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THE OPTIMAL PRODUCTION PLAN UNDER LIMITED PRODUCTION CAPACITY AT ANY POINT IN TIME

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Abstract For the production planning decision maker needing to fit new orders into a pre-existing production plan with limited production capacity, he must estimate how much unused capacity exists at any given point in time in the prearranged production plan. In this study, we assume that the available production capacity at each point in time is estimated and known, and it is intended to build this issue into a mathematical model that is concrete for discussion with the purpose of minimizing total costs, where the total costs is the sum of production operating costs and inventory costs at any given point in time during the production process. Seeking an optimal solution and a sensitivity analysis of optimal solutions are the main parts of this study. From the results of this study, we can provide a decision procedure for an optimal production control plan for a new order for the production planners as a decision reference.

Keywords: Optimization, limited production capacity, capacity utilization rate, production plan, inventory management

1. Introduction

As Chen & Lan [1,2] pointed out in their papers, after accepting a new order for a production planning decision maker, the decision for the production starting point in time and the control of production rate at any point in time are the effective methods in expanding production profit or reducing total production costs for a manufacturer. At each unit time in the production process of a manufacturer, the available production capacity (referred to as maximum production capacity in this study) is related to the existing production plan of orders still in the production process. Under pressure of delivery commitments from existing orders, whether or not a new order with a specific quantity and delivery time can be accepted, is the issue we are going to deal with in this study. Without impairing the ongoing production plan for existing orders, the production planning decision maker must evaluate in advance what available production capacity at any point in time during the production period is, and then revise the processing plan or add new capacity by expanding the plan in order that the production target of new order may be achieved effectively. The pressure of the new order for the production planning decision maker of a manufacturer, in addition to the production operating costs, inventory costs, delivery time, and delivery quantity, still has the limitation of production capacity at any given point in time.

Generally, production capacity limits the production quantity and it limits the production rate at each point in time. In many published models, quite often the production capacity assumed to be fixed or it may be supplied sufficiently because it is quite difficult to increase the production capacity on short period. The reality for a production planning decision maker that production capacity is limited at each point in time is something one has to face from time to time. Sox & Muckstadt [3] emphasized in their work that an efficient production plan must basically employ the production capacity effectively and sufficiently. If the production capacity supplied is sufficient or it can be expanded on demand at any given point in time during the production time interval, there would not be problems related to the limitation of production capacity at that given point in time.

Suppose that the production capacity is not large at any point in time within the production period and the quantity to be delivered for new order is above a certain level in the future, the production planning decision maker may risk being unable to deliver the quantity on time if he accepts the new order hastily. This risk includes, not only the financial loss of delay in delivery or having to reject the new order at a later time after having accepted it. This risk could incur intangible losses in good will, and in the worst scenario, permanent loss of the customer. Accepting the order that should not have been accepted or rejecting the order that should have been accepted, either way could result losses to the firm. Therefore, when a new order comes in, the production planning decision maker will have to make a primary assessment whether to accept the new order based on the relationship between the production ceiling capacity, and the time and production quantity to be delivered.

When a production planning decision maker accepts a new order and after making a primary assessment, the production activities must be conducted within the assessed production capacity. If production starts too early or the production rate is too fast, it will, in effect, raise the production operating costs and it will also result in an accumulation of inventory costs for the manufacturer. Therefore, it is less than ideal for a manufacturer to produce the products earlier or faster. Yet, if the starting time of production is too late or the production rate is too slow, there will be risk not being able to deliver on time due to the limitation of production capacity. As was pointed out in the paper of Soroush [4], it is the most pleasant target in production plan management to meet the delivery time. As such, a production planning decision maker must decide at what point in time to start production, at what production rate to produce and at what point in time to start adopting Full Capacity Production (FCP, the capacity utilization rate is 100%) after accepting a new order, so that the new order may be delivered on time, and the total costs, including the production operating costs and the inventory costs may be minimized.

Because production capacity cannot be accumulated or saved for future applications, it has to be used to the maximum at any given point in time. But, production capacity cannot be over utilized because the manufacturer runs the risk of incremental buildup in production operating costs and inventory costs. Therefore, the mathematical model constructed in this study is to express the characteristics that the production operating costs in capacity utilization rate at any point in time is different. This study is based on the assessment of maximum production capacity at any point in time during the period of production in the future. Upon accepting an order after a primary assessment by production planning decision maker has been made, we intend to study the issue how the production planning decision maker determines the production plan, including the production plan function, production starting point in time and FCP starting point in time of a new order prior to the initiation of production with the consideration that the production rate at any point in time is limited to and does not exceed the maximum production capacity. In this study, we build this issue into a mathematical model that is concrete for discussion with the purpose of seeking optimal solutions of this production model and the sensitivity analysis for optimal solutions. Then, we will provide a decision procedure of the optimal production plan decision model for the production planners as a decision reference when a new order comes in.

This study focuses on the ways of production and delivery, not the kind of products. We only concern "single delivery" in our study. Products are all stored and then delivered only at the delivery point in time. Currently, the cost of human resources and products storing are very high in Taiwan. Most of the Taiwanese companies adopt a cost-saving strategy: taking orders in Taiwan, producing in China and making delivery in Taiwan. As long as the said business operation measure is taken, the delivery of an order, regardless of the types of products, can only be made in accordance with the sailing or flight schedule of transport companies. Shippers in Taiwan have almost no other choices but accept the shipping dates set by receivers in China. Under this circumstance, the order must be delivered in boodle.

2. Notations and Assumptions

In this study, the notations, meanings and assumptions of parameters or decision variables used in the mathematical model are as follows:

- T : The length of time between the starting point in time of accepting a new order and the point in time for delivery. That is, the production period for a new order is $[0,\ T]$ and has to deliver at point in time T.
- B : The quantity of merchandise that manufacturer is required to deliver at point in time T.
- $u(t)$: The production capacity ceiling at each point in time t during the production period $[0,\ T]$, where $u(t) > 0,\ \forall\, t \in [0,\ T]$.
- c_2 : The unit inventory cost of merchandise at a unit time, where $c_2 > 0$ and it is a constant.
- $x(t)$: The accumulated quantity of merchandise that manufacturer is produced during the time interval $[0,\ t\,]$, where $x(0) = 0$ and $x(T) = B$. In this study, $x(t)$ is the production plan function, and it is a decision function.
- $x'(t)$: The production rate of a production plan function x at point in time t, where $0 \le x'(t) \le u(t),\ \forall\, t \in [0,\ T]$. In this study, $x'(t)/u(t)$ is the capacity utilization rate at point in time t, and $0 \le x'(t)/u(t) \le 1,\ \forall\, t \in [0,\ T]$.
- t_x : The production starting point in time of production plan function x, and it is a decision variable, where $t_x = Max\ \{\, t \mid x(t) = 0,\ t \in [\,0,\ T\,]\ \}$, i.e., $x(t) = 0,\ 0 \le t \le t_x$ and $x(t) > 0,\ t_x < t \le T$.
- T_x : The FCP starting point in time of production plan function x, and it is a decision variable.
  If the set $\{\, t \mid t_x \le t \le T,\ \ and\ x'(t) = u(t)\ \}$ is not an empty set, then T_x is defined as $T_x = Min\ \{\, t \mid t_x \le t \le T,\ and\ x'(t) = u(t)\ \}$; otherwise, defined it as $T_x = T$.
- $(m)^+$: The meaning of this notation is $(m)^+ = \begin{cases} m, & m > 0 \\ 0, & m \le 0 \end{cases}$.

In this study, the new order form currently being discussed is not the standard order form that is used in the production process, but it is a "rush-order" order form. This rush-order order form can create conflicting problems for the production planning decision maker, especially, when the production planning decision maker must readjust the prearranged production plan with limited available production capacity. This can result in the increment in production operating costs intangibly [6]. So, the production operating costs vary according to different capacity utilization rate at different point in time of this order form. In this study, we assume that the production operating costs C_t linearly increases with the capacity utilization rate $x'(t)/u(t)$ at point in time t, i.e., $C_t = c_1 \cdot (x'(t)/u(t))$, where $c_1 > 0$ is a constant.

3. Model Construction and Solving Procedure

Based on the above assumptions and the meaning of the notations, we can find that the inventory costs and production operating costs at point in time t of unit time are $c_2 x(t)$ and $c_1(x'(t))^2/u(t)$ respectively; so, within the time interval $[t_x,\ T]$, the total inventory costs and total production operating costs are $\int_{t_x}^{T} c_2 x(t)\, dt$ and $\int_{t_x}^{T} c_1(x'(t))^2/u(t)\, dt$ respectively. Hence, the mathematical model for minimizing the total costs $C(x)$ that corresponds to production plan x can be indicated as following:

$$\textbf{(I)} \begin{cases} \underset{x}{Min}\ C(x) = \displaystyle\int_{t_x}^{T} \left\{ c_1(x'(t))^2 \big/ u(t) + c_2 x(t) \right\} dt \\ s.t. \quad x(t_x) = 0, \ \ x(T) = B, \ \ 0 \le x'(t) \le u(t), \ \ \forall\, t \in [t_x,\, T] \end{cases}$$

As the production rate $x'(t)$ at point in time t must satisfy $0 \le x'(t) \le u(t)$, so when the equation $x'(t)/u(t) = 1$ is attained, the manufacturer adopts FCP at point in time t. This is possible when the point in time t is very close to T, which would make the length of time interval $[\,t,\ T\,]$ very short, so that the producer must increase the production rate within $[\,t,\ T\,]$ to be able to accomplish the required quantity B before the point in time T.

From the restriction in **Model (I)**, it can be found that when $\int_0^T u(s)\, ds < B$, then **Model (I)** has no feasible solution; when $\int_0^T u(s)\, ds = B$, then $x(t) = \int_0^t u(s)\, ds$ is the only feasible solution which turns to be the optimal solution. Hence, hereafter, in this study we assume that the inequality $\int_0^T u(s)\, ds > B$ holds true. Supposing that the optimal solutions of **Model (I)** exists and the notations x^*, t_{x^*} and T_{x^*} is the optimal production plan function, the optimal production starting point in time, and the optimal FCP starting point in time respectively; and then we have the following corollary:

Corollary:

(1) The $x^*(t)$ must be the increasing function within the time interval $[\,t_{x^*},\ T\,]$, which means that within $[t_{x^*}, T]$, $x^{*\prime}(t) > 0$ almost everywhere.
(That is to say, except for some countable points, $x^{*'}(t) > 0$ is always sustained)

(2) If $Min\ \{\, t \mid t_x \le t \le T,\ \ and\ x'(t) = u(t)\ \}$ existed, then $x^{*\prime}(t) = u(t)$, $\forall\, t \in [\,T_{x^*},\ T\,]$.

From the above **Corollary**, we can see that the optimal solutions of **Model (I)** must be one of the following situations, which are mutually exclusive:

Case (1): $x^{*\prime}(t) < u(t)$, $\forall\, t \in [\,t_{x^*}\,,\ T\,)$.

Case (2): There exists T_{x^*}, $0 < T_{x^*} < T$, such that $x^{*\prime}(t) < u(t)$, $\forall\, t \in [\,0,\ T_{x^*}\,)$ and $x^{*\prime}(t) = u(t)$, $\forall\, t \in [\,T_{x^*},\ T\,]$.

Based on the result of **Corollary**-(1), the optimal solution $x^*(t)$ of **Model (I)** must satisfy the Euler equation [5, p.16]. Hence, by the Euler equation and the boundary condition $x^*(t_{x^*}) = 0$, we have $x^*(t)$ is as follows:

$$x^*(t) = \int_{t_{x^*}}^{t} \left(\frac{c_2}{2c_1} \cdot s + \frac{k}{2c_1} \right) \cdot u(s)\, ds, \quad \forall\, t \in [\,t_{x^*},\ T\,], \tag{3.1}$$

where k is an integration constant.

We will discuss the optimal solutions that belong to **Case (1)** and **Case (2)** separately, and then link them up in the following.

3.1. If the optimal solutions belong to Case (1)

With $x^*(T) = B$, we can obtain the integration constant k in equation (3.1) as the following:

$$k = \frac{c_2}{\int_{t_{x^*}}^{T} u(s)\, ds} \cdot \left(\frac{2c_1 B}{c_2} - \int_{t_{x^*}}^{T} s \cdot u(s)\, ds \right). \tag{3.2}$$

Case (1.1): Assuming $t_{x^*} = 0$, then

From equations (3.1) and (3.2), we have $x^*(t)$ as follows:

$$x^*(t) = \int_0^t \left(\frac{c_2}{2c_1} \cdot s + \frac{k}{2c_1} \right) \cdot u(s)\, ds, \quad \forall\, t \in [0,\, T], \tag{3.3}$$

where

$$k = \frac{c_2}{\int_0^T u(s)\, ds} \cdot \left(\frac{2c_1 B}{c_2} - \int_0^T s \cdot u(s)\, ds \right). \tag{3.4}$$

From equation (3.3) and **Corollary**, we have the sufficient and necessary condition of $x^{*\prime}(t) \geq 0$ is $k \geq 0$, $\forall\ t \in [\,0\ ,\ T\,]$. Further from equation (3.4), the sufficient and necessary condition of $k \geq 0$ is the inequality $\frac{2c_1B}{c_2} \geq \int_0^T s \cdot u(s)\, ds$.

Case (1.2): Assuming $t_{x^*} > 0$, then

Now, t_{x^*} is a decision variable. By the Transversality Condition [5, p.57], we have

$$x^{*\prime}(t_{x^*}) = 0. \tag{3.5}$$

From equations (3.1) and (3.5), we have the integration constant $k = -c_2 t_{x^*}$. Substituting this into equations (3.1) and (3.2) respectively, we have

$$x^*(t) = \int_{t_{x^*}}^{t} \frac{c_2}{2c_1} \cdot (s - t_{x^*}) \cdot u(s)\, ds, \quad \forall\, t \in [\,t_{x^*},\, T\,] \tag{3.6}$$

and t_{x^*} must satisfy the following relationship

$$\int_{t_{x^*}}^{T} (s - t_{x^*}) \cdot u(s)\, ds = \frac{2c_1 B}{c_2}. \tag{3.7}$$

First, from **Corollary** and equation (3.5), we have that the inequality $x^{*\prime}(t) \geq 0$ holds true, $\forall\ t \in [\,t_{x^*},\ T\,]$. Next, from equation (3.7), we can prove that the sufficient and necessary condition of $t_{x^*} > 0$ is the inequality $\int_0^T s \cdot u(s)\, ds > \frac{2c_1B}{c_2}$.

Concluding from the above **Case (1.1)** and **Case (1.2)**, if the optimal solutions of **Model (I)** belong to **Case (1)**, then it has the following property:

(1) If $\displaystyle\int_0^T s \cdot u(s)\, ds \leq \frac{2c_1B}{c_2}$, then $t_{x^*} = 0$, and the optimal production plan function x^* is as seen in equations (3.3) and (3.4).

(2) If $\displaystyle\int_0^T s \cdot u(s)\, ds > \frac{2c_1B}{c_2}$, then $t_{x^*} > 0$ and t_{x^*} is determined by equation (3.7) solely, and the optimal production plan function x^* is as seen in equation (3.6).

3.2. If the optimal solutions belong to Case (2)

From equation (3.1), we have

$$x^*(t) = \int_{t_{x^*}}^{t} \left(\frac{c_2}{2c_1} \cdot s + \frac{k}{2c_1} \right) \cdot u(s)\, ds, \quad \forall\, t \in [\, t_{x^*},\ T_{x^*}\,],$$

where T_{x^*} is the optimal FCP starting point in time, and thus, it is a decision variable that satisfies the following relationship:

$$x^*(T_{x^*}) = B - \int_{T_{x^*}}^{T} u(s)\, ds. \tag{3.8}$$

By the Transversality Condition [5, p.71], we have

$$x^{*\prime}(T_{x^*}) = u(T_{x^*}). \tag{3.9}$$

With equation (3.9), we have the integration constant $k = 2c_1 - c_2 T_{x^*}$, and $x^*(t)$ as the following:

$$x^*(t) = \int_{t_{x^*}}^{t} \left(1 - \frac{c_2}{2c_1} \cdot (T_{x^*} - s) \right) \cdot u(s)\, ds, \quad t \in [\, t_{x^*},\ T_{x^*}\,]. \tag{3.10}$$

By equations (3.8) and (3.10), we find that t_{x^*} and T_{x^*} must satisfy the following relationship:

$$\int_{t_{x^*}}^{T_{x^*}} (T_{x^*} - s) \cdot u(s)\, ds = \frac{2c_1}{c_2} \cdot \left\{ \int_{t_{x^*}}^{T} u(s)\, ds - B \right\}. \tag{3.11}$$

Furthermore, we can prove that the following equality is always true

$$t_{x^*} = 0. \tag{3.12}$$

From equations (3.11) and (3.12), we see that T_{x^*} is determined by the following equation:

$$\int_{0}^{T_{x^*}} (T_{x^*} - t) \cdot u(t)\, dt = \frac{2c_1}{c_2} \cdot \left\{ \int_{0}^{T} u(t)\, dt - B \right\}. \tag{3.13}$$

From equation (3.13), we can prove that if the optimal solutions of **Model (I)** belong to **Case (2)**, then, the sufficient and necessary condition is the following inequality must be sustained:

$$\int_{0}^{T} (T - t) \cdot u(t)\, dt - \frac{2c_1}{c_2} \cdot \left\{ \int_{0}^{T} u(t)\, dt - B \right\} > 0.$$

And now, T_{x^*} is determined solely by equation (3.13), and $x^*(t)$ as follows:

$$x^*(t) = \int_{0}^{t} \left(1 - \frac{c_2}{2c_1} \cdot (T_{x^*} - s)^+ \right) \cdot u(s)\, ds, \quad \forall\, t \in [\, 0,\ T\,]. \tag{3.14}$$

3.3. The optimal solutions of Model (I)

From the results in Section 3.2, we find that if the following inequality holds, then the optimal solutions belong to **Case (1)**.

$$\int_{0}^{T} (T - t) \cdot u(t)\, dt - \frac{2c_1}{c_2} \cdot \left\{ \int_{0}^{T} u(t)\, dt - B \right\} \le 0$$

So, combining Section 3.1 and Section 3.2, we can get the optimal solutions of **Model (I)** for a given $u(t)$ as the following:

(I) When $\int_0^T (T-t)\cdot u(t)\,dt - \frac{2c_1}{c_2}\cdot\left\{\int_0^T u(t)\,dt - B\right\} > 0$, then $t_{x^*}=0$ and $0 < T_{x^*} < T$, where T_{x^*} is determined by equation (3.13) solely, and $x^*(t)$ is as seen in equation (3.14).

(II) When $\int_0^T (T-t)\cdot u(t)\,dt - \frac{2c_1}{c_2}\cdot\left\{\int_0^T u(t)\,dt - B\right\} \le 0$, then we have

(1) If $\int_0^T s\cdot u(s)\,ds - \frac{2c_1 B}{c_2} \le 0$, then $t_{x^*}=0$, and $x^*(t)$ is as seen in equations (3.3) and (3.4).

(2) If $\int_0^T s\cdot u(s)\,ds - \frac{2c_1 B}{c_2} > 0$, then $t_{x^*} > 0$ and t_{x^*} is determined by equation (3.7) solely, and $x^*(t)$ is as seen in equation (3.6).

4. Implications and Sensitivity Analyses

In this study, the optimal production plan includes the optimal production plan function x^*, the optimal production starting point in time t_{x^*} and the optimal FCP starting point in time T_{x^*}. For a production planning decision maker, controlling the information of t_{x^*} and T_{x^*} are vital key points to making production plan decisions, and we will focus on the analysis of these two decision variables in this section. For the purpose of convenience, when $t_{x^*}=0$, we call the optimal production the Immediate Production (IP), and when $t_{x^*} > 0$, we call it the Deferred Production (DP). Also, we make DF_1, DF_2 and DF_3 as the decision criterion functions for a given $u(t)$, and they follow respectively:

$$DF_1(B,T) = \int_0^T u(t)dt - B \tag{4.1}$$

$$DF_2(c_1,c_2,B,T) = \int_0^T (T-t)\cdot u(t)dt - \frac{2c_1}{c_2}\cdot\left\{\int_0^T u(t)dt - B\right\} \tag{4.2}$$

$$DF_3(c_1,c_2,B,T) = \int_0^T t\cdot u(t)dt - \frac{2c_1 B}{c_2} \tag{4.3}$$

From the restriction of **Model (I)** and the results of Section 3, we can see that DF_1 is the criterion function for accepting a new order, whereas, DF_2 is the criterion function for adopting FCP after accepting a new order, and DF_3 is the criterion function for adopting IP when the FCP is totally ruled out.

When a production planning decision maker is facing a new order, under the assessed confirmed capacity ceiling $u(t)$, he must first evaluate the value of function DF_1 to preliminarily determine whether or not the new order is to be accepted. After accepting a new order, production may or may not be involved with FCP and the decision criterion is the value of functionDF_2. Production without involvement of FCP is the first consideration for the production planning decision maker, and the decision criterion to adopt IP depends on the value of function DF_3. So, when the production planning decision maker faces a new order and the anticipated production target is not attainable without adopting the FCP, the production planning decision maker will have to backtrack to adopt partial or total FCP. If this new approach still lags behind the anticipated production target, the last thing that needs to be done is to discuss how to adjust the parameters or how to expand the capacity level to achieve the production target. The total decision procedure and flow chart of the optimal production plan model in this study is shown in **Appendix A**.

From the optimal solutions of Section 3, there will be no issue based on the decision of T_{x^*} when without adopting FCP, and the decision is made based only on the value of t_{x^*}; when the FCP is adopted, the decision is made based only on the value of T_{x^*} because $t_{x^*} = 0$ is always true (see equation (3.12)). For a given capacity ceiling $u(t)$, both decision variables t_{x^*} and T_{x^*} are related to parameters c_1, c_2, B and T, where c_1 and c_2 are the internal cost factors of a manufacturer, while B and T are external order demand factors of a manufacturer, and all these factors change as the customer's order changes. Once the production planning decision maker is faced with a new order, all parameters become known. The production planning decision maker can make decisions based upon the parameters and the production capacity ceiling $u(t)$ to calculate the related values of t_{x^*} or T_{x^*} through the decision criterion functions in order to adopt the optimal production plan function x^*.

Because the decision criterion functions are functions of parameters c_1, c_2, B and T; so, we will study what impact would be on the decision criterion functions if one of the parameters is changed in section 4.1. In section 4.2, we will study what the impact would be on t_{x^*} or T_{x^*} if one of the parameters is changed. In section 4.3, we will study the impact on the whole production decision process with overall changes in the production capacity ceiling $u(t)$.

4.1. Implications and parameter analyses of decision criterion functions

From the optimal solutions of **Model (I)**, we can further reach the following five decision models (DM) and their implications according to the parameter conditions of decision criterion functions.

1. When the function value $DF_1 < 0$ (referred to as DM1):
   This means that the capacity level $u(t)$ assessed by the manufacturer will not be able to attain the production target of delivering B quantity at point in time T, so that there is no optimal production plan function and the manufacturer should abandon the new order. Under the above circumstances, the solution for the manufacturer is to expand capacity level $u(t)$ because the function value is determined completely by B, T and $u(t)$.
2. When the function value $DF_1 = 0$ (referred to as DM2):
   Here, we have $t_{x^*} = T_{x^*} = 0$. This means that the optimal production plan is IP, and the production process must adopt FCP from the very beginning of production. The decision at this time relies totally on B, T and $u(t)$ and is irrelevant to cost factors c_1 and c_2.
   Now, the optimal production plan function is $x^*(t) = \int_0^t u(s)\, ds$, $\forall\, t \in [0\ ,\ T]$.
3. When the function values $DF_1 > 0$ and $DF_2 > 0$ (referred to as DM3):
   Here, $t_{x^*} = 0$ and $0 < T_{x^*} < T$, where T_{x^*} is determined by equation (3.13) solely. This means that the optimal production plan is IP and it should adopt FCP at point in time T_{x^*} after in the production process.
   Now, the optimal production plan function x^* is as seen in equation (3.14).
4. When the function values $DF_1 > 0$, $DF_2 \le 0$ and $DF_3 \le 0$ (referred to as DM4):
   Here, $t_{x^*} = 0$. This means that the optimal production plan is IP and FCP should not be adopted during the entire production process.
   Now, the optimal production plan function x^* is as seen in equations (3.3) and (3.4).
5. When the function values $DF_1 > 0$, $DF_2 \le 0$ and $DF_3 > 0$ (referred to as DM5):
   Here, $t_{x^*} > 0$, and t_{x^*} is determined by equation (3.7) solely. This means that the optimal production plan is DP and that production should be started only at point in time t_{x^*} after accepting a new order, but without adopting FCP during the entire

production process.

Now, the optimal production plan function x^* is as seen in equation (3.6).

From the above implications, we can find that the order of the five decision models from good to poor is DM5, DM4, DM3, DM2, and DM1. Which decision model the production planning decision maker adopts depends entirely on the values of the decision criterion functions. For a given $u(t)$, we obtained the changes of the decision criterion functions to the changes of parameters are as shown in Table 1.

Table 1: The sensitivity analyses of decision criterion functions

Parameters / Decision Criterion Functions	c_1	c_2	B	T	reference
DF_1	#	#	$-$	$+$	equation (4.1)
DF_2	$-$	$+$	$+$	$+$	equation (4.2)
DF_3	$-$	$+$	$-$	$+$	equation (4.3)

"+": Decision criterion function is an increasing function of the parameter.
"−": Decision criterion function is a decreasing function of the parameter.
"#": Decision criterion function is irrelevant to the parameter.

From Table 1, we have the following results: First, the decision criterion function DF_1 is irrelevant to the values of c_1 and c_2. Thus, the primary decision whether accepting a new order or not is not related to production operating costs or inventory costs, but whether accepting a new order or not is completely determined by the value of quantity B and the delivery time T. As a whole, when B value is smaller and T value is larger, it is easier to get the value of function DF_1 larger than 0, and it is more advantageous to accept a new order. Second, the decision criterion function DF_2 is related to all the values of c_1, c_2, B and T. This means that whether FCP is adopted depends on the interaction of these parameters' values. As a whole, when c_1 value is larger, c_2 value is smaller, and when B value is smaller, T value is smaller, thus, it is easy to make the value of function DF_2 smaller than 0, and more likely to cause the production planning decision maker not to adopt FCP decision. Third, the decision criterion function DF_3 is related to all the values of c_1, c_2, B and T. This means, the adoption of FCP is completely unnecessary under this condition, and the adoption of IP depends on the interaction of these parameter values. As a whole, when c_1 value is larger, c_2 value is smaller, and when B value is larger, T value is smaller, so, it is easier to make the value of function DF_3 smaller than 0, and the production planning decision maker tends to easily adopt IP decision.

4.2. Parameter analyses of decision variables

In this section, we focus on the analysis of decision variables t_{x^*} and T_{x^*}. From section 4.1, we have that the two decision variables are depend on the values of the decision criterion functions. For a given $u(t)$, only when the function values $DF_1 > 0$, $DF_2 \leq 0$ and $DF_3 > 0$, is the value of t_{x^*} related with the values of c_1, c_2, B and T; and only when the function values $DF_1 > 0$ and $DF_2 > 0$, is the value of T_{x^*} related to the values of c_1, c_2, B and T. We can find the changes of t_{x^*} and T_{x^*} with respect to parameters under the conditions of decision criterion functions as shown in Table 2.

From Table 2, we can find the following results: First, when the function values $DF_1 > 0$, $DF_2 \leq 0$ and $DF_3 > 0$; under such circumstances, it is for DP and it is not necessary to adopt FCP completely. When c_1 value is larger, c_2 value is smaller, and when B value is

Table 2: The sensitivity analyses of decision variables

Parameters / Decision Variables	c_1	c_2	B	T	Conditions	Reference
T_{x^*}	+	−	−	+	$DF_1 > 0$,$DF_2 > 0$	equation (3.13)
t_{x^*}	−	+	−	+	$DF_1 > 0$,$DF_2 \leq 0$,$DF_3 > 0$	equation (3.7)

"+": Decision variable is increasing with the parameter.
"−": Decision variable is decreasing with the parameter.

larger, T value is smaller; it is easier to cause the value of t_{x^*} to approach 0. Thus, this makes the time length of DP shortened and more likely to prompt the production planning decision maker to adopt the IP decision. Second, when the function values $DF_1 > 0$ and $DF_2 > 0$; under such circumstances, it is for IP and it is necessary to adopt FCP part of the time. When c_1 value is larger, c_2 value is smaller, and when B value is smaller, T value is larger; it is more likely to cause the value of T_{x^*} to approach T, and therefore, it is more likely to make the optimal FCP starting point in time postponed to a later time.

4.3. Impact of changes in capacity

When the production planning decision maker is facing a new order, if the anticipated production target of the new order cannot be achieved under the assessed capacity level, but the decision maker doesn't want to lose the order. If the decision maker wants to achieve the anticipated production target of the new order, he must to readjust the parameters moderately. However, the demand factors of a customer are rather difficult to change, but the internal factors of manufacturer are relatively easy to change.

Under the circumstances where the production operating costs and inventory costs are invariant, and when the capacity ceiling $u(t)$ is increased with m as a result of increased equipment or adoption of other measures that help to increase production capacity, the capacity ceiling will shift to $u_1(t)$, where $u_1(t) = u(t) + m$, $m > 0$, $\forall\ t \in [0,\ T]$. What would the impact on decision criterion functions DF_1, DF_2 and DF_3 be? We denote the decision criterion functions as DF_1^m, DF_2^m and DF_3^m after the capacity level shifts from $u(t)$ to $u_1(t)$. From the analysis of these decision criterion functions against parameter m, we can find the following results: First, the function DF_1^m increases with the increase in m. This means that through the strategy of increasing production capacity to raise the ability to accept a new order. Second, the function DF_2^m decreases with the increase in m. This means that through the strategy of raising production capacity to increase the possibility of not having to adopt FCP. Third, the function DF_3^m increases with the increase in m. This means that through the strategy of raising production capacity, it can relax the pressure of having to adopt IP.

From the results of the above analysis, it shows that raising the production capacity level can increase the ability to accepting a new order and cutting costs. However, the economic effectiveness of raising the production capacity level in contrast to the cost of raising production capacity level needs further evaluation.

5. Conclusions

In this study, we assumed that the unit production operating cost increases linearly along with any given increases in the capacity utilization rate, and the unit inventory cost remains constant at any given point in time. We aimed to minimize total costs (including production operating costs and inventory costs) under the condition that the production capacity was limited at any given point in time. The conclusions we drew are as follows:

1. With the presence of a new order, the production planning decision maker may base on the value of function DF_1 (see equation (4.1)) to make a decision on whether or not to accept the new order. After production capacity is determined, the new order may be accepted only if $DF_1 \geq 0$, and when $DF_1 < 0$, the order shall be either abandoned or be accepted after raising production capacity and re-assessing the situation. In this part, there are no production and inventory cost factors under consideration.
2. After accepting a new order, the production planning decision maker may not need to adopt FCP decision during the whole production process. When it is necessary to adopt FCP decision, it can prove that the optimal FCP interval must occur during the last stage of the whole production process (see **Corollary**).
3. Based on the assumption in constructing the mathematical model, we use the capacity utilization rate to describe the unit production operating cost, thus, adopting FCP is rather costly and whether or not to adopt FCP decision requires the value of decision tool DF_2 (see equation (4.2)). At $DF_2 \leq 0$, it is not necessary to adopt FCP decision to achieve the anticipated production target and at $DF_2 > 0$, it is necessary to adopt FCP decision at certain point in time after production has started.
4. In the optimal production plan, t_{x^*} is the optimal production starting point in time. When it needs to adopt FCP decision, then $t_{x^*} = 0$ certainly (see equation (3.12)) and the optimal FCP starting point in time T_{x^*} is determined by equation (3.13) solely.
5. When it is not necessary to adopt FCP decision during the entire production process, whether or not to adopt IP or DP depends on the value of decision tool DF_3 (see equation (4.3)). When $DF_3 \leq 0$, it needs to adopt IP and when $DF_3 > 0$, it is necessary to adopt DP and the optimal DP point in time t_{x^*} is determined by equation (3.7) solely.
6. Under the condition that the production operating costs and inventory costs are invariant, when the capacity ceiling $u(t)$ is raised for the whole manufacturing process due to increased equipment or adopting other measures to increase capacity, the possibility being able to accept a new order increases; and it may delay the optimal FCP starting point in time, or FCP may not even be adopted at all. When it is not necessary to adopt FCP decision, the optimal production starting point in time will also be delayed.
7. There are five decision models of the optimal production plan in this study, and they can be constructed into decision procedure and flow chart for the production planners as a decision reference, as shown in **Appendix A**.

For the production planning decision maker of a manufacturer, controlling First Time is a must-have concept and must-have ability under this present ever-changing, dynamic environment. It is vitally important to get a hold of information to make timely planning decisions. The decision models provided in this study may be converted into computer programs and become a useful tool for the production planning decision maker to be at the right time and the right place. For a new order (rush-order), through rapid assessment of internal and external factors and to execute it, it will not only satisfy the order requirement of customers, but it will also bring about maximum efficiency for the Company. However, for the convenience of this study, we assumed that the unit production operating cost at any given point in time is a linearly increasing function of the capacity utilization rate at that specified point in time. In realistic situations, this assumption may not satisfy the requirements. This is the restriction inherent in this study.

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Appendix A

Procedure of Decision Models

Step 1: Presence of new order for production planning decision maker.
Step 2: Calculate the value of function DF_1.
Step 3: Preliminary assessment whether to accept the new order or not.
- If $DF_1 < 0$: If the production capacity can be adjusted, then return to Step 2, Otherwise, abandon the order and stop (DM1).
- If $DF_1 = 0$: Accept the new order and adopt IP and FCP immediately after accepting order, stop (DM2).
  The optimal production plan function is
  $x^*(t) = \int_0^t u(s)\, ds, \forall\, t \in [0\,, T]$.
- If $DF_1 > 0$: Accept the new order and go to Step 4.

Step 4: Calculate the value of function DF_2.
Step 5: Decide whether to adopt FCP and the optimal FCP starting point in time.
- If $DF_2 > 0$: Adopt IP after accepting the new order and adopt FCP at T_{x^*}, stop (DM3).
  The optimal production plan function x^* is as seen in equation (3.14).
- If $DF_2 \leq 0$: No FCP is required and go to Step 6.

Step 6: Calculate the value of function DF_3.
Step 7: Decide whether to adopt IP or DP and the optimal production starting point in time of DP.
- If $DF_3 \leq 0$: Adopt IP after accepting the new order, stop (DM4).
  The optimal production plan function x^* is as seen in equations (3.3) and (3.4).
- If $DF_3 > 0$: Adopt DP and the optimal production starting point in time is t_{x^*}, stop (DM5).
  The optimal production plan function x^* is as seen in equation (3.6).

Note: DF_1, DF_2, and DF_3 are as seen in equations (4.1), (4.2), and (4.3) respectively.
T_{x^*} is determined by equation (3.13) solely.
t_{x^*} is determined by equation (3.7) solely.

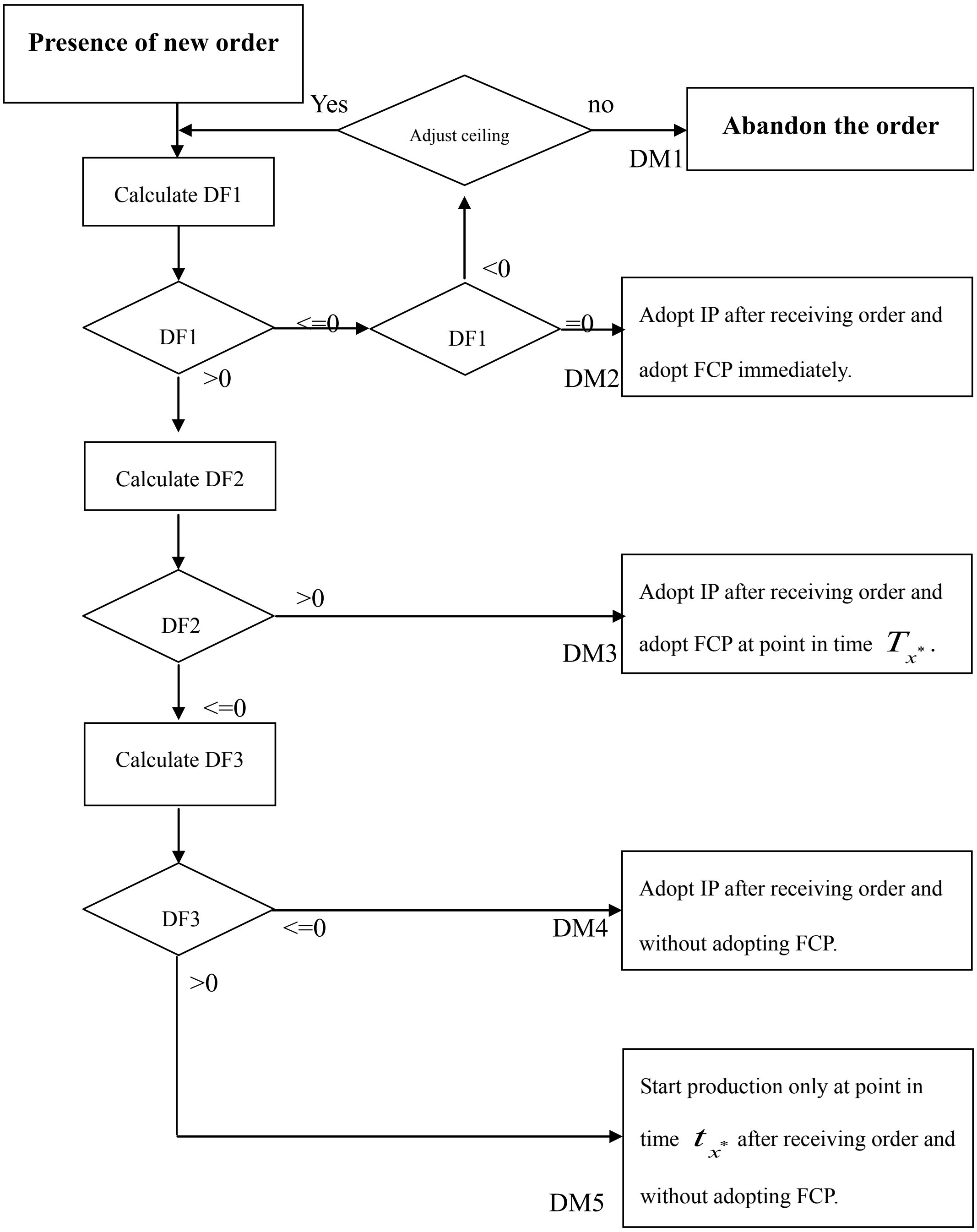

Figure 1: Flow chart of decision models

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