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SENSITIVITY TO SERVICE TIMES IN INFINITE-SERVER SYSTEMS

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Abstract The stationary distribution of the number of customers in the infinite-server system with non-Poissonian arrivals is dependent on the form of the service time distribution. In particular, when the interarrival time is a hyperexponential random variable that is more variable than an exponential random variable, the stationary state distribution becomes stochastically less variable, as the service time becomes more variable. On the other hand, when the interarrival time is an Erlang random variable that is less variable than an exponential random variable, the more variable service time yields the more variable stationary state distribution.

Keywords: Queue, $H_2/G/\infty$, $E_n/G/\infty$, sensitivity, service time, stochastically more variable

1. Introduction

It is well-known that the $M/G/\infty$ has the insensitivity to the service time distribution G [9, 12]. The stationary state distribution of the number of customers in system $\{p_n, n = 1, 2, \ldots\}$ is a Poisson distribution with parameter λ/μ, independent of the form of the service time distribution, where λ^{-1} and μ^{-1} are respectively the mean interarrival time and the mean service time. Thus, $\{p_n, n = 1, 2, \ldots\}$ depends only on the mean of service times. It is quite interesting what happens when the arrival process is not Poisson. Although the homogeneity of Poisson arrivals may be weakened [5, 7], the generalization to renewal arrivals will disappear the insensitivity. We may not substitute the elementary system $GI/M/\infty$ for the general system $GI/G/\infty$ as the equivalent random theory [11] does.

The main object of this paper is to study the sensitivity of the service time distribution in the $GI/G/\infty$. For this, we introduce some stochastic order being stochastically more or less variable between random variables [9]. We consider whether or not the number of customers in system becomes more variable, as the service time becomes more variable and expect as follows:

(i) When the interarrival time is a hyperexponential random variable that is more variable than an exponential random variable, often called burst arrivals, the stationary state distribution becomes stochastically less variable as the service time becomes more variable. That means that the $H_n/D/\infty$ with deterministic services is the worst on performance in $H_n/G/\infty$ systems with the burst arrivals.

(ii) When the interarrival time is an Erlang random variable that is less variable than an exponential random variable, often called smooth arrivals, the stationary state distribution becomes stochastically more variable as the service time random variable becomes more variable. The $E_n/D/\infty$ is the best on performance in $E_n/G/\infty$ systems with the smooth arrivals.

In order to prove these expectations, some conditions for arrival and service processes must be added. In section 2, the variance $Var(GI/D/\infty)$ of the number of customers in the $GI/D/\infty$ and the variance $Var(GI/M/\infty)$ of the $GI/M/\infty$ are compared. In particular, a hyperexponential distribution with two phases is considered for the interarrival times of burst arrivals. Then, it is shown that $Var(H_2/D/\infty)$ is greater than $Var(H_2/M/\infty)$ for the fixed mean service times. However, the generalization to the case with n-phases ($n > 2$) is not difficult. For smooth arrivals, n-th Erlang distributions ($n = 2, 3,$) are considered. It is also proved that $Var(E_n/M/\infty)$ is greater than $Var(E_n/D/\infty)$.

In section 3, we go towards the $H_2/G/\infty$ and $E_n/G/\infty$. Of interests are the impact of the second moment of service times on the system variance. When the random variable S of the service time is more variable than or equal to the exponential one, the distribution $S(x) = 1-p_1e^{-\mu_1 x}-p_2e^{-\mu_2 x}$ $(p_1+p_2 = 1,\ p_1 \geq 0,\ p_2 \geq 0)$ with the mean $1/\mu = p_1/\mu_1+p_2/\mu_2$ is considered. We may write the $GI/G/\infty$ as the $GI/M_1, M_2/\infty$ or $GI/H_2/\infty$. If S is less variable than the exponential one, the distribution $S(x) = 1 - p_1e^{-\mu x} - p_2I_{\{x\leq\mu^{-1}\}}$ $(p_1 + p_2 = 1,\ p_1 \geq 0,\ p_2 \geq 0)$ is considered, where $I_{\{\cdot\}}$ is an index function. We may write the $GI/GI/\infty$ as the $GI/M, D/\infty$. Under these conditions, Expectations (i) and (ii) are proved. When the arrivals are burst, we have

$$Var(GI/D/\infty) > Var(GI/M, D/\infty) > Var(GI/M/\infty) > Var(GI/M_1, M_2/\infty).$$

In particular, when the variance of the service times is sufficiently large and $p_1/\mu_1 << p_2/\mu_2$ (extremely unbalanced condition), $Var(GI/H_2/\infty)$ is close to $Var(M/G/\infty)$. When arrivals are smooth, we have

$$Var(GI/D/\infty) < Var(GI/M, D/\infty) < Var(GI/M/\infty) < Var(GI/M_1, M_2/\infty).$$

2. $GI/D/\infty$ and $GI/M/\infty$

We study the variance of the number of customers in the $GI/D/\infty$. Let $t_n = n\mu^{-1}(n = 0, 1, \ldots)$ be a sequence of times with an initial time point $t_0 = 0$ for the mean service time μ^{-1}. A stationary state distribution $\{p_j^{(D)}, j \geq 0\}$ of the number of customers in the system at $\lim_{n\to\infty} t_n$ is considered. We note that this distribution is equal to the stationary state distribution at an arbitrary time point. The mean and variance are given by

$$E(GI/D/\infty) = \sum_{j=1}^{\infty} jp_j^{(D)}$$

and

$$Var(GI/D/\infty) = \sum_{j=1}^{\infty} j^2p_j^{(D)} - \left(\sum_{j=1}^{\infty} jp_j^{(D)}\right)^2.$$

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent nonnegative random variables with X_1 having distribution $B(\cdot)$ and X_n having distribution $A(\cdot), n > 1$. Let $S_0 = 0,\ S_n = \sum_{i=1}^{n} X_i,\ n \geq 1$, and define a delayed renewal process

$$N_D(t) = \sup\{n : S_n \leq t\}.$$

Since the observed process may be in the equilibrium at t_0, X_1 has the equilibrium distribution of $A(x)$, that is,

$$B(x) = \lambda \int_0^x [1 - A(y)]dy$$

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and

$$B^*(s) = \frac{\lambda[1 - A^*(s)]}{s}, \tag{2.1}$$

where

$$\lambda^{-1} = \int_0^\infty [1 - A(y)]dy.$$

Since the number of customers in the system at $t_{n+1} = (n+1)\mu^{-1}$ is equal to the number of arriving customers in $(t_n, t_{n+1}]$, we have for $\mu^{-1} = t_{n+1} - t_n$

$$p_j^{(D)} = P\{N_D(\mu^{-1}) = j\}.$$

We have now, using convolution notation,

$$P\{N_D(t) = j\} = P\{S_j \leq t\} - P\{S_{j+1} \leq t\} = B * A^{(j-1)*}(t) - B * A^{j*}(t).$$

Let

$$E[N_D(t)] = \sum_{j=1}^{\infty} jP\{N_D(t) = j\}$$

and

$$Var[N_D(t)] = \sum_{j=1}^{\infty} j^2 P\{N_D(t) = j\} - \left(\sum_{j=1}^{\infty} jP\{N_D(t) = j\}\right)^2.$$

Then, it is easily shown that

$$E[N_D(t)] = \sum_{j=1}^{\infty} B * A^{(j-1)*}(t)$$

and

$$Var[N_D(t)] = 2\sum_{j=1}^{\infty} jB * A^{(j-1)*}(t) - E[N_D(t)] - \{E[N_D(t)]\}^2.$$

Taking the Laplace transforms of these equations, we have

$$\int_0^\infty e^{-st} dE[N_D(t)] = \frac{B^*(s)}{1 - A^*(s)} = \frac{\lambda}{s}$$

and

$$\int_0^\infty e^{-st} dVar[N_D(t)] = \frac{2B^*(s)}{\{1 - A^*(s)\}^2} - \frac{\lambda}{s} - \left(\frac{\lambda}{s}\right)^2.$$

Equation (2.1) gives

$$\frac{B^*(s)}{\{1 - A^*(s)\}^2} = \frac{\lambda}{s\{1 - A^*(s)\}}.$$

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Thus, we have

$$E[N_D(t)] = \lambda t$$

and

$$Var[N_D(t)] = 2\ \lambda \int_0^t \mathcal{L}_u^{-1}\left[\frac{1}{s\{1 - A^*(s)\}}\right] du - \lambda t - (\lambda t)^2, \tag{2.2}$$

where $\mathcal{L}_u^{-1}$ denotes the Laplace inversion from s to u. When $A(\cdot)$ is an exponential distribution with λ^{-1}, we have

$$Var[N_D(t)] = \lambda t.$$

Let $A(\cdot)$ be a hyperexponential distribution with the mean λ^{-1} that is more variable than an exponential distribution with the mean λ^{-1}, that is ,

$$A(x) = 1 - k_1 e^{-\lambda_1 x} - k_2 e^{-\lambda_2 x}, \tag{2.3}$$

the Laplace transform is given by

$$A^*(s) = \frac{k_1\lambda_1}{s + \lambda_1} + \frac{k_2\lambda_2}{s + \lambda_2}. \tag{2.4}$$

Such an arrival process which is more variable than an exponential one is often called a bursty arrival process. We will say that arrivals are bursty. We have now

$$\frac{1}{s[1 - A^*(s)]} = \frac{(s + \lambda_2)(s + \lambda_1)}{s^2(s + k_1\lambda_2 + k_2\lambda_1)} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s + k_1\lambda_2 + k_2\lambda_1}, \tag{2.5}$$

where

$$a = \lambda,$$

$$b = \lambda(\lambda_1 + \lambda_2 - \lambda)/\lambda_1\lambda_2 = \frac{1}{2}\left(\frac{Var(A)}{[E(A)]^2} + 1\right)$$

and

$$c = 1 - (\lambda_1 + \lambda_2)\lambda/\lambda_1\lambda_2 + \lambda^2/\lambda_1\lambda_2 = \frac{1}{2}\left(1 - \frac{Var(A)}{[E(A)]^2}\right).$$

Finally, we have from (2.2) and (2.5)

$$Var[N_D(t)] = \frac{Var(A)}{[E(A)]^2}\lambda t + \left\{1 - \frac{Var(A)}{[E(A)]^2}\right\} \frac{1 - e^{-(k_1\lambda_2 + k_2\lambda_1)t}}{(k_1\lambda_2 + k_2\lambda_1)t}\lambda t. \tag{2.6}$$

The variance of the number of customers in the $GI/D/\infty$ is equal to $Var[N_D(\mu^{-1})]$. When GI arrivals is H_2 ones, we have from (2.6)

$$Var(H_2/D/\infty) = \frac{\lambda}{\mu}\left[\frac{Var(A)}{[E(A)]^2} + \left\{1 - \frac{Var(A)}{[E(A)]^2}\right\} \frac{1 - e^{-(k_1\lambda_2 + k_2\lambda_1)/\mu}}{(k_1\lambda_2 + k_2\lambda_1)/\mu}\right]. \tag{2.7}$$

We next consider the mean and variance for the number of customers in an $H_2/M/\infty$. Those are respectively given by [1, 10]

$$E(H_2/M/\infty) = \frac{\lambda}{\mu}$$

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and

$$Var(H_2/M/\infty) = \frac{\lambda}{\mu}\frac{1}{1-A^*(\mu)} - \left(\frac{\lambda}{\mu}\right)^2. \tag{2.8}$$

From (2.4), we have for (2.8)

$$Var(H_2/M/\infty) = \frac{\lambda}{\mu}\left[\frac{1}{2}\left(\frac{Var(A)}{[E(A)]^2}+1\right)+\frac{1}{2}\left(1-\frac{Var(A)}{[E(A)]^2}\right)\frac{\mu}{\mu+k_1\lambda_2+k_2\lambda_1}\right]. \tag{2.9}$$

From (2.7) and (2.9), we have for $a = k_1\lambda_2 + k_2\lambda_2$

$$\begin{aligned} f(\mu) &\equiv Var(H_2/D/\infty) - Var(H_2/M/\infty) \\ &= \frac{\lambda}{\mu}\left(\frac{Var(A)}{[E(A)]^2}-1\right)\left(\frac{1}{2}-\frac{1-e^{-a/\mu}}{a/\mu}+\frac{1}{2}\frac{1}{1+a/\mu}\right). \end{aligned}$$

Since

$$\frac{Var(A)}{[E(A)]^2} \geq 1$$

and

$$\frac{1}{2}-\frac{1-e^{-a/\mu}}{a/\mu}+\frac{1}{2}\frac{1}{1+a/\mu} \geq 0,$$

we have

$$Var(H_2/D/\infty) \geq Var(H_2/M/\infty)$$

with equality if and only if A is exponential.

Let us consider the case in which the interarrival distribution is Erlang or Gamma type one with parameter $(n, n\lambda)$, that is,

$$A(t) = \int_0^t \frac{n\lambda e^{-n\lambda x}(n\lambda x)^{n-1}}{(n-1)!}dx \tag{2.10}$$

and

$$A^*(s) = \left(\frac{n\lambda}{s+n\lambda}\right)^n,$$

where $E(A) = 1/\lambda$ and $Var(A) = 1/n\lambda^2$ for the random variable A. This random variable A is written as E_n and the $E_n/D/\infty$ will be studied. The variance for the number of customers in the $E_n/D/\infty$ is given by

$$Var(E_n/D/\infty) = 2\lambda\int_0^{1/\mu}\mathcal{L}_t^{-1}\left[\frac{1}{s\{1-A^*(s)\}}\right]dt - \frac{\lambda}{\mu} - \left(\frac{\lambda}{\mu}\right)^2, \tag{2.11}$$

where

$$\frac{1}{1-A^*(s)} = \frac{(s+n\lambda)^n}{(s+n\lambda)^n-(n\lambda)^n}. \tag{2.12}$$

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An equation of the n-th degree

$$(s+n\lambda)^n-(n\lambda)^n=0$$

has n different roots

$$-n\lambda(1-e^{2\pi ik/n}),\ k=0,1,\ldots,n-1$$

for $i^2=-1$. Using these roots, we have from (2.11) and (2.12)

$$\begin{aligned}
&Var(E_n/D/\infty)=2\lambda\int_0^{1/\mu}\mathcal{L}_t^{-1}\left[\frac{1}{s\{1-A^*(s)\}}\right]dt-\frac{\lambda}{\mu}-\left(\frac{\lambda}{\mu}\right)^2\\
=&\ 2\lambda\int_0^{1/\mu}\mathcal{L}_t^{-1}\left[\frac{(s+n\lambda)^n}{s^2\sum_{k=1}^{n-1}\ {}_nC_k s^{n-k-1}(n\lambda)^k}\right]dt-\frac{\lambda}{\mu}-\left(\frac{\lambda}{\mu}\right)^2\\
=&\ 2\lambda\int_0^{1/\mu}\mathcal{L}_t^{-1}\left[\frac{(s+n\lambda)^n}{s^2\prod_{k=1}^{n-1}\{s+n\lambda(1-e^{2\pi ik/n})\}}\right]dt-\frac{\lambda}{\mu}-\left(\frac{\lambda}{\mu}\right)^2\\
=&\ \frac{\lambda}{n\mu}\\
+&\ 2\lambda\int_0^{1/\mu}\mathcal{L}_t^{-1}\left[\sum_{k=1}^{n-1}\frac{(n\lambda e^{2\pi ik/n})^n}{\{n\lambda(1-e^{2\pi ik/n})\}^2\prod_{l\neq k}\{n\lambda(e^{2\pi ik/n}-e^{2\pi il/n})\}}\frac{1}{s+n\lambda(1-e^{2\pi ik/n})}\right]dt\\
=&\ \frac{\lambda}{n\mu}+2\lambda\int_0^{1/\mu}\sum_{k=1}^{n-1}\frac{e^{-n\lambda(1-e^{2\pi ik/n})t}}{(1-e^{2\pi ik/n})^2\prod_{l\neq k}(e^{2\pi ik/n}-e^{2\pi il/n})}dt.
\end{aligned}$$

Separating this equation into conjugate parts, we have the following forms:
For even n,

$$\begin{aligned}
Var(E_n/D/\infty)\ =&\ \frac{\lambda}{n\mu}+2\lambda\int_0^{1/\mu}\sum_{k=1}^{n/2-1}\frac{e^{-n\lambda(1-e^{2\pi ki/n})t}}{(1-e^{2\pi ki/n})^2\prod_{l\neq k}(e^{2\pi ki/n}-e^{2\pi li/n})}dt\\
+&\ 2\lambda\int_0^{1/\mu}\sum_{k=1}^{n/2-1}\frac{e^{-n\lambda(1-e^{-2\pi ki/n})t}}{(1-e^{-2\pi ki/n})^2\prod_{l\neq k}(e^{-2\pi ki/n}-e^{-2\pi li/n})}dt\\
+&\ 2\lambda\int_0^{1/\mu}\frac{e^{-2n\lambda t}}{4\prod_{l\neq n/2}(-1-e^{2\pi li/n})}dt
\end{aligned}$$

or

$$\begin{aligned}
Var(E_n/D/\infty)\ =&\ \frac{\lambda}{\mu}\left[\frac{1}{n}+2\sum_{k=1}^{n/2-1}\frac{\{1-e^{-n\lambda(1-e^{2\pi ki/n})/\mu}\}\{n\lambda(1-e^{2\pi ki/n})\}^{-1}\mu}{(1-e^{2\pi ki/n})^2\prod_{l\neq k}(e^{2\pi ki/n}-e^{2\pi li/n})}\right.\\
+&\ 2\sum_{k=1}^{n/2-1}\frac{\{1-e^{-n\lambda(1-e^{-2\pi ki/n})/\mu}\}\{n\lambda(1-e^{-2\pi ki/n})\}^{-1}\mu}{(1-e^{-2\pi ki/n})^2\prod_{l\neq k}(e^{-2\pi ki/n}-e^{-2\pi li/n})}\\
+&\ \left.2\frac{(1-e^{-2n\lambda/\mu})(2n\lambda)^{-1}\mu}{4\prod_{l\neq n/2}(-1-e^{2\pi li/n})}\right].
\end{aligned}\tag{2.13}$$

Note that the second and third terms of the right-hand-side in (2.13) are conjugate each other.

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For odd n,

$$Var(E_n/D/\infty) = \frac{\lambda}{n\mu} + 2\lambda\int_0^{1/\mu}\sum_{k=1}^{(n-1)/2}\frac{e^{-n\lambda(1-e^{2\pi ik/n})t}}{(1-e^{2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{2\pi ik/n}-e^{2\pi il/n})}dt$$
$$+\ 2\lambda\int_0^{1/\mu}\sum_{k=1}^{(n-1)/2}\frac{e^{-n\lambda(1-e^{-2\pi ik/n})t}}{(1-e^{-2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{-2\pi ik/n}-e^{-2\pi il/n})}dt$$

or

$$Var(E_n/D/\infty) = \frac{\lambda}{\mu}\left[\frac{1}{n} + 2\sum_{k=1}^{(n-1)/2}\frac{\{1-e^{-n\lambda(1-e^{2\pi ki/n})/\mu}\}\{n\lambda(1-e^{2\pi ki/n})\}^{-1}\mu}{(1-e^{2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{2\pi ik/n}-e^{2\pi il/n})}\right.$$
$$\left.+\ 2\sum_{k=1}^{(n-1)/2}\frac{\{1-e^{-n\lambda(1-e^{-2\pi ki/n})/\mu}\}\{n\lambda(1-e^{-2\pi ki/n})\}^{-1}\mu}{(1-e^{-2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{-2\pi ik/n}-e^{-2\pi il/n})}\right]. \tag{2.14}$$

The second and third terms of the right-hand-side in (2.14) are also conjugate each other.

When the service times are exponential, we have

$$Var(E_n/M/\infty) = \frac{\lambda}{\mu\{1-A^*(\mu)\}} - \left(\frac{\lambda}{\mu}\right)^2$$
$$= \frac{1}{2}\left(1+\frac{1}{n}\right)\frac{\lambda}{\mu} + \lambda\sum_{k=1}^{n-1}\frac{\{\mu+n\lambda(1-e^{2\pi ik/n})\}^{-1}}{(1-e^{2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{2\pi ik/n}-e^{2\pi il/n})}.$$

For even n,

$$\begin{aligned}Var(E_n/M/\infty) &= \frac{\lambda}{\mu}\left[\frac{1}{2}\left(1+\frac{1}{n}\right) + \sum_{k=1}^{n/2-1}\frac{\{\mu+n\lambda(1-e^{2\pi ik/n})\}^{-1}\mu}{(1-e^{2\pi ki/n})^2\prod_{l\neq k}(e^{2\pi ki/n}-e^{2\pi li/n})}\right.\\ &+ \sum_{k=1}^{n/2-1}\frac{\{\mu+n\lambda(1-e^{-2\pi ik/n})\}^{-1}\mu}{(1-e^{-2\pi ki/n})^2\prod_{l\neq k}(e^{-2\pi ki/n}-e^{-2\pi li/n})}\\ &\left.+ \frac{(\mu+2n\lambda)^{-1}\mu}{4\prod_{l\neq n/2}(-1-e^{2\pi li/n})}\right].\end{aligned} \tag{2.15}$$

Note that the second and third terms of the right-hand-side in (2.15) are conjugate each other. For odd n,

$$\begin{aligned}Var(E_n/M/\infty) &= \frac{\lambda}{\mu}\left[\frac{1}{2}\left(1+\frac{1}{n}\right) + \sum_{k=1}^{(n-1)/2}\frac{\{\mu+n\lambda(1-e^{2\pi ik/n})\}^{-1}\mu}{(1-e^{2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{2\pi ik/n}-e^{2\pi il/n})}\right.\\ &\left.+ \sum_{k=1}^{(n-1)/2}\frac{\{\mu+n\lambda(1-e^{-2\pi ik/n})\}^{-1}\mu}{(1-e^{-2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{-2\pi ik/n}-e^{-2\pi il/n})}\right].\end{aligned} \tag{2.16}$$

The second and third terms of the right-hand-side in (2.16) are also conjugate each other.

Noting that all the coefficients of $\{\mu+n\lambda(1-e^{2\pi ik/n})\}^{-1}\mu$ in (2.15) or (2.16) and $\{1-e^{-n\lambda(1-e^{2\pi ki/n})/\mu}\}\{n\lambda(1-e^{2\pi ki/n})\}^{-1}\mu$ in (2.13) or (2.14) are common for $k=1,2,\ldots,n/2-1$ or $k=1,2,\ldots,(n-1)/2$ and taking the difference between (2.15) and (2.13) or (2.16) and (2.14), we obtain for any $\mu>0$

$$f(\mu) \equiv Var(E_n/M/\infty) - Var(E_n/D/\infty) > 0.$$

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Then, we have

$$Var(E_n/D/\infty) < Var(E_n/M/\infty) < Var(M/G/\infty) \ \left(= \frac{\lambda}{\mu}\right).$$

Let us now give examples for $n = 2, 3, 4$. When $n = 2$, we have

$$Var(E_2/D/\infty) = \frac{\lambda}{\mu}\left(\frac{1}{2} + \frac{1}{2}\,\frac{1 - e^{-4\lambda/\mu}}{4\lambda/\mu}\right),$$

$$Var(E_2/M/\infty) = \frac{\lambda}{\mu}\left(\frac{3}{4} + \frac{1}{4}\,\frac{1}{1 + 4\lambda/\mu}\right)$$

and then

$$Var(E_2/D/\infty) < Var(E_2/M/\infty) < Var(M/G/\infty) \ \left(= \frac{\lambda}{\mu}\right).$$

When $n = 3$, we have

$$Var(E_3/D/\infty) = \frac{\lambda}{\mu}\left(\frac{1}{3} + \frac{2}{3}\,\frac{1 - e^{-9\lambda/2\mu}\cos\frac{3\sqrt{3}\lambda}{2\mu}}{9\lambda/2\mu}\right),$$

$$Var(E_3/M/\infty) = \frac{\lambda}{\mu}\left(\frac{2}{3} + \frac{1}{3}\,\frac{1 + 6\lambda/\mu}{1 + 9\lambda/\mu + 27(\lambda/\mu)^2}\right)$$

and then

$$Var(E_3/D/\infty) < Var(E_3/M/\infty) < Var(M/G/\infty) \ \left(= \frac{\lambda}{\mu}\right).$$

When $n = 4$, we have

$$Var(E_4/D/\infty) = \frac{\lambda}{\mu}\left(\frac{1}{4} + \frac{1}{2}\,\frac{1 - e^{-4\lambda/\mu}\cos\frac{4\lambda}{\mu}}{4\lambda/\mu} + \frac{1}{4}\,\frac{1 - e^{-8\lambda/\mu}}{8\lambda/\mu}\right),$$

$$Var(E_4/M/\infty) = \frac{\lambda}{\mu}\ \left(\frac{5}{8} + \frac{1}{4}\,\frac{1 + 8\lambda/\mu}{1 + 8\lambda/\mu + 32(\lambda/\mu)^2} + \frac{1}{8}\,\frac{1}{1 + 8\lambda/\mu}\right)$$

and then

$$Var(E_4/D/\infty) < Var(E_4/M/\infty) < Var(M/G/\infty) \ \left(= \frac{\lambda}{\mu}\right).$$

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3. Towards $GI/G/\infty$

We study a $GI(A(x))/G(S(x))/\infty$ in which $A(x)$ is an interarrival time distribution and $S(x)$ is a service time distribution. In particular, when the random variable A is more variable than or equal to the exponential one, the distribution $A(\cdot)$ is supposed to be hyperexponential such as (2.3). When A is less variable than the exponential type, $A(\cdot)$ is supposed to be Erlang or Gamma type such as (2.10). If the random variable S is more variable than or equal to the exponential one, let the distribution be $S(x) = 1-p_1e^{-\mu_1 x}-p_2e^{-\mu_2 x}$ $(p_1+p_2 = 1)$. Otherwise, we consider $S(x) = 1 - p_1e^{-\mu_1 x} - p_2I_{\{x\le\mu_2^{-1}\}}$ $(p_1 + p_2 = 1)$ which is a mixture of exponential and deterministic distributions, where $I_{\{\cdot\}}$ is an index function. Each parameter may be determined by moment matchings. In the $GI(A(x))/G(S(x))/\infty$, an arrival has the service time distribution $1 - e^{-\mu_1 x}$ with probability p_1 and has the service time distribution $1 - e^{-\mu_2 x}$ or the service time distribution $1 - I_{\{x\le\mu_2^{-1}\}}$ with probability p_2. Then, the arrival stream is decomposed into two streams called a p_1-decomposed stream selected with probability p_1 and a p_2-decomposed stream selected with probability p_2. We call its decomposition (p_1, p_2)-decomposition. Letting A_i $(i = 1, 2)$ denote the random variable of interarrival times of a p_i-decomposed stream, the $GI(A_i(x))/M(1-e^{-\mu_1 x})/\infty$ and $GI(A_i(x))/M(1-e^{-\mu_2 x})/\infty$ or the $GI(A_i(x))/M(1-e^{-\mu_1 x})/\infty$ and $GI(A_i(x))/D(1-I_{\{x\le\mu_2^{-1}\}})/\infty$ are marginal systems of the $GI(A(x))/G(S(x))/\infty$. Hereafter, we will study these two marginal systems and the correlation between them.

The Laplace transform of the interarrival time distribution $A_i(x)$ $(i = 1, 2)$ of p_i-decomposed arriving customers who have a service time distribution $1 - e^{-\mu_i x}$ is given by

$$A_i^*(s) = \int_0^\infty e^{-st} dA_i(x) = \frac{p_i A^*(s)}{1-(1-p_i)A^*(s)}.$$

Note that

$$E(A_i) = \frac{E(A)}{p_i},$$

$$E(A_i^2) = \frac{E(A^2)}{p_i} + \frac{2(1-p_i)[E(A)]^2}{p_i^2}$$

and

$$\frac{E(A_i^2)}{2[E(A_i)]^2} = p_i\frac{E(A^2)}{2[E(A)]^2} + 1 - p_i. \tag{3.1}$$

If

$$\frac{E(A^2)}{2[E(A)]^2} \ge 1,$$

$$1 \le \frac{E(A_i^2)}{2[E(A_i)]^2} \le \frac{E(A^2)}{2[E(A)]^2}. \tag{3.2}$$

If

$$\frac{E(A^2)}{2[E(A)]^2} \le 1,$$

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$$1 \geq \frac{E(A_i^2)}{2[E(A_i)]^2} \geq \frac{E(A^2)}{2[E(A)]^2}.$$

When a random variable A is more variable than an exponential random variable, the (p_1, p_2)-decompositions make both A_1 and A_2 less variable than A. However, these random variables are still more variable than an exponential one and have decreasing arrival rates. The decreasing arrival rate is called the decreasing failure rate in [9]. Burst is weakened due to the (p_1, p_2)-decompositions. If the random variable A is less variable than an exponential random variable, the (p_1, p_2)-decompositions make each of A_1 and A_2 more variable than A. We note that these random variables still have increasing arrival rates. The increasing arrival rate is called the increasing failure rate in [9]. Smoothness is weakened due to the (p_1, p_2)-decompositions.

Let us consider the p_1-decomposed (p_2-decomposed) sequence of nonnegative independent random variables with the common distribution $A_1(\cdot)$ ($A_2(\cdot)$). Then, we have the following theorem.

Theorem 3.1 *When A is a decreasing (increasing) arrival rate random variable, then A_1 and A_2 are also decreasing (increasing) arrival rate random variables and the superposed sequence of A_1 and A_2 sequences has the positive (negative) correlation.*

Proof. Since A is a non-negative random variable with $\lambda^{-1} = E(A)$, A is a decreasing (increasing) arrival rate random variable if and only if A is more (less) variable than or equal to an exponential random variable with the rate λ. If A is more (less) variable than or equal to an exponential random variable with the rate λ,

$$A^*(s) \geq \ (\leq) \ \frac{\lambda}{s+\lambda} \quad \text{for any } s \geq 0,$$

we have

$$\frac{1 - A^*(s)}{A^*(s)} \leq \ (\geq) \ \frac{s}{\lambda} \quad \text{for any } s \geq 0.$$

This gives for $i = 1, 2$

$$A_i^*(s) = \frac{p_i}{\{1 - A^*(s)\}\{A^*(s)\}^{-1} + p_i} \geq \ (\leq) \ \frac{p_i\lambda}{s + p_i\lambda} \quad \text{for any } s \geq 0.$$

Hence, A_i is more (less) variable than or equal to an exponential random variable with the rate $p_i\lambda$ and have decreasing (increasing) arrival rates.

Next, we define the following arrival rate functions:

$$\lambda(t) = \frac{A'(t)}{1 - A(t)},$$

$$\lambda_1(t) = \frac{A_1'(t)}{1 - A_1(t)},$$

and

$$\lambda_2(t) = \frac{A_2'(t)}{1 - A_2(t)},$$

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where

$$\lambda(t) = \lambda_1(t) + \lambda_2(t)$$

and

$$\lambda_i(0) = p_i\lambda(0) \ (i = 1, 2).$$

Let 0 denote a p_1-decomposed arrival time point. Let $-a_2$ denote the last p_2-decomposed arrival time point before 0. In addition, let $-b$ denote the last arrival point of a p_1-decomposed or a p_2-decomposed arrival time point. Note that $-a_2 \leq -b$ in which b is the interarrival time. Then, if there are no arrivals $(0, t]$, the arrival rate of p_2-customers at time t is given by $\lambda_2(a_2 + t)$. Since A_2 is a decreasing arrival rate random variable, we have

$$\lambda_2(a_2 + t) < \lambda_2(t).$$

It follows that

$$\exp\left[-\int_0^t \lambda_2(a_2 + x)dx\right] = \exp\left[-\int_{a_2}^{a_2+t} \lambda_2(x)dx\right] > \exp\left[-\int_0^t \lambda_2(x)dx\right]. \quad (3.3)$$

The left-hand-side is increasing in a_2. As the interarrival time b increases, a_2 also increases. The relationship (3.3) signifies that the longer a_2 yields the more delay of the next p_2-decomposed arrival after 0. Thus, the succeeding interarrival times have the positive correlation.

At the similar manner, letting $-a_1$ denote the last p_1-decomposed arrival time point before 0 at which a p_2-decomposed arrival occurs, we have

$$\exp\left[-\int_0^t \lambda_1(a_1 + x)dx\right] = \exp\left[-\int_{a_1}^{a_1+t} \lambda_1(x)dx\right] > \exp\left[-\int_0^t \lambda_1(x)dx\right]. \quad (3.4)$$

The left-hand-side is increasing in a_1. The relationship (3.4) signifies that the longer a_1 yields the more delay of the next p_1-decomposed arrival after 0. The positive correlation of the succeeding interarrival times can be also derived.

Although each of the p_1-arrival process and the p_2-arrival process is an independent renewal process, the two processes depend on each other and their superposition is not renewal. The succeeding interarrival times have the positive correlation.

Inversely, when A, A_1 and A_2 are increasing arrival rate random variables, the succeeding interarrival times of the superposed process have the negative correlation. Q.E.D.

We note that if A is a constant arrival rate random variable, that is, an exponential random variable, A_1 and A_2 are also exponential ones and then the correlation is never brought, since

$$\int_0^t (\lambda - \lambda_1 - \lambda_2)dx = 0.$$

Their superpositions turn out to be a renewal process. Three processes must be Poisson.

We now consider the $GI(A(x))/H_2(S(x))/\infty$ with $A(x) = 1 - k_1e^{-\lambda_1 x} - k_2e^{-\lambda_2 x}$ $(k_1 + k_2 = 1)$ and $S(x) = 1 - p_1e^{-\mu_1 x} - p_2e^{-\mu_2 x}$ $(p_1 + p_2 = 1)$ that may be more specially written as the $H_2(A(x))/H_2(S(x))/\infty$. This system is decomposed into two systems $GI(A_1(x))/M(1-$

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$e^{-\mu_1 x})/\infty$ and $GI(A_2(x))/M(1-e^{-\mu_2 x})/\infty$ due to the (p_1, p_2)-decompositions. Note that we now consider the case in which the random variable A is more variable than an exponential random variable. Let us now show that the random variables A_1 and A_2 are burst and also 2-hyperexponential. The arrival process with a 2-hyperexponential interarrival time A is an interrupted Poisson process introduced by Kuczura [3]. Let λ_A denote the Poisson arrival rate on an on-state period of this Kuczura's interrupted Poisson process. Let γ_A^{-1} and ω_A^{-1} denote the mean on-state period length and off-state period length, respectively. Picking out the arrivals with probability p_i, we can get a new interrupted Poisson process with the Poisson arrival rate $\lambda_{A_i} = p_i \lambda_A$ on an on-state period, the mean on-state period length $\gamma_{A_i}^{-1} = \gamma_A^{-1}$ and the mean off-state period length $\omega_{A_i}^{-1} = \omega_A^{-1}$. Thus, the interarrival time A_i follows a 2-hyperexponential distribution. That means that we may consider two decomposed systems written as $H_2(A_1(x))/M(1-e^{-\mu_1 x})/\infty$ and $H_2(A_2(x))/M(1-e^{-\mu_2 x})/\infty$. The concrete representation is given as follows: Since

$$A^*(s) = \sum_{i=1}^{2} \frac{k_i \lambda_i}{s+\lambda_i} \quad (\lambda_1 < \lambda_2) \ ,$$

we have

$$\begin{aligned} A_1^*(s) &= \frac{p_1\{k_1\lambda_1(s+\lambda_2)+k_2\lambda_2(s+\lambda_1)\}}{(s+\lambda_1)(s+\lambda_2)-(1-p_1)\{k_1\lambda_1(s+\lambda_2)+k_2\lambda_2(s+\lambda_1)\}} \\ &= \frac{p_1\{(k_1\lambda_1+k_2\lambda_2)s+\lambda_1\lambda_2\}}{s^2+\{\lambda_1+\lambda_2-(1-p_1)(k_1\lambda_1+k_2\lambda_2)\}s+p_1\lambda_1\lambda_2} \\ &=: \frac{g(s)}{f(s)} \ . \end{aligned}$$

Since

$$f(0) = p_1\lambda_1\lambda_2 > 0,$$

$$f(-\lambda_1) = -(1-p_1)k_1\lambda_1(\lambda_2-\lambda_1) < 0$$

and

$$f(-\lambda_2) = -(1-p_1)k_2\lambda_2(\lambda_1-\lambda_2) > 0,$$

we can write as

$$f(s) = (s+\gamma_1)(s+\gamma_2) \quad (0 < \gamma_1 < \lambda_1 < \gamma_2 < \lambda_2),$$

where $-\gamma_1$ and $-\gamma_2$ are real roots of $f(s) = 0$ and given by

$$(-\gamma_1, -\gamma_2) = \frac{-b \pm \sqrt{b^2-4c}}{2},$$

where

$$b = \lambda_1 + \lambda_2 - (1-p_1)(k_1\lambda_1 + k_2\lambda_2)$$

and

$$c = p_1\lambda_1\lambda_2.$$

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Thus, we obtain

$$A_1^*(s) = \frac{g(s)}{(s+\gamma_1)(s+\gamma_2)} = \frac{g(-\gamma_1)}{(-\gamma_1+\gamma_2)(s+\gamma_1)} + \frac{g(-\gamma_2)}{(-\gamma_2+\gamma_1)(s+\gamma_2)}.$$

The Laplace inversion gives

$$A_1(x) = 1 - l_1 e^{-\gamma_1 x} - l_2 e^{-\gamma_2 x},$$

where

$$l_1 = \frac{-p_1(k_1\lambda_1 + k_2\lambda_2)\gamma_1 + p_1\lambda_1\lambda_2}{\gamma_1(\gamma_2 - \gamma_1)} = \frac{-p_1(k_1\lambda_1 + k_2\lambda_2) + \gamma_2}{\gamma_2 - \gamma_1} > 0$$

and

$$l_2 = \frac{-p_1(k_1\lambda_1 + k_2\lambda_2)\gamma_2 + p_1\lambda_1\lambda_2}{\gamma_2(\gamma_1 - \gamma_2)} = \frac{-p_1(k_1\lambda_1 + k_2\lambda_2) + \gamma_1}{\gamma_1 - \gamma_2} > 0.$$

Distribution $A_1(x)$ is hyperexponential and then A_1 is a decreasing arrival rate random variable.

Distribution $A_2(x)$ is given by

$$A_2(x) = 1 - m_1 e^{-\omega_1 x} - m_2 e^{-\omega_2 x} \ (0 < \omega_1 < \lambda_1 < \omega_2 < \lambda_2)$$

where

$$(-\omega_1, -\omega_2) = \frac{-b \pm \sqrt{b^2 - 4c}}{2},$$

for

$$b = \lambda_1 + \lambda_2 - (1 - p_2)(k_1\lambda_1 + k_2\lambda_2),$$

$$c = p_2\lambda_1\lambda_2,$$

$$m_1 = \frac{-p_2(k_1\lambda_1 + k_2\lambda_2)\omega_1 + p_2\lambda_1\lambda_2}{\omega_1(\omega_2 - \omega_1)} = \frac{-p_2(k_1\lambda_1 + k_2\lambda_2) + \omega_2}{\omega_2 - \omega_1} > 0$$

and

$$m_2 = \frac{-p_2(k_1\lambda_1 + k_2\lambda_2)\omega_2 + p_2\lambda_1\lambda_2}{\omega_2(\omega_1 - \omega_2)} = \frac{-p_2(k_1\lambda_1 + k_2\lambda_2) + \omega_1}{\omega_1 - \omega_2} > 0.$$

Distribution $A_2(x)$ is also hyperexponential and the random variable A_2 is a decreasing arrival rate random variable.

We now write the variance of the number of customers in the $GI/G/\infty$ with the interarrival time distribution $A(x)$ and the service time distribution $S(x)$ such as

$$Var[GI(A(x))/G(S(x))/\infty].$$

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Here, that may be represented by

$$Var[H_2(A(x))/H_2(S(x))/\infty.$$

This variance is given by

$$\begin{aligned}
& Var[H_2(A(x))/H_2(S(x))/\infty] \\
= \; & Var[H_2(A_1(x))/M(1-e^{-\mu_1 x})/\infty] + Var[H_2(A_2(x))/M(1-e^{-\mu_2 x})/\infty] \\
+ \; & 2Cov[H_2(A_1(x))/M(1-e^{-\mu_1 x})/\infty, H_2(A_2(x))/M(1-e^{-\mu_2 x})/\infty].
\end{aligned}$$

We may denote this such that

$$\begin{aligned}
& Var[H_2(k_1,k_2,\lambda_1,\lambda_2)/H_2(p_1,p_2,\mu_1,\mu_2)/\infty] \\
= \; & Var[H_2(l_1,l_2,\gamma_1,\gamma_2)/M(\mu_1)/\infty] + Var[H_2(m_1,m_2,\omega_1,\omega_2)/M(\mu_2)/\infty] \\
+ \; & 2Cov[H_2(l_1,l_2,\gamma_1,\gamma_2)/M(\mu_1)/\infty, H_2(m_1,m_2,\omega_1,\omega_2)/M(\mu_2)/\infty]. \qquad (3.5)
\end{aligned}$$

Let $N^{(A_i)}(t)$ $(i=1,2)$ denote the number of p_i-decomposed arrivals in $(0,t)$ for sufficiently large t. Since A_1 and A_2 have the decreasing arrival rates, Theorem 1 implies

$$Var\{N^{(A_1)}(t) + N^{(A_2)}(t)\} > Var\{N^{(A_1)}(t)\} + Var\{N^{(A_2)}(t)\}.$$

Therefore, the covariance part in (3.5) is positive. Intuitively, when $\mu_1 = \mu_2$, the covariance takes the maximum value. Then, the variance $Var[H_2(k_1,k_2,\lambda_1,\lambda_2)/H_2(p_1,p_2,\mu_1,\mu_2)/\infty]$ itself is also the largest. The following arguments derive this: Since the random variable $A/E(A)$ is more variable than or equal to $A_i/E(A_i), i=1,2$ as shown in (3.2), the variance is maximized at the case of $E(A^2)/[E(A)]^2 = E(A_1^2)/[E(A_1)]^2$ or $E(A^2)/[E(A)]^2 = E(A_2^2)/[E(A_2)]^2$. If A is not an exponential random variable, that may be impossible except for $p_1 = 1$ or $p_2 = 1$. Only the system $H_2(k_1,k_2,\lambda_1,\lambda_2)/M(\mu)/\infty$ satisfies the condition, that is,

$$Var[H_2(k_1,k_2,\lambda_1,\lambda_2)]/H_2(p_1,p_2,\mu_1,\mu_2)/\infty] \leq Var[H_2(k_1,k_2,\lambda_1,\lambda_2)]/M(\mu)/\infty].$$

After all, we have the following inequalities:

$$\begin{aligned}
& Var[H_2(l_1,l_2,\gamma_1,\gamma_2)/M(\mu_1)/\infty] + Var[H_2(m_1,m_2,\omega_1,\omega_2)/M(\mu_2)/\infty] \\
\leq \; & Var[H_2(k_1,k_2,\lambda_1,\lambda_2)/H_2(p_1,p_2,\mu_1,\mu_2)/\infty] \\
\leq \; & Var[H_2(k_1,k_2,\lambda_1,\lambda_2)/M(\mu)/\infty].
\end{aligned}$$

We note that the equivalent random theory [11] that was predominant method for traffic engineering of telephone networks neglected the correlation. It seems that the effect of the correlation was sufficiently small for dimensioning the number of circuits.

As the coefficient of variation of the service time distribution in the

$$H_2(k_1,k_2,\lambda_1,\lambda_2)/H_2(p_1,p_2,\mu_1,\mu_2)/\infty$$

becomes larger, how does the variances of the system behave? In order to investigate this, let us assume that the hyperexponential service time distribution is symmetric [8], that is,

$$\frac{p_1}{\mu_1} = \frac{p_2}{\mu_2} = \frac{1}{2\mu}.$$

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We have from (2.9) and (3.1)

$$\begin{aligned}
& Var[H_2(l_1,l_2,\gamma_1,\gamma_2)/M(\mu_1)/\infty] + Var[H_2(m_1,m_2,\omega_1,\omega_2)/M(\mu_2)/\infty] \\
= \; & \frac{p_1\lambda}{\mu_1}\frac{\lambda\mu_1/\lambda_1\lambda_2 + E(A_1^2)/2E[(A_1)]^2}{\lambda\mu_1/\lambda_1\lambda_2+1} + \frac{p_2\lambda}{\mu_2}\frac{\lambda\mu_2/\lambda_1\lambda_2 + E(A_2^2)/2E[(A_2)]^2}{\lambda\mu_1/\lambda_1\lambda_2+1} \\
= \; & \frac{p_1\lambda}{\mu_1}\frac{\lambda\mu_1/\lambda_1\lambda_2 + p_2 + p_1E(A^2)/2E[(A)]^2}{\lambda\mu_1/\lambda_1\lambda_2+1} + \frac{p_2\lambda}{\mu_2}\frac{\lambda\mu_2/\lambda_1\lambda_2 + p_1 + p_2E(A^2)/2E[(A)]^2}{\lambda\mu_1/\lambda_1\lambda_2+1} \\
= \; & \frac{\lambda}{2\mu}\frac{\lambda\mu_1/\lambda_1\lambda_2 + p_2 + p_1E(A^2)/2E[(A)]^2}{\lambda\mu_1/\lambda_1\lambda_2+1} + \frac{\lambda}{2\mu}\frac{\lambda\mu_2/\lambda_1\lambda_2 + p_1 + p_2E(A^2)/2E[(A)]^2}{\lambda\mu_1/\lambda_1\lambda_2+1} \\
= \; & \frac{\lambda}{2\mu}\left[2 + \frac{1}{2}\left(\frac{E(A^2)}{2[E(A)]^2} - 1\right)\left(\frac{\mu_1/\mu}{\lambda\mu_1/\lambda_1\lambda_2+1} + \frac{\mu_2/\mu}{\lambda\mu_2/\lambda_1\lambda_2+1}\right)\right]. \qquad (3.6)
\end{aligned}$$

For the coefficient of variation of service times C_s, it can be written that

$$\mu_1 = \left[1 + \sqrt{1 - 2/(1+C_s^2)}\right]\mu \equiv (1+\gamma)\mu$$

and

$$\mu_2 = \left[1 - \sqrt{1 - 2/(1+C_s^2)}\right]\mu \equiv (1-\gamma)\mu\ .$$

Thus, (3.6) yields

$$\begin{aligned}
& Var[H_2(l_1,l_2,\gamma_1,\gamma_2)/M(\mu_1)/\infty] + Var[H_2(m_1,m_2,\omega_1,\omega_2)/M(\mu_2)/\infty] \\
= \; & \frac{\lambda}{2\mu}\left[2 + \frac{1}{2}\left(\frac{E(A^2)}{2[E(A)]^2} - 1\right)\left(\frac{1+\gamma}{\lambda\mu(1+\gamma)/\lambda_1\lambda_2+1} + \frac{1-\gamma}{\lambda\mu(1-\gamma)/\lambda_1\lambda_2+1}\right)\right].
\end{aligned}$$

The total variance is decreasing in γ or C_s. The covariance is also decreasing in γ or C_s. Note that the decreasing quantity in C_s can be derived if the ratio $(p_1/\mu_1)/(p_2/\mu_2)$ is fixed. We can now conclude that as the service times of $H_2/H_2/\infty$ systems become more variable, the system variances become less. Using the $G/H_n/\infty$ theory [4], we can numerically certify this conclusion. In that paper [4], the variance is given by

$$Var[H_2(k_1,k_2,\lambda_1,\lambda_2)/H_2(p_1,p_2,\mu_1,\mu_2)/\infty] = 2B^{(2)} + \frac{\lambda}{\mu} - \left(\frac{\lambda}{\mu}\right)^2,$$

where

$$B^{(2)} = \frac{p_1\lambda}{2\mu_1}\left(p_1 + \frac{2\mu_1p_2}{\mu_1+\mu_2}\right)\frac{A^*(\mu_1)}{1-A^*(\mu_1)} + \frac{p_2\lambda}{2\mu_2}\left(p_2 + \frac{2\mu_2p_1}{\mu_1+\mu_2}\right)\frac{A^*(\mu_2)}{1-A^*(\mu_2)}.$$

When the service time distribution has a heavy tail [6], where is the variance going? If $p_2 \approx 0$ and p_2-decompositions rarely occur, the following holds [2]:

$$\frac{E(A_2^2)}{2[E(A_2)]^2} \approx 1.$$

A Poisson stream with the rate $\lambda_2 = p_2\lambda$ is separated from the original stream by (p_1,p_2)-decompositions. The service time follows an exponential distribution with extremely large mean $1/\mu_2$. Then,

$$\frac{E(A_1^2)}{2[E(A_1)]^2} \approx \frac{E(A^2)}{2[E(A)]^2}.$$

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or

$$A_1 \approx A.$$

We have now independently decomposed two systems, an $H_2(l_1, l_2, \gamma_1, \gamma_2)/M(\mu_1)/\infty$ and an $M(\lambda_2)/M(\mu_2)/\infty$. The variance of the former is given by

$$\begin{aligned} Var[H_2(l_1, \gamma_1, \gamma_2)/M(\mu_1)/\infty] &= \frac{\gamma}{\mu_1} \frac{\gamma\mu_1/\gamma_1\gamma_2 + E(A_1^2)/2[E(A_1)]^2}{\gamma\mu_1/\gamma_1\gamma_2 + 1} \\ &\approx \frac{\lambda}{\mu_1} \frac{\lambda\mu_1/\lambda_1\lambda_2 + E(A^2)/2[E(A)]^2}{\lambda\mu_1/\lambda_1\lambda_2 + 1} \end{aligned}$$

The variance of the latter is λ_2/μ_2. The direct sum of their two variances is equal to $Var[H_2(k_1, k_2, \lambda_1, \lambda_2)/H_2(p_1, p_2, \mu_1, \mu_2)/\infty]$. If $\lambda_2/\mu_2 \approx \lambda/\mu$ and $\lambda_1/\mu_1 \approx 0$, the system $H_2(k_1, k_2, \lambda_1, \lambda_2)/H_2(p_1, p_2, \mu_1, \mu_2)/\infty$ behaves like an $M(\lambda)/M(\mu)/\infty$ and then the variance is almost equal to λ/μ. The service time follows an extremely unbalanced hyperexponential distribution such that $p_1 \approx 1$, $p_2 \approx 0$, $\mu_1 \gg 0$, $\mu_2 \approx 0$ and $p_1/\mu_1(\approx 0) \ll p_2/\mu_2$. We have now the condition under which the $H_2/H_2/\infty$ may be considered as the $M/M/\infty$.

When the service time random variable is stochastically less variable than an exponential random variable, its distribution may be given by a mixture of exponential and deterministic distributions. That is,

$$S(x) = 1 - p_1 e^{-\mu_1 x} - p_2 I_{\{x \le \mu_2^{-1}\}} \quad (p_1 + p_2 = 1),$$

where $I_{\{\cdot\}}$ is an index function. Three parameter p_1, μ_1^{-1} and μ_2^{-1} are given by the mean service time μ^{-1}, the coefficient of variation of service times $C_f(\le 1)$ and the boundary condition under which $S(x)$ is exponential if $p_1 = 1$ or $S(x)$ is deterministic if $p_1 = 0$ as follows:

$$p_1 = C_f^2$$

and

$$\left(\frac{1}{\mu_1}, \frac{1}{\mu_2}\right) = \left(\frac{1}{\mu}, \frac{1}{\mu}\right)$$

and afterwards, we call these services, MD services. Note that when $p_1(= C_f^2) = 1/n$, the coefficient of variation of the MD services is equal to that of the n-Erlang services. That is the reason why we define $p_1 = C_f^2$ instead of $p_1 = C_f$. The variance of the number of customers in the system is given by

$$\begin{aligned} & Var[H_2(k_1, k_2, \lambda_1, \lambda_2)/MD(p_1, p_2, \mu_1, \mu_2)/\infty] \\ = \ & Var[H_2(l_1, l_2, \gamma_1, \gamma_2)/M(\mu_1)/\infty] + Var[H_2(m_1, m_2, \omega_1 x, \omega_2)/D(\mu_2)/\infty] \\ + \ & 2Cov[H_2(l_1, l_2, \gamma_1, \gamma_2)/M(\mu_1)/\infty, H_2(m_1, m_2, \omega_1 x, \omega_2)/D(\mu_2)/\infty]. \end{aligned} \quad (3.7)$$

The covariance term is positive and then we have

$$\begin{aligned} & Var[H_2(k_1, k_2, \lambda_1, \lambda_2)/M(\mu)/\infty] \\ \le \ & Var[H_2(l_1, l_2, \gamma_1, \gamma_2)/M(\mu_1)/\infty] + Var[H_2(m_1, m_2, \omega_1, \omega_2)/D(\mu_2)/\infty] \\ \le \ & Var[H_2(k_1, k_2, \lambda_1, \lambda_2)/MD(p_1, p_2, \mu_1, \mu_2)/\infty], \\ \le \ & Var[H_2(k_1, k_2, \lambda_1, \lambda_2)/D(\mu)/\infty]. \end{aligned}$$

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It is found that as the coefficient of variation C_f for service times decreases, the variance of the number of customers in the system increases. In $H_2/G/\infty$ systems, the $H_2/D/\infty$ is the worst on performance. We may conclude that when the arrival process is more variable than Poisson one, that is, when the arrival process is bursty, the $GI/D/\infty$ is the worst on performance in the $GI/G/\infty$ systems.

On the contrary, when the arrival process is smooth, what happens? Let us consider the case in which a random variable A is Gamma(=Erlang) with parameters $(n, n\lambda)$. Then, (p_1, p_2)-decomposed random variables A_1 and A_2 are also smooth but more variable than A. The Laplace transforms $A_1^*(s)$ and $A_2^*(s)$ are given by

$$A_j^*(s) = \frac{p_j(n\lambda)^n}{(s+n\lambda)^n - (1-p_j)(n\lambda)^n}, \quad j = 1, 2.$$

Each of equations

$$(s+n\lambda)^n - (1-p_j)(n\lambda)^n = 0, \ j = 1, 2$$

has n different roots

$$-n\lambda\{1-(1-p_j)^{1/n}e^{2\pi ik/n}\}, \ k = 0, 1, \ldots, n-1,$$

when $0 < p_j < 1$. Using these roots, $A_j(s)$ is given by

$$A_j^*(s) = \frac{p_j(n\lambda)^n}{\prod_{k=0}^{n-1}[s+n\lambda\{1-(1-p_j)^{1/n}e^{2\pi ik/n}\}]}. \tag{3.8}$$

We call the random variable A_j, p_j-variant n-Erlang and write as $E_n^{(p_j)}$. It can be easily shown that $E_n^{(p_j)}$ is smooth and has the increasing arrival rate. The Laplace inversion of (3.8) gives the distribution

$$A_j(x) = \frac{n\lambda p_j(1-p_j)^{1/n}}{1-p_j}\int_0^x \sum_{k=0}^{n-1} \frac{\exp[-n\lambda\{1-(1-p_j)^{1/n}e^{2\pi ik/n}\}t]}{\prod_{l=0\ (l\neq k)}^{n-1}(e^{2\pi ik/n}-e^{2\pi il/n})}dt.$$

As shown in Theorem 1, the superposed sequence of random variables $E_n^{(p_1)}$ and $E_n^{(p_2)}$ has the negative correlation.

For (3.5), we have

$$\begin{aligned} & Var[E_n(n,n\lambda)/H_2(p_1,p_2,\mu_1,\mu_2)/\infty] \\ = \ & Var[E_n^{(p_1)}/M(\mu_1)/\infty] + Var[E_n^{(p_2)}/M(\mu_2)/\infty] \\ + \ & 2Cov[E_n^{(p_1)}/M(\mu_1)/\infty, E_n^{(p_2)}/M(\mu_2)/\infty]. \end{aligned}$$

and since the covariance part is negative, we have

$$\begin{aligned} & Var[E_n^{(p_1)}/M(\mu_1)/\infty] + Var[E_n^{(p_2)}/M(\mu_2)/\infty] \\ \geq \ & Var[E_n(n,n\lambda)/H_2(p_1,p_2,\mu_1,\mu_2)/\infty] \\ \geq \ & Var[E_n(n,n\lambda)/M(\mu)/\infty]. \end{aligned}$$

For (3.7), we have

$$\begin{aligned} & Var[E_n(n,n\lambda)/MD(p_1,p_2,\mu_1,\mu_2)/\infty] \\ = \ & Var[E_n^{(p_1)}/M(\mu_1)/\infty] + Var[E_n^{(p_2)}/D(\mu_2)/\infty] \\ + \ & 2Cov[E_n^{(p_1)}/M(\mu_1)/\infty, E_n^{(p_2)}/D(\mu_2)/\infty], \end{aligned}$$

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and

$$
\begin{aligned}
& Var[E_n(n,n\lambda)/M(\mu)/\infty] \\
\geq\ & Var[E_n^{(p_1)}/M(\mu_1)/\infty] + Var[E_n^{(p_2)}/D(\mu_2)/\infty] \\
\geq\ & Var[E_n(n,n\lambda)/MD(p_1,p_2,\mu_1,\mu_2)/\infty], \\
\geq\ & Var[E_n(n,n\lambda)/D(\mu)/\infty].
\end{aligned}
$$

Here,

$$
\begin{aligned}
& Var(E_n^{(p_2)}/D(\mu_2)/\infty) = 2p_2\lambda\int_0^{1/\mu_2}\mathcal{L}_t^{-1}\left[\frac{1}{s\{1-A_2^*(s)\}}\right]dt - \frac{p_2\lambda}{\mu_2} - \left(\frac{p_2\lambda}{\mu_2}\right)^2 \\
=\ & 2p_2\lambda\int_0^{1/\mu_2}\mathcal{L}_t^{-1}\left[\frac{(s+n\lambda)^n - p_1(n\lambda)^n}{s^2\sum_{k=1}^{n-1}{}_nC_k s^{n-k-1}(n\lambda)^k}\right]dt - \frac{p_2\lambda}{\mu_2} - \left(\frac{p_2\lambda}{\mu_2}\right)^2 \\
=\ & 2p_2\lambda\int_0^{1/\mu_2}\mathcal{L}_t^{-1}\left[\frac{(s+n\lambda)^n - p_1(n\lambda)^n}{s^2\prod_{k=1}^{n-1}\{s+n\lambda(1-e^{2\pi ik/n})\}}\right]dt - \frac{p_2\lambda}{\mu_2} - \left(\frac{p_2\lambda}{\mu_2}\right)^2 \\
=\ & \frac{p_2^2\lambda}{n\mu_2} + \frac{p_2(1-p_2)\lambda}{\mu_2} \\
+\ & 2p_2\lambda\int_0^{1/\mu_2}\mathcal{L}_t^{-1}\left[\sum_{k=1}^{n-1}\frac{\{(n\lambda e^{2\pi ik/n})^n - p_1(n\lambda)^n\}\{s+n\lambda(1-e^{2\pi ik/n})\}^{-1}}{\{n\lambda(1-e^{2\pi ik/n})\}^2\prod_{l=1\ (l\neq k)}^{n-1}\{n\lambda(e^{2\pi ik/n}-e^{2\pi il/n})\}}\right]dt \\
=\ & \frac{p_2^2\lambda}{n\mu_2} + \frac{p_2(1-p_2)\lambda}{\mu_2} \\
+\ & 2p_2^2\lambda\int_0^{1/\mu_2}\sum_{k=1}^{n-1}\frac{e^{-n\lambda(1-e^{2\pi ik/n})t}}{(1-e^{2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{2\pi ik/n}-e^{2\pi li/n})}dt.
\end{aligned}
$$

For the $E_n^{(p_j)}/M(\mu_j)/\infty$, we have

$$
\begin{aligned}
& Var(E_n^{(p_j)}/M(\mu_j)/\infty) = \frac{p_j\lambda}{\mu_j\{1-A_i^*(\mu_j)\}} - \left(\frac{p_j\lambda}{\mu_j}\right)^2 \\
=\ & \left(1-\frac{p_j}{2}+\frac{p_j}{2n}\right)\frac{p_j\lambda}{\mu_j} + p_j^2\lambda\sum_{k=1}^{n-1}\frac{\{\mu_j+n\lambda(1-e^{2\pi ik/n})\}^{-1}}{(1-e^{2\pi ik/n})^2\prod_{l=1\ (l\neq k)}^{n-1}(e^{2\pi ik/n}-e^{2\pi il/n})}.
\end{aligned}
$$

We may conclude that the $Smooth\ Arrivals/D/\infty$ is the best in $Smooth\ Arrivals/GI/\infty$ systems on performance. As the service time random variable becomes stochastically more variable, the number of customers in the system also becomes stochastically more variable.

4. Further Studies

It is quite interesting that the $Bursty\ arrivals/D/\infty$ is the worst on performance in the $Bursty\ arrivals/G/\infty$ systems. The departure process from the $Bursty\ arrivals/D/\infty$ is stochastically equivalent to the bursty arrival process. That is, the arrival burst is completely conserved. The service time variations weaken the burst of departures. At the result, we can have the better system with more variable service times on performance. How about the waiting $Bursty\ arrivals/G/1$ systems? Under the light traffic condition, that is,

$$P(A > S) \approx 1,$$

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where A and S are the interarrival time and service time random variables, respectively, the arrival process of the $Bursty\ arrivals/D/1$ is equivalent to the departure process such as the $Bursty\ arrivals/D/\infty$. Since the service time variations weaken the burstness of departures, the $Bursty\ arrivals/D/1$ is the worst on performance in the $Bursty\ arrivals/G/1$ systems. However, under the heavy traffic condition such as $E(A) \approx E(S)$, the inter-departure times become more variable, as the service times become more variable. Thus, the $Bursty\ arrivals/D/1$ is the best on performance in the $Bursty\ arrivals/G/1$ systems. Further studies on the intermediate traffic condition still remain.

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