+1;\boldsymbol{p}_2) \le u, \end{cases}$$

where

$$H_{4,1}(v;\boldsymbol{p}_2) = -2(v-1)^{-1}\{r_{3,1}(v+1;\boldsymbol{p}_2)\}^{-v+1} + p_2\{r_{3,1}(v+1;\boldsymbol{p}_2)\}^{-v} \\ + H_{3,1}(v+1;\boldsymbol{p}_1)vr_{3,1}(v+1;\boldsymbol{p}_2)$$

$$H_{4,2}(v;\boldsymbol{p}_2) = -(v-1)^{-1}\{r_{3,2}(v+1;\boldsymbol{p}_2)\}^{-v+1} + 2p_1\{r_{3,2}(v+1;\boldsymbol{p}_2)\}^{-v}.$$

Also,

$$V_4^2(u,v,\boldsymbol{p}_4) = \begin{cases} 7u(v-1)^{-1} + 4p_4 + 4p_3 + 2p_2 + p_1 + H_{3,1}(v;\boldsymbol{p}_1)u^v, \\ \qquad\qquad\qquad\qquad \text{if } 0 < u < r_{2,1}(v+1;\boldsymbol{p}_1) \\ 6u(v-1)^{-1} + 4p_4 + 4p_3 + 2p_2 + 2p_1, \qquad \text{if } r_{2,1}(v+1;\boldsymbol{p}_1) \le u, \end{cases}$$

$$V_4^3(u,v,\boldsymbol{p}_4) = 5u(v-1)^{-1} + 4p_4 + 4p_3 + 4p_2 + p_1,$$

and

$$V_4^4(u,v,\boldsymbol{p}_4) = 4u(v-1)^{-1} + 4p_4 + 4p_3 + 4p_2 + 4p_1.$$

(S_4) means $r_{4,1}(v;\boldsymbol{p}_3) < r_{3,1}(v;\boldsymbol{p}_2)$ and (Q_3) means $r_{3,1}(v;\boldsymbol{p}_2) < r_{3,1}(v+1;\boldsymbol{p}_2)$, $r_{4,1}(v;\boldsymbol{p}_3)$ is the unique root of the following equation of u;

$$10u(v-1)^{-1} + 4p_4 + 3p_3 + 2p_2 + p_1 + H_{4,1}(v;\boldsymbol{p}_2)u^v - H_{4,2}(v;\boldsymbol{p}_2)u^{v+1} \\ = 7u(v-1)^{-1} + 4p_4 + 4p_3 + 2p_2 + p_1 + H_{3,1}(v;\boldsymbol{p}_1)u^v,$$

that is,

$$-p_3 + 3u(v-1)^{-1} + (H_{4,1}(v;\boldsymbol{p}_2) + H_{3,1}(v;\boldsymbol{p}_1))u^v - H_{4,2}(v;\boldsymbol{p}_2)u^{v+1} = 0.$$

Also, if $u > r_{2,1}(v+1;\boldsymbol{p}_1) = p_1 v$, then

$$V_4^2(u,v,\boldsymbol{p}_4) - V_4^3(u,v,\boldsymbol{p}_4) = \frac{p_1 v}{v-1} - (2p_2 - p_1) > 0$$

and this means that $r_{4,2}(v;\boldsymbol{p}_3)$ is the unique root of the following equation of u;

$$7u(v-1)^{-1} + 4p_4 + 4p_3 + 2p_2 + p_1 + H_{3,1}(v;\boldsymbol{p}_1)u^v = 5u(v-1)^{-1} + 4p_4 + 4p_3 + 4p_2 + p_1,$$

that is,

$$-2p_2 + 2u(v-1)^{-1} + H_{3,1}(v;\boldsymbol{p}_1)u^v = 0$$

which means

$$-2p_2 + 2u(v-1)^{-1} + \left[-(v-1)^{-1}\{r_{2,1}(v+1;\boldsymbol{p}_1)\}^{-v+1} + p_1\{r_{2,1}(v+1;\boldsymbol{p}_1)\}^{-v}\right]u^v = 0.$$

Table 1: Values of $r_{3,1}(v, p_1, p_2)$ for $p_1 = 1.00$, $p_2 = 0.99, 0.98, \cdots, 0.90$, and $v = 2, 3, \cdots, 10$

(p_1, p_2)	$v=2$	$v=3$	$v=4$	$v=5$	$v=6$	$v=7$	$v=8$	$v=9$	$v=10$
(1.00,0.99)	0.530	1.009	1.495	1.985	2.477	2.971	3.466	3.960	4.455
(1.00,0.98)	0.524	0.998	1.479	1.965	2.452	2.941	3.431	3.920	4.410
(1.00,0.97)	0.519	0.988	1.464	1.944	2.427	2.911	3.396	3.880	4.365
(1.00,0.96)	0.513	0.977	1.449	1.924	2.402	2.881	3.360	3.840	4.320
(1.00,0.95)	0.507	0.967	1.433	1.904	2.377	2.851	3.325	3.800	4.275
(1.00,0.94)	0.501	0.956	1.418	1.884	2.352	2.821	3.290	3.760	4.230
(1.00,0.93)	0.496	0.946	1.403	1.864	2.327	2.791	3.255	3.720	4.185
(1.00,0.92)	0.490	0.935	1.387	1.843	2.302	2.761	3.220	3.680	4.140
(1.00,0.91)	0.484	0.925	1.372	1.823	2.276	2.731	3.185	3.640	4.095
(1.00,0.90)	0.479	0.914	1.357	1.803	2.251	2.701	3.150	3.600	4.050

By using $p_1 = 1$ and $r_{2,1}(v+1; \boldsymbol{p}_1) = v$, this equation is rewritten as follows:

$$(u/v)^v - 2v(u/v) + 2p_2(v-1) = 0.$$

Furthermore, if $v > 4/3$, we have

$$r_{4,3}(v; \boldsymbol{p}_3) = 3p_1(v-1) > r_{2,1}(v+1; \boldsymbol{p}_1).$$

The values of $r_{4,1}(v; \boldsymbol{p}_3)$ and $r_{4,2}(v; \boldsymbol{p}_3)$ are calculated and tabulated in Table 2 and Table 3, respectively.

6. Properties of A_n.

Now we consider the properties of A_n. Let $\boldsymbol{p}_n^* = (p_1^*, p_2^*, \cdots, p_n^*)$ satisfy

$$kp_{n-k}^* \leq (k+1)p_{n-k-1}^* - p_1^* \qquad (1 \leq k \leq n-2) \tag{6.1}$$

and

$$(i+1)p_{i+1}^* \geq \sum_{j=1}^{i} p_j^* \qquad (1 \leq i \leq n-2). \tag{6.2}$$

In (6.1), the value of p_i^* for $i = 2, 3, \cdots, n-1$ depends on p_{i-1}^* and p_1^*. Especially, the value of p_2^* depends on only the value of p_1^* and we have

$$(n-2)p_2^* \leq (n-1)p_1^* - p_1^*,$$

that is,

$$p_2^* \leq p_1^*.$$

As $p_1^* \geq p_2^* \geq \cdots \geq p_{n-1}^*$, two cases, $p_2^* = p_1^*$ and $p_2^* \leq p_1^*$, have to be considered. In both cases, we restrict our attention to the case

$$(n-i)p_i^* = (n-i+1)p_{i-1}^* - p_1^*$$

for $3 \leq i \leq n-1$ under constraint (6.2).

In the case of $p_2^* = p_1^*$, we have $p_3^* = p_1^*$. Assume that $p_{i-1}^* = p_1^*$ for $3 \leq i \leq n-1$, then we have $p_i^* = p_1^*$. As $p_1^* \geq p_2^* \geq \cdots \geq p_{n-1}^*$, we have, without loss of generality, $p_1^* = 1$, which

Table 2: Values of $r_{4,1}(v, p_1, p_2, p_3)$ for $p_1 = 1.00$, $p_2 = 0.99, 0.98, \cdots, 0.92$, $p_3 = 2p_2 - 1.0, 2p_2 - 1.01, \cdots, 2p_2 - 1.04$, and $v = 2, 3, \cdots, 10$

(p_1, p_2, p_3)	$v = 2$	$v = 3$	$v = 4$	$v = 5$	$v = 6$	$v = 7$	$v = 8$	$v = 9$	$v = 10$
(1.00,0.99,0.98)	0.325	0.641	0.963	1.289	1.617	1.946	2.275	2.604	2.933
(1.00,0.99,0.97)	0.322	0.634	0.953	1.276	1.601	1.927	2.253	2.578	2.904
(1.00,0.99,0.96)	0.319	0.628	0.944	1.264	1.585	1.908	2.230	2.552	2.874
(1.00,0.99,0.95)	0.315	0.622	0.935	1.251	1.569	1.888	2.208	2.526	2.845
(1.00,0.99,0.94)	0.312	0.616	0.925	1.239	1.554	1.869	2.185	2.500	2.815
(1.00,0.98,0.96)	0.319	0.628	0.944	1.263	1.584	1.907	2.229	2.552	2.874
(1.00,0.98,0.95)	0.315	0.622	0.934	1.250	1.569	1.888	2.207	2.526	2.844
(1.00,0.98,0.94)	0.312	0.615	0.925	1.238	1.553	1.869	2.184	2.500	2.815
(1.00,0.98,0.93)	0.309	0.609	0.915	1.225	1.537	1.849	2.162	2.474	2.785
(1.00,0.98,0.92)	0.305	0.603	0.906	1.213	1.521	1.830	2.139	2.448	2.756
(1.00,0.97,0.94)	0.312	0.615	0.924	1.237	1.552	1.868	2.184	2.499	2.814
(1.00,0.97,0.93)	0.309	0.609	0.915	1.225	1.536	1.849	2.161	2.473	2.785
(1.00,0.97,0.92)	0.305	0.602	0.906	1.212	1.521	1.829	2.138	2.447	2.755
(1.00,0.97,0.91)	0.302	0.596	0.896	1.200	1.505	1.810	2.116	2.421	2.726
(1.00,0.97,0.90)	0.299	0.590	0.887	1.187	1.489	1.791	2.093	2.395	2.696
(1.00,0.96,0.92)	0.305	0.602	0.905	1.212	1.520	1.829	2.138	2.447	2.755
(1.00,0.96,0.91)	0.302	0.596	0.896	1.199	1.504	1.810	2.115	2.421	2.726
(1.00,0.96,0.90)	0.299	0.590	0.886	1.186	1.488	1.790	2.093	2.394	2.696
(1.00,0.96,0.89)	0.295	0.583	0.877	1.174	1.472	1.771	2.070	2.368	2.666
(1.00,0.96,0.88)	0.292	0.577	0.867	1.161	1.456	1.752	2.047	2.342	2.637
(1.00,0.95,0.90)	0.299	0.589	0.886	1.186	1.487	1.790	2.092	2.394	2.696
(1.00,0.95,0.89)	0.295	0.583	0.876	1.173	1.472	1.771	2.069	2.368	2.666
(1.00,0.95,0.88)	0.292	0.577	0.867	1.161	1.456	1.751	2.047	2.342	2.636
(1.00,0.95,0.87)	0.289	0.570	0.858	1.148	1.440	1.732	2.024	2.316	2.607
(1.00,0.95,0.86)	0.285	0.564	0.848	1.135	1.424	1.712	2.001	2.289	2.577
(1.00,0.94,0.88)	0.292	0.576	0.867	1.160	1.455	1.751	2.046	2.341	2.636
(1.00,0.94,0.87)	0.289	0.570	0.857	1.147	1.439	1.731	2.023	2.315	2.607
(1.00,0.94,0.86)	0.285	0.564	0.848	1.135	1.423	1.712	2.001	2.289	2.577
(1.00,0.94,0.85)	0.282	0.557	0.838	1.122	1.407	1.693	1.978	2.263	2.547
(1.00,0.94,0.84)	0.279	0.551	0.829	1.109	1.391	1.673	1.955	2.236	2.518
(1.00,0.93,0.86)	0.285	0.563	0.847	1.134	1.423	1.711	2.000	2.289	2.577
(1.00,0.93,0.85)	0.282	0.557	0.838	1.122	1.407	1.692	1.977	2.262	2.547
(1.00,0.93,0.84)	0.279	0.551	0.828	1.109	1.391	1.673	1.955	2.236	2.517
(1.00,0.93,0.83)	0.275	0.544	0.819	1.096	1.375	1.653	1.932	2.210	2.488
(1.00,0.93,0.82)	0.272	0.538	0.809	1.083	1.359	1.634	1.909	2.184	2.458
(1.00,0.92,0.84)	0.279	0.551	0.828	1.108	1.390	1.672	1.954	2.236	2.517
(1.00,0.92,0.83)	0.275	0.544	0.819	1.096	1.374	1.653	1.931	2.210	2.487
(1.00,0.92,0.82)	0.272	0.538	0.809	1.083	1.358	1.633	1.909	2.183	2.458
(1.00,0.92,0.81)	0.269	0.532	0.800	1.070	1.342	1.614	1.886	2.157	2.428
(1.00,0.92,0.80)	0.266	0.525	0.790	1.058	1.326	1.594	1.863	2.131	2.398

Table 3: Values of $r_{4,2}(v, p_1, p_2, p_3)$ for $p_1 = 1.00$, $p_2 = 0.99, 0.98, \cdots, 0.90$, and $v = 2, 3, \cdots, 10$

(p_1, p_2)	$v=2$	$v=3$	$v=4$	$v=5$	$v=6$	$v=7$	$v=8$	$v=9$	$v=10$
(1.00,0.99)	1.157	2.169	3.166	4.159	5.150	6.140	7.129	8.118	9.106
(1.00,0.98)	1.143	2.142	3.127	4.107	5.085	6.063	7.040	8.016	8.993
(1.00,0.97)	1.129	2.115	3.087	4.056	5.022	5.987	6.953	7.918	8.883
(1.00,0.96)	1.116	2.089	3.049	4.005	4.959	5.914	6.867	7.821	8.775
(1.00,0.95)	1.102	2.062	3.010	3.955	4.898	5.841	6.784	7.727	8.670
(1.00,0.94)	1.088	2.036	2.972	3.905	4.837	5.769	6.701	7.634	8.566
(1.00,0.93)	1.074	2.010	2.935	3.856	4.777	5.698	6.620	7.542	8.464
(1.00,0.92)	1.061	1.985	2.898	3.808	4.718	5.629	6.540	7.451	8.364
(1.00,0.91)	1.047	1.959	2.861	3.760	4.660	5.560	6.460	7.362	8.264
(1.00,0.90)	1.034	1.934	2.824	3.713	4.602	5.491	6.382	7.274	8.166

means that $p_1^* = p_2^* = \cdots = p_n^* = 1$, that is, $(1, 1, \cdots, 1) \in A_n$. This is the undiscounted case and the optimal strategy is obtained in Hamada [5] for the case of gamma distribution.

Now, we consider the case of $p_2^* < p_1^*$ and

$$(n-i)p_i^* = (n-i+1)p_{i-1}^* - p_1^*$$

for $3 \leq i \leq n-1$ under constraint (6.2). As $p_1^* = 1$, let $p_2^* = \alpha$ for $0 < \alpha < 1$, then

$$p_i^* = \frac{(n-i+1)p_{i-1}^* - 1}{n-i}$$

for $3 \leq i \leq n-1$. The values of $p_2^*, \cdots, p_{n-2}^*$ and p_{n-1}^* depend on α and n, and we use $p_{i,n}^*(\alpha)$ in place of p_i^* for $i = 2, 3, \cdots, n-1$ and let $p_{2,n}^*(\alpha) = \alpha$. Now we have

$$(n-j)p_{j,n}^*(\alpha) = (n-j+1)p_{j-1,n}^*(\alpha) - 1$$

for $3 \leq j \leq n-1$. Then, add this equation side by side for $3 \leq j \leq i$, we have

$$\sum_{j=3}^{i}(n-j)p_{j,n}^*(\alpha) = \sum_{j=3}^{i}(n-j+1)p_{j-1,n}^*(\alpha) - (i-2)$$

for $3 \leq i \leq n-1$, that is,

$$(n-i)p_{i,n}^*(\alpha) = (n-2)p_{2,n}^*(\alpha) - (i-2),$$

which means that

$$p_{i,n}^*(\alpha) = \frac{(n-2)\alpha - (i-2)}{n-i} \tag{6.3}$$

for $3 \leq i \leq n-1$. Then,

$$p_{i,n}^*(\alpha) - p_{i,n-1}^*(\alpha) = \frac{(i-2)(1-\alpha)}{(n-i)(n-i-1)}.$$

As $i \geq 3$ and $0 < \alpha < 1$, $p_{i,n}^*(\alpha) > p_{i,n-1}^*(\alpha)$ for $3 \leq i \leq n-1$, and by the same way, we have

$$p_{i,n}^*(\alpha) > p_{i,n-1}^*(\alpha) > \cdots > p_{i,i+1}^*(\alpha)$$

Table 4: Values of $\bar{n}(\alpha)$ for $\alpha = 0.99, 0.98, \cdots, 0.60$

α	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.90
$\bar{n}(\alpha)$	101	51	35	26	21	18	16	14	13	11
α	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.80
$\bar{n}(\alpha)$	11	10	9	9	8	8	7	7	7	6
α	0.79	0.78	0.77	0.76	0.75	0.74	0.73	0.72	0.71	0.70
$\bar{n}(\alpha)$	6	6	6	6	5	5	5	5	5	5
α	0.69	0.68	0.67	0.66	0.65	0.64	0.63	0.62	0.61	0.60
$\bar{n}(\alpha)$	5	5	5	4	4	4	4	4	4	4

and therefore

$$\min_{i+1 \leq k \leq n} p^*_{i,k}(\alpha) = p^*_{i,i+1}(\alpha).$$

Since (6.3) holds for $i = n - 1$,

$$p^*_{n-1,n}(\alpha) = \alpha - (n-3)(1-\alpha),$$

from which $p^*_{n-1,n}(\alpha) > 0$ means that $\alpha/(1-\alpha) > n - 3$, that is,

$$n < 3 + \frac{\alpha}{1-\alpha}.$$

Let $\bar{n}(\alpha)$ be the largest n that satisfies $n < 3 + \alpha/(1-\alpha)$. Then, we have the following theorem.

Theorem 6.1 *For* $0 < \alpha < 1$,
(i) $\bar{n}(\alpha) \geq 3$ *and*
(ii) $\bar{n}(\alpha) = m$ *if* $\dfrac{m-3}{m-2} < \alpha \leq \dfrac{m-2}{m-1}$.

Proof. From the definition of $\bar{n}(\alpha)$, $\bar{n}(\alpha) \geq 3$ is trivial and we also have

$$m < 3 + \frac{\alpha}{1-\alpha} \leq m + 1$$

that is

$$m - 3 < \frac{\alpha}{1-\alpha} \leq m - 2$$

from which

$$\frac{m-3}{m-2} < \alpha \leq \frac{m-2}{m-1}. \qquad \square$$

Values of $\bar{n}(\alpha)$ for $\alpha = 0.99, 0.98, \cdots, 0.60$ are tabulated in Table 4.

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