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INVENTORY MODELS WITH A NEGATIVE EXPONENTIAL CRASHING COST TAKING TIME VALUE INTO ACCOUNT

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Abstract This paper discusses the inventory replenishment policy over an infinite planning horizon with a negative exponential lead time crashing cost, taking time value into account. Our work is based on the paper of Ben-Daya and Raouf that has been cited 65 times. We extend their model to incorporate the time value of money and then find the criterion to decide the optimal solution. Numerical examples illustrate our findings to demonstrate that we provide an easy and efficient procedure to find the optimal solution.

Keywords: Inventory, lead time, present value

1. Introduction

For the traditional inventory model as described by Silver and Peterson [18], lead time is treated as a predetermined constant or a stochastic parameter. Liao and Shyu [11] first studied a probabilistic model creating the inventory system of linearly crashing lead time with normal demand. With predetermined order quantities, they treated the lead time as the only variable. Ben-Daya and Raouf [2] extended the Liao and Shyu [11] model by allowing variable ordering quantities. They also constructed a new model by changing the lead time crashing cost from piece-wise linear decreasing function to a negative exponential function. Ouyang, Yeh and Wu [15] studied the first model of Ben-Daya and Raouf [2] by adding the stock-out cost. During the stock-out period, the total amount of stock-out is considered as a mixture of backorders and lost sales. The inventory model with an infinite planning horizon taking time value into account was first studied by Trippi and Lewin [19]. The discount occurs only during the ordering point in that model. Ten years later, Gurnani [8] approached this model using a finite planning horizon taking into account the cost of the objective function with and without the discount of time value. Dohi, Kaio and Osaki [7] proposed a new inventory model with an infinite time span taking into account time value but suggesting that the inventory holding cost is continuously being discounted over time. There are 65 papers that have quoted Ben-Daya and Raouf [2] in their references. In view of the high level of citations of the original paper of Ben-Daya and Raouf [2], it seems important to address this problem accurately and completely to ensure the successful application of their proposed method by others. However to save the precious space of this journal, we only list those papers that are closely related to our extension. The following papers consider that the lead time crashing cost is constructed as a piecewise linear function: Chang [3], Chang, Ouyang, Wu and Ho [4], Chu, Yang and Chen [6], Lan, Chu and Chung [9], Lee, Wu and Hsu [10], Ouyang and Chuang [13], Ouyang and Wu [14], Wu [20], Yang, Ronald and Chu [21]. On the other hand, Chu, Chung and Lan [5] found the solution by the Newton-Raphson method. From the perspective of a firm's usable fund, an increase of inventory in a business firm's balance sheet is a de facto usage of fund. As the time value of money serves as the foundation for all other notions in finance, it influences business finance regarding the concept of interest by learning to calculate present and future value This means that a dollar in our possession today is preferred over a dollar we expect to receive at some point in the future. Thus, decision makers must take the time value of money into consideration when they are making inventory decisions. This is traditionally done by restating money values through time with time value of money calculations; given that the cost of money is fluctuating over time in the market, a more realistic inventory model thus should incorporate this factor into it. (See, for example, Ross, Westerfield, Jaffe and Jordan [17]). In the past, the two properties of lead time exponential crashing cost and the time value of money have received attention separately, but have never considered together simultaneously. The purpose of this paper is to develop an inventory model with a negative exponential lead time crashing cost taking into account time value. The inventory holding cost is continuously discounted. On the other hand, the set-up cost and the lead time crashing cost are discretely discounted. Under two acceptable assumptions, we find the criterion to decide the optimal lead time and order quantity. Numerical examples are included to illustrate our model and the solution procedure.

2. Assumptions and Notation

To develop the proposed models the following assumptions and notation of Ben-Daya and Raouf [2], and Chu, Chung and Lan [5] are used.

1. The deterministic lead time L and the demand follow a normal distribution with mean D and standard derivation σ.
2. The reorder point r = expected demand during lead time + safety stock (SS) and SS = k × (standard deviation of lead time demand) where k is known as a safety factor.
3. The total crashing cost is related to the lead time by a function of the form
$$R(L) = \alpha\, e^{-\beta L},$$
where α is the cost for reducing lead time to negligible, so α is the scale parameter and β is the shape parameter.
4. The ordering cost is denoted by A and h is the holding cost per item per year.
5. The total relevant cost is denoted by $C(Q, L)$ where Q is the order quantity.
6. The interest rate is θ per year.

In Silver and Peterson [18], the reorder point is given by $r = DL + k\sigma\sqrt{L}$ where DL is the expected demand during lead time and $k\sigma\sqrt{L}$ is a safety stock. The expected net inventory immediately before and after the receipt of an order of size Q is $r - DL$ and $Q + r - DL$, respectively. Therefore, expected average inventory level of one cycle for $t \in [0, Q/D]$, equals $Q + k\sigma\sqrt{L} - Dt$. Hence the present value of the inventory carrying cost for the first cycle is

$$\int_{t=0}^{Q/D} \left[Q + k\sigma\sqrt{L} - Dt\right]\, h\, e^{-\theta t} dt. \tag{1}$$

Equation (1) can be simplified as

$$hk\sigma\sqrt{L}\frac{1-e^{-\theta\frac{Q}{D}}}{\theta}+h\frac{D}{\theta^2}\left(e^{-\theta\frac{Q}{D}}-1+\theta\frac{Q}{D}\right). \tag{2}$$

We adopt the discounted cash flow approach of Moon and Yun [12]. There will be cash outflows for the ordering cost and lead time crashing cost at the beginning of each cycle. Therefore, the total cost for the first cycle is

$$\left(A+\alpha\, e^{-\beta L}\right)+\left\{hk\sigma\sqrt{L}\frac{1-e^{-\theta\frac{Q}{D}}}{\theta}+h\frac{D}{\theta^2}\left(e^{-\theta\frac{Q}{D}}-1+\theta\frac{Q}{D}\right)\right\}. \tag{3}$$

Consequently, the present value of the expected total cost over infinite time horizon, $C\left(Q,L\right)$, is given by

$$\frac{1}{1-e^{-\theta\frac{Q}{D}}}\left(A+\alpha\, e^{-\beta L}\right)+\frac{1}{1-e^{-\theta\frac{Q}{D}}}\left\{\frac{hk\sigma}{\theta}\sqrt{L}\left(1-e^{-\theta\frac{Q}{D}}\right)+h\frac{D}{\theta^2}\left(e^{-\theta\frac{Q}{D}}-1+\theta\frac{Q}{D}\right)\right\}. \tag{4}$$

For $L\in[0,\infty)$ and $Q\in(0,\infty)$, with $f\left(L\right)=A+\alpha\, e^{-\beta L}$, we rewrite $C\left(Q,L\right)$ as

$$\frac{f\left(L\right)}{1-e^{-\theta\frac{Q}{D}}}+\frac{hk\sigma}{\theta}\sqrt{L}+h\frac{D}{\theta^2}\frac{e^{-\theta\frac{Q}{D}}-1+\theta\frac{Q}{D}}{1-e^{-\theta\frac{Q}{D}}}. \tag{5}$$

According to Rachamadugu [16], in order to compare our result with the previous model of Ben-Daya and Raouf [2], we use $A\left(Q,L\right)=\theta C\left(Q,L\right)$, an alternate but equivalent measure. $A\left(Q,L\right)$ represents the equivalent uniform cash flow stream that generates the same $C\left(Q,L\right)$. From $\lim\limits_{\theta\to 0}A\left(Q,L\right)=\frac{AD}{Q}+\frac{hQ}{2}+hk\sigma\sqrt{L}+\frac{D}{Q}\alpha\, e^{-\beta L}$, which is the Equation (7) of Ben-Daya and Raouf [2]. Hence, we extend the Ben-Daya and Raouf's model.

3. Mathematical Formulation

By Equation (5), we have that

$$\frac{\partial C\left(Q,L\right)}{\partial Q}=\frac{-\theta}{D}\frac{f\left(L\right)}{\left(1-e^{-\theta\frac{Q}{D}}\right)^2}e^{-\theta\frac{Q}{D}}+\frac{h}{\theta}\frac{1-e^{-\theta\frac{Q}{D}}-\theta\frac{Q}{D}e^{-\theta\frac{Q}{D}}}{\left(1-e^{-\theta\frac{Q}{D}}\right)^2} \tag{6}$$

and

$$\frac{\partial^2 C\left(Q,L\right)}{\partial Q^2}=f\left(L\right)\frac{\theta^2}{D^2}\frac{\left(1+e^{-\theta\frac{Q}{D}}\right)e^{-\theta\frac{Q}{D}}}{\left(1-e^{-\theta\frac{Q}{D}}\right)^3}+\frac{h}{D}e^{-\theta\frac{Q}{D}}\frac{\left(2+\theta\frac{Q}{D}\right)e^{-\theta\frac{Q}{D}}-2+\theta\frac{Q}{D}}{\left(1-e^{-\theta\frac{Q}{D}}\right)^3}. \tag{7}$$

From [16], Rachamadugu derived that $e^{-x}>\frac{2-x}{2+x}$ for $x>0$. Hence, $C\left(Q,L\right)$ is convex for $Q\in(0,\infty)$. From Equation (6), it yields that

$$\frac{\partial C\left(Q,L\right)}{\partial Q}=\frac{h}{\theta}\frac{e^{-\theta\frac{Q}{D}}}{\left(1-e^{-\theta\frac{Q}{D}}\right)^2}\left[e^{\theta\frac{Q}{D}}-1-\theta\frac{Q}{D}-\frac{\theta^2}{Dh}\left(A+\alpha\, e^{-\beta L}\right)\right]. \tag{8}$$

Since $e^{\theta\frac{Q}{D}}-1-\theta\frac{Q}{D}$ is strictly increasing in Q, given a L, there exists a unique Q satisfying $e^{\theta\frac{Q}{D}}-1-\theta\frac{Q}{D}=\frac{\theta^2}{Dh}\left(A+\alpha\, e^{-\beta L}\right)$. We denote the unique positive root of $\frac{\partial C(Q,L)}{\partial Q}=0$ as $Q^{\wedge}\left(L\right)$. Using the convexity of $C\left(Q,L\right)$ in Q, we derive that the minimum problem

$$\min\left\{C\left(Q,L\right):0\le L<\infty,\ 0<Q<\infty\right\} \tag{9}$$

is equivalent to the minimum problem

$$\min\left\{C\left(Q^{\wedge}\left(L\right),L\right):0\leq L<\infty\right\}. \tag{10}$$

To simplify the expression, let $\phi\left(L\right)=C\left(Q^{\wedge}\left(L\right),L\right)$ for $0\leq L<\infty$.

Here, we write an implicit form of $\phi\left(L\right)$ as

$$\phi\left(L\right)=A+\alpha e^{-\beta L}+\frac{h}{\theta}Q^{\wedge}\left(L\right)+\frac{hk\sigma\sqrt{L}}{\theta}, \tag{11}$$

with

$$e^{\frac{\theta}{D}Q^{\wedge}(L)}-1-\frac{\theta}{D}Q^{\wedge}\left(L\right)=\frac{\theta^{2}}{Dh}\left(A+\alpha\, e^{-\beta L}\right). \tag{12}$$

In the remainder of this paper, we only consider $Q^{\wedge}\left(L\right)$, except for the Equation (23) so we use Q to denote $Q^{\wedge}\left(L\right)$ to simplify the expression. Since

$$\frac{dQ}{dL}=\frac{-\alpha\beta\theta}{h}\frac{e^{-\beta L}}{e^{\theta\frac{Q}{D}}-1}, \tag{13}$$

we have

$$\frac{d}{dL}\phi\left(L\right)=-\alpha\,\beta\, e^{-\beta L}\frac{e^{\theta\frac{Q}{D}}}{e^{\theta\frac{Q}{D}}-1}+\frac{hk\sigma}{2\theta\sqrt{L}}. \tag{14}$$

After cross multiplication, we get

$$\phi'\left(L\right)=\frac{\alpha\beta\, e^{-\beta L}}{\sqrt{L}\left(1-e^{-\theta\frac{Q}{D}}\right)}\left[\frac{hk\sigma}{2\theta\alpha\beta}\left(1-e^{-\theta\frac{Q}{D}}\right)e^{\beta L}-\sqrt{L}\right]. \tag{15}$$

We are motivated by Chu, Chung and Lan [5] to decompose $\phi'\left(L\right)$ as the multiplication of a positive function and a convex function. First, we define

$$V\left(L\right)=\frac{\alpha\beta\, e^{-\beta L}}{\sqrt{L}\left(1-e^{-\theta\frac{Q}{D}}\right)} \tag{16}$$

and

$$W\left(L\right)=\left[\frac{hk\sigma}{2\theta\alpha\beta}\left(1-e^{-\theta\frac{Q}{D}}\right)e^{\beta L}-\sqrt{L}\right]. \tag{17}$$

It is apparent that $V\left(L\right)>0$. Hence, our goal is to verify that $W\left(L\right)$ is a convex function for $L\in[0,\infty)$.

After we express $\phi'\left(L\right)$ *as* $V\left(L\right)W\left(L\right)$ where $V\left(L\right)$ is a positive function and $W\left(L\right)$ is a convex function, then it implies that the solution of $\phi'\left(L\right)=0$ is the solution for $W\left(L\right)=0$ such that we can locate the solution of $\phi'\left(L\right)=0$ according to different cases in the next section.

4. Lemmas and Theorem

For technical reasons, we will assume two extra conditions: $Dh>100(A+\alpha)\theta^{2}$ and $A>(0.779)\alpha$. In the next lemma, we will provide detailed explanation to show that why our two extra assumptions are necessary and acceptable.

Lemma 1. If $Dh > 100(A+\alpha)\theta^2$ and $A > (0.779)\alpha$, then $W(L)$ is a convex function for $L \in [0,\infty)$ and $W'(L) = 0$ has a unique positive solution, say L_0.

Proof of Lemma 1.
We have

$$W'(L) = \frac{hk\sigma}{2\theta\alpha\beta}\left[\left(1-e^{-\theta\frac{Q}{D}}\right)\beta e^{\beta L} - \frac{\theta^2\alpha\beta}{Dh}\frac{e^{-\theta\frac{Q}{D}}}{e^{\theta\frac{Q}{D}}-1}\right] - \frac{1}{2\sqrt{L}} \tag{18}$$

and

$$W''(L) = \frac{hk\sigma\beta}{2D^2h^2\theta\alpha\left(e^{\theta\frac{Q}{D}}-1\right)^3}\left[D^2h^2e^{\beta L}\left(1-e^{-\theta\frac{Q}{D}}\right)\left(e^{\theta\frac{Q}{D}}-1\right)^3 - Dh\theta^2\alpha\, e^{-\theta\frac{Q}{D}}\left(e^{\theta\frac{Q}{D}}-1\right)^2 - \theta^4\alpha^2e^{-\beta L}\left(2-e^{-\theta\frac{Q}{D}}\right)\right] + \frac{1}{4\sqrt{L^3}}. \tag{19}$$

To simplify the expression, we use $\Omega = D^2h^2e^{\beta L}\left(1-e^{-\theta\frac{Q}{D}}\right)\left(e^{\theta\frac{Q}{D}}-1\right) - Dh\theta^2\alpha\, e^{-\theta\frac{Q}{D}}$ and $\Delta = \theta^4\alpha^2e^{-\beta L}\left(2-e^{-\theta\frac{Q}{D}}\right)$. To prove $W''(L) > 0$, it is sufficient to show that

$$\left(e^{\theta\frac{Q}{D}}-1\right)^2\Omega - \Delta > 0. \tag{20}$$

For some constant, say ϕ, we want the next inequality being satisfied

$$\frac{\theta^2}{Dh}\left(A+\alpha\, e^{-\beta L}\right) < \frac{A+\alpha}{Dh}\theta^2 < \phi. \tag{21}$$

Since $e^{\theta\frac{Q}{D}} - 1 - \theta\frac{Q}{D} = \sum_{k=2}^{\infty}\frac{1}{k!}\theta^k\frac{Q^k}{D^k} > \frac{1}{2}\theta^2\frac{Q^2}{D^2}$, Equations (12) and (21), it follows that $\phi > \frac{1}{2}\theta^2\frac{Q^2}{D^2}$ so we derive

$$\sqrt{2\phi} > \theta\frac{Q}{D}. \tag{22}$$

From the Taylor's series expansion, it implies that for a given Q, there exists a $Q^{\#}$, with $0 < Q^{\#} < Q$, satisfying

$$e^{\theta\frac{Q}{D}} - 1 - \theta\frac{Q}{D} = \frac{1}{2}\frac{\theta^2Q^2}{D^2} + \frac{1}{6}e^{\theta\frac{Q^{\#}}{D}}\frac{\theta^3Q^3}{D^3}. \tag{23}$$

We need another condition to imply the following inequality being valid, hence for some constant, say μ, with

$$e^{\sqrt{2\phi}} < 6\mu. \tag{24}$$

Using $\sqrt{2\phi} > \theta\frac{Q}{D}$, hence $\frac{1}{6}e^{\theta\frac{Q^{\#}}{D}}\frac{\theta^3Q^3}{D^3} < \frac{1}{6}e^{\sqrt{2\phi}}\frac{\theta^2Q^2}{D^2} < \mu\frac{\theta^2Q^2}{D^2}$. Therefore, it follows that $\left(\frac{1}{2}+\mu\right)\frac{\theta^2Q^2}{D^2} > e^{\theta\frac{Q}{D}} - 1 - \theta\frac{Q}{D} = \frac{\theta^2}{Dh}\left(A+\alpha\, e^{-\beta L}\right)$ so

$$\left(\frac{1}{2}+\mu\right)Q^2 > \frac{D}{h}\left(A+\alpha\, e^{-\beta L}\right). \tag{25}$$

Now, we evaluate Ω as follows:

$$\Omega > \left[D^2h^2e^{\beta L}\left(e^{\theta\frac{Q}{D}} - 2 + e^{-\theta\frac{Q}{D}}\right) - Dh\theta^2\alpha\right] > \left[D^2h^2e^{\beta L}\theta^2\frac{Q^2}{D^2} - Dh\theta^2\alpha\right]. \tag{26}$$

By Equation (25), we improve Equation (26) as

$$\Omega > Dh\theta^2 \left[e^{\beta L} \frac{2}{1+2\mu} \left(A + \alpha\, e^{-\beta L} \right) - \alpha \right] = Dh\theta^2 \left(\frac{2}{1+2\mu} A e^{\beta L} + \frac{1-2\mu}{1+2\mu} \alpha \right). \tag{27}$$

Second, we compute $\left(e^{\theta \frac{Q}{D}} - 1 \right)^2 Dh\theta^2 A e^{\beta L} - \Delta$ as follows:

$$\left(e^{\theta \frac{Q}{D}} - 1 \right)^2 Dh\theta^2 A e^{\beta L} - \Delta > \left(\theta \frac{Q}{D} \right)^2 Dh\theta^2 A e^{\beta L} - 2\theta^4 \alpha^2 e^{-\beta L} = \theta^4 \left(\frac{Q^2}{D} A h e^{\beta L} - 2\alpha^2\, e^{-\beta L} \right). \tag{28}$$

Using Equation (25) again, we revise Equation (28) as

$$\left(e^{\theta \frac{Q}{D}} - 1 \right)^2 Dh\theta^2 A e^{\beta L} - \Delta > \theta^4 \left[\left(A + \alpha\, e^{-\beta L} \right) \frac{2}{1+2\mu} A e^{\beta L} - 2\alpha^2\, e^{-\beta L} \right]. \tag{29}$$

We face the problem to show that

$$\left[\left(A + \alpha\, e^{-\beta L} \right) \frac{2}{1+2\mu} A e^{\beta L} - 2\alpha^2\, e^{-\beta L} \right] \tag{30}$$

is positive?

With $y = A e^{\beta\, L}$, we may rewrite (30) as

$$\left[\left(A + \alpha\, e^{-\beta L} \right) \frac{2}{1+2\mu} A e^{\beta L} - 2\alpha^2\, e^{-\beta L} \right] = e^{-\beta\, L} \frac{2}{1+2\mu} \left(y^2 + \alpha\, y - (1+2\mu)\, \alpha^2 \right). \tag{31}$$

If we solve the quadratic equation $y^2 + \alpha\, y - (1+2\mu)\, \alpha^2 = 0$, then it yields that $y = \frac{-\alpha \pm \alpha\sqrt{5+8\mu}}{2}$. Hence, to obtain $y^2 + \alpha\, y - (1+2\mu)\, \alpha^2 > 0$, we need the following condition

$$y > \frac{\sqrt{5+8\mu} - 1}{2} \alpha. \tag{32}$$

It means that if we try to derive a property for all $L > 0$, then we need the next condition, that is our third goal,

$$A > \frac{\sqrt{5+8\mu} - 1}{2} \alpha. \tag{33}$$

We combine all extraditions that we require in the following:

$$\frac{\theta^2}{Dh} \left(A + \alpha\, e^{-\beta L} \right) < \phi, \quad e^{\sqrt{2\phi}} < 6\mu, \quad \text{and} \quad \frac{\sqrt{5+8\mu} - 1}{2} \alpha < A.$$

From the numerical example of Ben-Daya and Raouf [2] with the following data: $D = 600$ units/year, $A = \$200$ per order, $h = \$20$, $\sigma = 6$ units/week, $R\,(L) = \alpha\, e^{-\beta L}$, with L in weeks, $\alpha = 156$, and $\beta = 1$.

For different values of ϕ, we compute $\delta = \frac{1}{6} e^{\sqrt{2\,\phi}}$ and $\frac{\sqrt{5+8\delta}-1}{2}$ to list them in the next table.

We know that $\frac{\theta^2}{Dh} \left(A + \alpha\, e^{-\beta L} \right) \le \frac{\theta^2}{Dh} \left(A + \alpha \right) \approx 2.967 \times 10^{-4}$. Hence, we begin our search for $\phi = 10^{-3}$.

Table 1: Estimation for $\delta = \frac{1}{6}e^{\sqrt{2\,\phi}}$ and $\frac{\sqrt{5+8\delta}-1}{2}$

ϕ	10^{-3}	10^{-2}	10^{-1}	0.2	0.3	0.4	0.5	
$\delta = \frac{1}{6}e^{\sqrt{2\,\phi}}$	0.174	0.192	0.261	0.314	0.362	0.408	0.453	0.5
$\frac{\sqrt{5+8\delta}-1}{2}$	0.764	0.778	0.831	0.870	0.905	0.937	0.968	1

From previous Table 1, if we can make the value of ϕ as small as possible, then we will imply a small value of $\frac{\sqrt{5+8\delta}-1}{2}$.

Consequently, if we assume that $\phi = 0.01$ and $\mu = 0.192$, then

$$\frac{1}{6}e^{\sqrt{2\phi}} = 0.19198 < \mu = 0.192$$

such that the inequality $e^{\sqrt{2\phi}} < 6\mu$ is valid.

Moreover, we will want another assumption $(0.779)\alpha < A$. Since, if $(0.779)\alpha < A$, with $\mu = 0.192$ then $\frac{\sqrt{5+8\mu}-1}{2}\alpha = (0.77828)\alpha < A$ such that $\frac{\sqrt{5+8\mu}-1}{2}\alpha < A$ holds.

Based on the above discussion, we will assume two extra assumptions to finish our proof:

$$\frac{\theta^2}{Dh}\left(A + \alpha\, e^{-\beta L}\right) < \frac{1}{100} \quad \text{and} \quad (0.779)\alpha < A.$$

From the data of Ben-Daya and Raouf [2], we know that

$$\frac{\theta^2}{Dh}\left(A + \alpha\, e^{-\beta L}\right) \le \frac{\theta^2}{Dh}(A+\alpha) \approx 3 \times 10^{-4} < \frac{1}{100}$$

and

$$(0.779)\,\alpha = 121.5 < A = 200.$$

If we follow the sensitivity analysis in Axsater [1], then in his Table 1, there are 11 examples. The first one is the base example. The rest 10 examples are vary the five problem parameters—A_R, λ, h, A_D, and ω—up and down, where A_R is the fixed cost of replenishing inventory; λ is the demand intensity; h is the holding cost per unit and time unit, A_D is the fixed cost of dispatching and ω is the customer waiting cost per unit and time unit. His base data are: $A_R = 125$, $\lambda = 10$, $h = 7$, $A_D = 50$, and $\omega = 10$. We list the ups and downs for each parameter in the next Table 2.

Table 2: The ups and downs for each parameter from Axsater [1]

	A_R	λ	h	A_D	ω
Base values	125	10	7	50	10
Ups	150	15	9	75	12
Downs	100	5	5	25	8

We find the average up may be equal to

$$\frac{1}{5}\left(\frac{150}{125} + \frac{15}{10} + \frac{9}{7} + \frac{75}{50} + \frac{12}{10}\right) = 1.337$$

and the average down may be equal to

$$\frac{1}{5}\left(\frac{100}{125} + \frac{5}{10} + \frac{5}{7} + \frac{25}{50} + \frac{8}{10}\right) = 0.663.$$

From the data of Ben-Daya and Raouf [4], for the base example, we know that

$$(0.779)\,\alpha = 121.5 < A = 200.$$

Next, we consider the ups and downs. We only examine the two unfavorable cases as (a) α increases to 133.7%, and (b) A decreases to 66.3%.

For case (a), with α increases to 133.7%,

$$(0.779)\,\alpha\,(1.337) = 162.4 < A = 200.$$

For case (b), with A decreases to 66.3%.

$$(0.779)\,\alpha = 121.5 < 132.6 = 200(0.663) = A(0.663).$$

From the above discussion, we may claim that $(0.779)\,\alpha < A$ is a reasonable assumption for this kind of inventory model.

It means that our extra assumptions may be accepted for this kind of inventory system.

Combining Equations (27) and (29), according to $\mu = 0.192$ so $\frac{2}{1+2\mu} > 1$ and $\frac{1-2\mu}{1+2\mu} > 0$, then we conclude $W''(L) > 0$. Hence, we prove that $W(L)$ is a convex function for $L \in [0,\infty)$, under the constrains $Dh > 100\,(A+\alpha)\,\theta^2$ and $A > (0.779)\,\alpha$.

Since $\lim\limits_{L\to 0} W'(L) = -\infty$ and $\lim\limits_{L\to\infty} W'(L) = \infty$, we know that $W'(L) = 0$ has a unique positive solution, say L_0.

Lemma 2. If $W(L_0) \geq 0$, then $L = 0$ is the minimum point of $\phi(L)$. On the other hand, if $W(L_0) < 0$, then $\phi'(L) = 0$ has two solutions, say L_1 and L_2, with $L_1 < L_0 < L_2$. Moreover, L_1 is a local maximum point and L_2 a local minimum point of $\phi(L)$.

Proof of Lemma 2.

We already implied that $V(L)$ is a positive function and $W(L)$ is a convex function, with minimum point L_0. Depending on the value of $W(L_0)$, it yields the following three cases: Case (a), $W(L_0) > 0$; Case (b), $W(L_0) = 0$; Case (c), $W(L_0) < 0$.

For Case (a), if $W(L_0) > 0$, then $W(L) > 0$ for $L \in (0,\infty)$. Since $\phi'(L) = V(L)\,W(L) > 0$ for $L \in (0,\infty)$ so $L = 0$ is the minimum point for $\phi(L)$.

For Case (b), if $W(L_0) = 0$, then $W(L) > 0$ for $L \in (0,L_0) \cup (L_0,\infty)$. Hence, $\phi'(L) > 0$ and $\phi(L)$ is an increasing function for $L \in (0,L_0) \cup (L_0,\infty)$. Therefore, we still imply that $L = 0$ is the minimum point for $\phi(L)$.

For Case (c), if $W(L_0) < 0$, using $W(0) > 0$ and $\lim\limits_{L\to\infty} W(L) = \infty$, we know that there are two points, say L_1 and L_2, satisfying $W(L_1) = 0 = W(L_2)$, with $L_1 < L_0 < L_2$. We have that $\phi'(L) > 0$for $L \in (0,L_1) \cup (L_2,\infty)$ and $\phi'(L) < 0$for $L \in (L_1,L_2)$. Hence, we conclude that L_1 is a local maximum point and L_2 a local minimum point of $\phi(L)$.

Combining Lemma 1 and Lemma 2, we state the main theorem of this paper.

Theorem 1. Under the constrain $Dh > 100\,(A+\alpha)\,\theta^2$ and $A > (0.779)\,\alpha$, there exists a unique L_0, with $W'(L_0) = 0$. If $W(L_0) \geq 0$, then $\phi(0)$ is the minimum value. Moreover, if

$W(L_0) < 0$, then the minimum value equals $\min\{\phi(0), \phi(L_2)\}$ where $W(L_1) = 0 = W(L_2)$ with $L_1 < L_0 < L_2$.

For completeness, for the condition of $W(L_0) < 0$, we sketch the graphs to express the relation among $L = 0$, $L = L_1$ and $L = L_2$ in Figure 1.

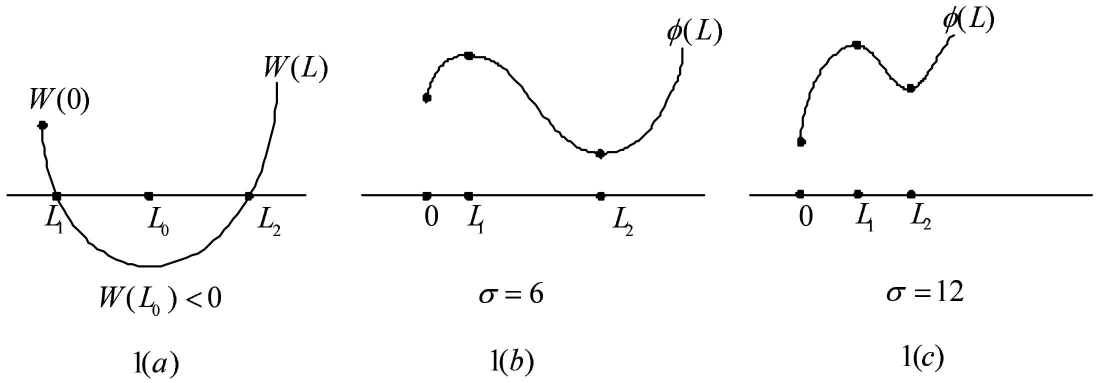

Figure 1: The graph of $\phi(L)$ when $W(L_0) < 0$

5. Solution Procedure

Next, we discuss how to use a search method to locate the optimal solution. Since $\theta\frac{Q}{D} << 1$, we will consider $e^{\theta\frac{Q}{D}} - 1 - \theta\frac{Q}{D}$ as $\frac{1}{2}\frac{\theta^2 Q^2}{D^2}$. Moreover, L_0 must exist. Therefore, we take the starting point as $L = 1$ and $Q = \sqrt{\frac{2AD}{h}}$ to find L_0 and $Q^\wedge(L_0)$.

If $W(L_0) \geq 0$, then we stop the search, taking $L = 0$ as the optimal solution for $\phi(L)$. On the other hand, if $W(L_0) < 0$, then we use $L = 2L_0$ and $Q = Q^\wedge(2L_0)$ as the starting points to locate L_2 and $Q^\wedge(L_2)$. Using $W(L_0) < 0$ and the convexity of $W(L)$, we know that there exist two roots as L_1 and L_2, with $L_1 < L_0 < L_2$. From our starting point, $2L_0$, the computation result will converge to the bigger solution L_2, since in MathCAD, they used the secant method to locate the solution. From Theorem 1, the minimum value equals $\min\{C(Q^\wedge(L_2), L_2), C(Q^\wedge(0), L = 0)\}$.

6. Numerical Examples

We consider the same problem in Ben-Daya and Raouf [2], with the variation of σsuch that three possible cases will be demonstrated. It is assumed that $D = 600$ units/year, $A = \$200$ per order, $h = \$20$, $\sigma = 6$ units/week, $k = 2.3$ and $R(L) = \alpha\, e^{-\beta L}$, with L in weeks, $\alpha = 156$, $\beta = 1$, and for the recent low interest rate period, namely $\theta = 5\%$. In the following, we examine the solution procedure for (1) $\sigma = 6$, (2) $\sigma = 12$ and (3) $\sigma = 18$. The starting points for finding L_0 and $Q^\wedge(L_0)$ are $L = 1$ and $Q = 109.5445$ for all three cases.

We solve Equation (12) that $e^{\frac{\theta}{D}Q^\wedge(L_0)} - 1 - \frac{\theta}{D}Q^\wedge(L_0) = \frac{\theta^2}{Dh}\left(A + \alpha\, e^{-\beta L_0}\right)$ where L_0 is the solution for $W'(L) = 0$ of Equation (18) and then check the value of $W(L_0)$ by Equation (17).

The computation results are listed in Table 3. It yields that when lead time is negligible as $L \to 0$, by Equation (12), $Q^\wedge(0) = 145.8545$.

Table 3: Summary of the optimal solutions, with $\theta = 5\%$

σ	L_0	$Q^\wedge(L_0)$	$W(L_0)$	L_2	$Q^\wedge(L_2)$	$\phi(Q^\wedge(L_2), L_2)$	$\phi(Q^\wedge(0), 0)$	Optimal cost
6	1.087	122.896	-0.508	2.1758	114.11	54004.04	58697.82	$\phi(Q^\wedge(L_2), L_2)$
12	0.646	129.780	-0.078	1.0272	123.696	60919.39	58697.82	$\phi(Q^\wedge(0), 0)$
18	0.434	134.154	0.252				58697.82	$\phi(Q^\wedge(0), 0)$

From the numerical examples, we may say that when the standard derivation of the lead time demand is big then the most economic way to run the inventory system is to reduce the lead time to as small as possible to attain the optimal value. When the standard derivation of the lead time demand is small, we point out that there is an optimal lead time, say L_2 that will attain the optimal value.

In the following, we try to provide a further explanation for different interest rates to help decision-makers execute the optimal replenishment policy.

Table 4: Summary of the optimal solutions, with $\sigma = 6$

θ	L_0	$Q^\wedge(L_0)$	$W(L_0)$	L_2	$Q^\wedge(L_2)$	$\phi(Q^\wedge(L_2), L_2)$	$\phi(Q^\wedge(0), 0)$	Optimal cost
0.01	1.084	123.107	-0.505	2.1689	114.29	269439	58697.82	$\phi(Q^\wedge(0), 0)$
0.05	1.087	122.896	-0.508	2.1758	114.11	54004.04	58697.82	$\phi(Q^\wedge(L_2), L_2)$
0.10	1.092	122.634	-0.512	2.1845	113.89	27074.81	58697.82	$\phi(Q^\wedge(L_2), L_2)$

With the base standard derivation $\sigma = 6$, when the interest rate is very low as $\theta = 1\%$, the best policy is to reduce the lead time to a negligible amount to obtain the biggest order quantity, $Q^\wedge(0)$, that satisfies Equation (12). It indicates that the first partial derivative system does not have an optimal solution. On the other hand, when the interest rate is relatively high as $\theta = 10\%$, there is an optimal lead time L_2 and an order quantity $Q^\wedge(L_2)$ that are the optimal solutions for the first partial derivative system.

7. Conclusion

We study the inventory model with a negative exponential crashing lead time cost. Under two reasonable assumptions, we find a criterion to determine whether or not there exists an interior local minimum point. From our decomposition of the first derivative of the objective function, the secant line search method is very easy to operate. Our findings not only provide the theoretical background for inventory models with a negative exponential crashing cost but also offer an easy algorithm to locate the optimal solution.

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