Journal of the Operations Research
Society of Japan
2007, Vol. 50, No. 4, 488-514

AN INVITATION TO MARKET-BASED OPTION PRICING AND ITS APPLICATIONS

Koichi Miyazaki
The University of Electro-Communication

(Received October 23, 2006)

Abstract This article provides an overview of market-based option pricing and its applications. First, two fundamental approaches for market-based option pricing from the literature are introduced. Then, three important new processes, the deterministic volatility model, the stochastic volatility model, and a model including jump, are discussed. Finally, several empirical analyses on the NIKKEI225 option market are provided as examples.

Keywords: Finance, market-based option pricing, deterministic volatility, implied tree, stochastic volatility, jump, model-free, characteristic function, fast Fourier transform, NIKKEI225 option

1. Introduction

Three decades have passed since the publication of the celebrated Black–Scholes option pricing model (Black and Scholes [7]; hereafter, BS model). Since then, the study of option pricing models has expanded in many directions, and several ways can be used to describe their development. One is to review a history of the customer's need for new products along with the valuation models proposed by researchers to meet those needs. For example, from the 1980s to the first half of the 1990s, exotic derivatives on equities and currencies were popular in financial engineering, while during the 1990s, term structure models of interest rates were rapidly developed. In the late 1990s, valuation models for credit risk were extensively examined, and since 2000, the pricing of insurance derivatives has followed an incomplete market model. Practitioners and researchers in the finance and insurance fields are very familiar with this kind of development.

The development of the market-based option pricing models was not like the development of the product-oriented model described above. My focus here is how option valuation models become more refined and how option markets mature with time. I do not discuss option pricing models for interest rates, credit risk, and insurance risk, but instead focus on relatively simple products, such as equities and currencies, and take a close look at pricing models for exotic options that incorporate market information, along with the nature of option markets.

For example, using the BS model to calculate the implied volatility of option market prices produces a feature called a volatility smile. In the volatility smile, the implied volatility of an at-the-money option (ATM) is usually lower than the implied volatility of an out-of-the-money (OTM) or an in-the-money (ITM) option. The volatility smile shows that a BS model that assumes a constant volatility is not appropriate for a market-based option pricing model. As a result, many researchers have been trying to modify the BS model to incorporate option market information. This article focuses on the methods and models that meet the necessities proposed by financial engineers' experiences.

The purpose of this article is not to provide a comprehensive survey of all market-based option pricing models, or to identify the origin of each model, but to concisely and simply present the most fundamental ideas, methods, and research directions that can be found in the literature. I hope that this article encourages many practitioners and researchers to look at market-based option pricing and its application in implied volatility analysis. Outside of the United States, theoretical or empirical research on implied volatility is not common, even though econometrics researchers extensively examine the volatility of assets with Autoregressive Conditional Heteroscedasticity (ARCH) and Generalized Autoregressive Conditional Heteroscedasticity (GARCH) types of models.

This article is organized into seven sections. In Section 2, two fundamental approaches to market-based option pricing are explained, along with a description of the historical development of market-based option pricing. This is followed by three sections that comprise the main body of the article; Section 3 discusses the deterministic volatility model and the implied tree, Section 4 introduces stochastic volatility models, and Section 5 reviews models that include the jump process and methods of option market analysis. In Section 6, several results from the author's research are presented, along with their implications for the NIKKEI225 option market. The last section summarizes the results and information and offers concluding remarks.

2. Important Early Market-based Option Pricing Models and Future Directions

This section discusses two important precursors to market-based option pricing, Breeden and Litzenberger [8] and Heston [29]. Directions in market-based option pricing models from the two approaches are then reviewed.

2.1. Breeden and Litzenberger [8]

Breeden and Litzenberger ([8]; hereafter, BL) proposed a method that evaluates a security based on the time–state preference model in the general equilibrium invented by Arrow [3] and Debreu [17]. The valuation placed importance on the price of an elementary claim on aggregate consumption. To make the model easier to understand, we will assume one-to-one mapping between the aggregate consumption and the value of the market portfolio on each day. Under such an assumption, the price of the elementary claim on the aggregate consumption is equal to the price of the elementary claim on the market portfolio. The elementary claim on the market portfolio is defined as the security that on the predetermined date, T, \$1 will be paid when the value of the market portfolio is equal to M. If the value is not equal to M, nothing will be paid. The price of the security is denoted as $P(M,T)$. BL therefore provided a method to evaluate a security based on the price of the elementary claim on the market portfolio, and also to derive the price of the elementary claim from the market price of European call options. Because of these two factors, I consider BL a precursor to market-based option pricing.

A discrete valuation of a security based on the price of the elementary claim: if security f has maturity t $(t \leq T)$ and payoff q_T^f over time periods [0, t]that are known to be functions of only the level of the market M_T at each date T, that is, $q_T^f = q_T^f(M_T)$, then its price V^fmust be

$$V^f = \sum_{T=1}^{t} \sum_{M_T=1}^{N} q_T^f(M_T)\, P(M_T,T) \tag{2.1}$$

where $P(M_T,T)$ (hereafter, we denote $P(M,T)$) is the price of the elementary claim on the market portfolio.

To evaluate the security using equation (2.1), the price of the elementary claim on the market portfolio $P(M,T)$ must be derived beforehand. BL derives the price $P(M,T)$ from the market prices of European call options on the market level M_T, as follows. Initially, suppose that the value of the market portfolio M_T takes discrete values of $M_T = \$1.00, \$2.00, \ldots, \$N$. $c(X,T)$is the price of a European call option on the market M_T with T periods to maturity, and with an exercise price of X. Figure 1A provides two payoff diagrams at the maturity of the two European call options. The payoff diagram of the option with maturity T and strike price X is ○ ; △ represents the other option, with maturity T and strike price$X+1$. In Figure 1B, the payoff diagram at maturity ▲ of the portfolio of the options $(c(X+1,T) - c(X,T))$ is obtained by subtracting diagram ○ from diagram △. In the same manner, Figure 2A provides two payoff diagrams at the maturity of two European call options. The option with maturity T and strike price of $X-1$ has a payoff diagram of □; ○ is the payoff diagram of the second option, with maturity T and the strike price X. In Figure 2B, the payoff diagram at maturity ■ of the portfolio of the options $(c(X,T) - c(X-1,T))$ is obtained by subtracting diagram □ from diagram ○. Subtracting the payoff diagram ■ in Figure 2B from the payoff diagram ▲ in Figure 1B, we can get the payoff diagram ● (Figure 3) at the maturity of the elementary claim $P(X,T)$, which states that on maturity T, \$1 will be paid when the value of the market portfolio is equal to M; if it is not, nothing will be paid. Thus, when the market level takes discrete values, the price of the elementary claim $P(X,T)$ equals the price of the portfolio of the call options $[c(X+1,T) - c(X,T)] - [c(X,T) - c(X-1,T)]$.

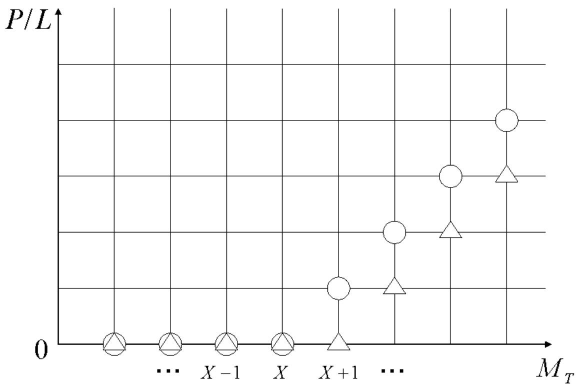

Figure 1A: Two payoff diagrams of the two European call options $(c(X+1,T), c(X,T)))$

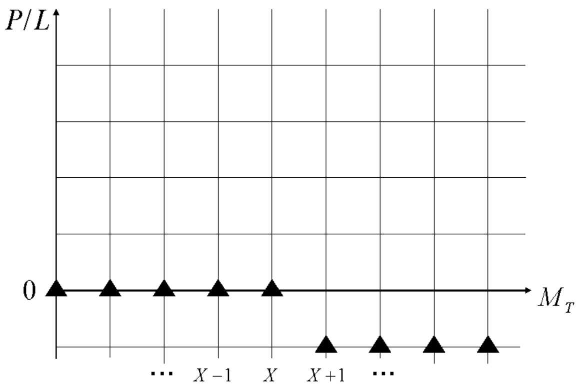

Figure 1B: The payoff diagram of the portfolio of the options $(c(X+1,T) - c(X,T))$

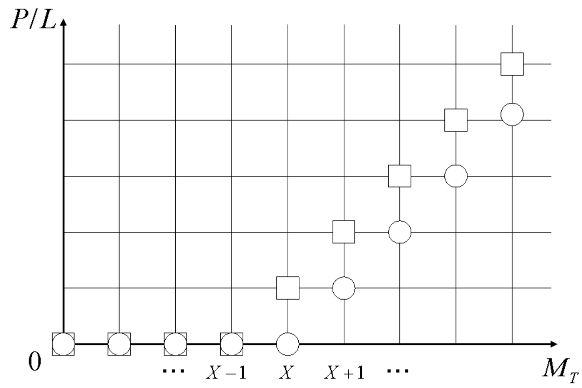

Figure 2A: Two payoff diagrams of the two European call options $(c\,(X,T)\,,c\,(X-1,T))$

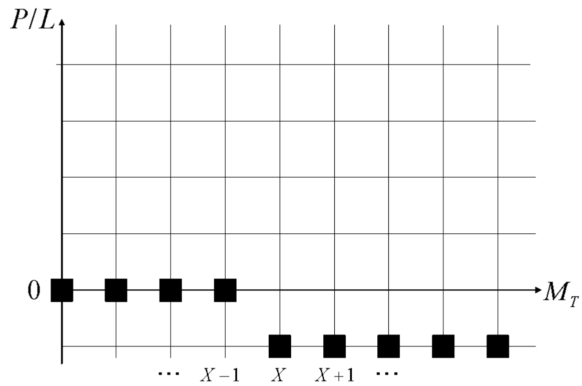

Figure 2B: The payoff diagram of the portfolio of the options $(c\,(X,T)-c\,(X-1,T))$

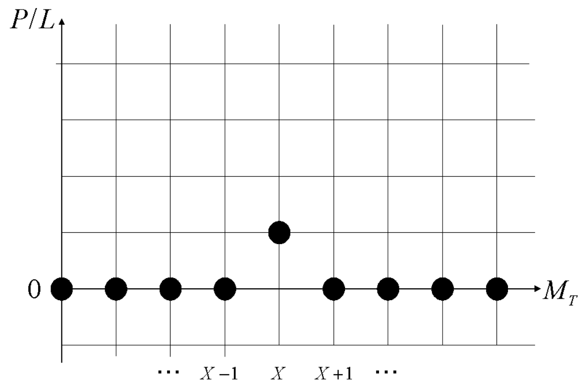

Figure 3: The payoff diagram of the elementary claim $P\,(X,T)$

Next, we try to generalize the \$1 mesh of the possible values of the market level M_T to the very small mesh ΔM. $P(X,T;\Delta M)$ is the price of elementary claim evaluated using the granular mesh. The definition of the elementary claim states that on the maturity T, \$1 will be paid when the value of the market portfolio is equal to M, and otherwise nothing will be paid. Based on this premise, we can derive the price of the elementary claim $P(X,T;\Delta M)$ by replacing +1 and –1 in the price of the portfolio of the call options $[c(X+1,T)-c(X,T)]-[c(X,T)-c(X-1,T)]$ by $+\Delta M$ and $-\Delta M$, respectively, then dividing it by ΔM, as follows:

$$P(X,T;\Delta M)=\frac{1}{\Delta M}\{[c(X+\Delta M,T)-c(X,T)]-[c(X,T)-c(X-\Delta M,T)]\} \quad (2.2)$$

Furthermore, dividing the both sides of equation (2.2) with ΔM, we can get

$$\frac{P(X,T;\Delta M)}{\Delta M}=\frac{1}{\Delta M}\left\{\frac{[c(X+\Delta M,T)-c(X,T)]}{\Delta M}-\frac{[c(X,T)-c(X-\Delta M,T)]}{\Delta M}\right\} \quad (2.3)$$

In equation (2.3), when the mesh ΔM approaches 0,

$$\lim_{\Delta M\to 0}\frac{P(X,T;\Delta M)}{\Delta M}=\frac{\partial^2 c(X,T)}{\partial X^2}=c_{XX}(X,T) \quad (2.4)$$

Equation (2.4) indicates that

$$\lim_{\Delta M\to 0}\frac{P(X=M,T;\Delta M)}{\Delta M}=c_{XX}(X=M,T) \quad (2.5)$$

The left side of equation (2.5) represents the probability density that the market level M_T at maturity T is equal to M. Thus, the discrete valuation formula given by equation (2.1) is more explicitly expressed in a continuous form as

$$V^f=\int_T\int_{M_T} q_T^f(M_T)\,c_{XX}(X=M,T)\,dM_T dT, \quad (2.6)$$

where $\int_T$ means the integration with respect to T on [0, t] and $\int_{M_T}$ express the integration with respect to M_T on [0, ∞]. The most notable feature of the BL valuation formula (2-6) is that it does not assume any stochastic process for the market level. The only assumption in the BL formula is that $c(X,T)$ is twice differentiable with respect to the strike price X; in the discrete case, this assumption can be ignored. The preferences of each individual investor and forecasts of market direction can be incorporated into option prices without restriction. These features of the BL formula helped lay the foundation for market-based option pricing and its applications.

2.2. Heston [29]

Heston's [29] paper (hereafter, Hes) famously describes the stochastic volatility model in option pricing. However, I do not think that this paper helped to originate market-based option pricing because of the stochastic volatility model. Stochastic volatility models, such as those of Hull and White [35], Wiggins [55], and Stein and Stein [53], were introduced in the literature before the publication of Hes.

Hes was valuable because it introduced three new approaches. First, it focused on the relationship between the stochastic process of the underlying asset (stochastic differential equation) and the partial differential equation (PDE) that the value of the derivative security satisfies (even the probability that the underlying asset becomes in-the-money is considered a derivative security). Second, in deriving an in-the-money probability, which is indispensable for the valuation of an option, [29] used the risk-neutral density function of the underlying asset to analytically solve the PDE that the characteristic function satisfies. Finally, he applied Fourier inversion to the function that was derived to obtain the in-the-money probability for the valuation of the option, based on the derived risk-neutral density function of the underlying asset.

Because of Heston's [29] approach, an option valuation that considers complicated stochastic processes such as jump and the general Levy method (Albanese and Kuznetsov, [1]; Carr and Wu, [16]; Huang and Wu, [33]) is available in a unified framework based on the characteristic function and its Fourier inversion. Furthermore, even when the option pricing takes into account the complicated stochastic processes mentioned above, we can easily derive the option price using fast Fourier transform (FFT). Thus, by fitting the option model price to the option market price, we can estimate the parameters of the complicated stochastic process and identify the process implicit in option price. Because of these contributions, I consider Hes the second significant precursor to market-based option pricing and its applications.

Heston's [29] stochastic volatility model

Under the risk-neutral measure, the equity and its variance follow the stochastic processes (2.7) and (2.8), respectively:

$$dS(t) = rSdt + \sqrt{v(t)}Sd\tilde{z}_1(t), \tag{2.7}$$

$$dv(t) = (\kappa[\theta - v(t)] - \lambda(S, v, t))\,dt + \sigma\sqrt{v(t)}d\tilde{z}_2(t) \tag{2.8}$$

where both $d\tilde{z}_1(t)$ and $d\tilde{z}_2(t)$ are standard Brownian motions, and their correlation is ρ.

Heston's [29] first approach

[29] first approach derived the PDE (2.9) that the derivative security $U(S, v, t)$ should satisfy when the underlying asset and its variance are given by equations (2.7) and (2.8), and then solved it under an appropriate boundary condition.

$$\begin{aligned}&\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S}\\&+\{\kappa[\theta - v(t)] - \lambda(S, v, t)\}\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0\end{aligned} \tag{2.9}$$

Using the analogy of the BS model, we have written the formula for the call option price $C(S, v, t)$ in the following form:

$$C(S, v, t) = SP_1(S, v, t) - Ke^{-r(T-t)}P_2(S, v, t). \tag{2.10}$$

The first part of equation (2.10), $SP_1(S, v, t)$, expresses the expected present value at time t of the underlying asset with the optimal exercise (in other words, at maturity T, the option is exercised when the price of the underlying asset is above the strike price). The second part of equation (2.10), $Ke^{-r(T-t)}P_2(S, v, t)$, indicates the expected present value at time t of the cash to be paid in the exercise of the option. [29] considered the right side of equation (2.10) to be a derivative product, and derived the PDE that each has to satisfy. For simplicity, the price of the market risk in the derivation is restricted as $\lambda(S, v, t) = \lambda_c v$, and the log of the underlying asset, $x = \ln S$, is used instead of the underlying S. From now on, we will discuss only the first part, $SP_1(S, v, t)$; the argument is directly applicable to the second part.

Applying Ito's lemma to SDE (2.7), we see that x follows

$$dx(t) = \left(r - \frac{v}{2}\right)dt + \sqrt{v(t)}d\tilde{z}_1(t). \tag{2.11}$$

And, with the restriction$\lambda(S, v, t) = \lambda_c v$, SDE (2.8) becomes

$$dv(t) = (\kappa[\theta - v(t)] - \lambda_c v(t))\,dt + \sigma\sqrt{v(t)}d\tilde{z}_2(t). \tag{2.12}$$

When the stochastic process of the underlying asset follows SDE (2.11) and (2.12), the PDE that $SP_1(S, v, t) = e^x P_1(x, v, t)$ should satisfy is given by

$$\frac{1}{2}v\frac{\partial^2 P_1}{\partial x^2} + \rho\sigma v\frac{\partial^2 P_1}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_1}{\partial v^2} + \left(r + \frac{1}{2}v\right)\frac{\partial P_1}{\partial x} + \{\kappa\theta - (\kappa + \lambda_c - \rho\sigma)v\}\frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t} = 0 \tag{2.13}$$

To attain the appropriate call option price, the PDE (2.13) should be solved under the following boundary condition,

$$P_1(x, v, T; \ln K) = 1_{\{x \geq \ln(K)\}} \tag{2.14}$$

$e^x P_1$ should satisfy the PDE (2.13) when variable x, v in equation (2.14) corresponds to $x(t)$ in equation (2.11) and $v(t)$ in equation (2.12), respectively.

Heston's [29] second approach

Heston's [29] second approach considers the PDE (2.13) that P_1 satisfies when the underlying asset process follows equation (2.15) and (2.16), rather than the PDE satisfied by $SP_1(S, v, t) = e^x P_1(x, v, t)$ when the underlying asset process follows equation (2.11) and (2.12):

$$dx(t) = \left(r + \frac{v}{2}\right)dt + \sqrt{v(t)}d\tilde{z}_1(t) \tag{2.15}$$

$$dv(t) = (\kappa\theta - (\kappa + \lambda_c - \rho\sigma)v)\,dt + \sigma\sqrt{v(t)}d\tilde{z}_2(t) \tag{2.16}$$

Putting

$$P_1(x, v, t; \ln K) = E\left[1_{\{x(T) > \ln(K)\}}\,|x(t) = x, v(t) = v\right] \tag{2.17}$$

P_1, as defined by equation (2.17), satisfies PDE (2.13) when $x(t)$ and $v(t)$ follow SDE (2.15) and (2.16), respectively. It is very difficult to directly derive the analytical solution of equation (2.17). In Heston's [29] third approach, the solution was derived using the characteristic function and its Fourier inversion.

Heston's [29] third approach

Heston's [29] third approach initially defines the characteristic function corresponding to $P_1(x, v, t)$ as

$$f_1(x, v, t; \phi) = E\left[e^{i\phi \cdot x(T)}\,|x(t) = x, v(t) = v\right] \tag{2.18}$$

The terminal condition is expressed as

$$f_1(x, v, T; \phi) = E\left[e^{i\phi \cdot x(T)}\,|x(T) = x, v(T) = v\right] = e^{i\phi \cdot x} \tag{2.19}$$

Thus, the characteristic function f_1 can be derived by solving PDE (2.13) under the boundary condition (2.14). [29] takes the solution form of the characteristic function f_1 to be

$$f_1(x, v, t; \phi) = \exp[C_1(T - t; \phi) + D_1(T - t; \phi)v + i\phi \cdot x] \tag{2.20}$$

then derives two ordinary differential equations (ODE) and solves them to analytically compute equation (2.20).

Next, he applies Fourier inversion to the characteristic functionf_1 that he obtains, and then derives, in order, the risk-neutral density function and the P_1, P_2 based upon it, and the call option price consisting of P_1, P_2. The risk-neutral distribution, which is indispensable for deriving the in-the-money probability P_1, is attained by the Fourier inversion of the characteristic function $f_1(x, v, t; \phi)$, $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi\cdot x} f_1(x, v, t; \phi)\, d\phi$.

Integrating from $\ln K$ to positive infinity with respect to x, P_1 is computed as

$$\begin{aligned} P_1(x, v, t; \ln K) &= \int_{\ln K}^{\infty} Re\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi\cdot x} f_1(x, v, t; \phi)\, d\phi\right\} dx \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left\{\frac{e^{-i\phi \ln K} f_1(x, v, t; \phi)}{i\phi} d\phi\right\} \end{aligned} \qquad (2.21)$$

2.3. Three major directions of the market-based option pricing from above two approaches

In this section, I review how market-based option pricing evolved from BL and Hes. A unified option valuation method and an interesting method to analyze the options market are proposed by examining the precision of three new approaches for market-based option pricing. In the three sections that follow (Sections 3–5), literature that is related to these three approaches is discussed in order.

Section 3 discusses a method of constructing the option valuation tree by incorporating the option market price (implied tree) and deterministic volatility models, and its related literature. Section 4 describes option pricing using stochastic volatility models, and related literature. Section 5 discusses literature related to option pricing using stochastic processes, including jump, and its applications. The first approach, based on the BL model, recovers the risk-neutral density function based on the option market prices; the second and third approaches, rooted in Hes, base option pricing on the characteristic function.

3. Deterministic Volatility Models and Implied Tree

In 1994, valuation models were proposed of exotic options. These options were based on the tree (implied tree) expressing the risk-neutral dynamics of the underlying asset, which incorporated the volatility smile observed in the option market. Dupire [23] was published first, followed by Rubinstein [49] and Derman and Kani [19]. The deterministic volatility models in equation (3.1) represent the models in which the implied trees are constructed in the three papers. These are often compared to the stochastic volatility models introduced in Section 4.

$$\frac{dS}{S} = r(t)\, dt + \sigma(S, t)\, d\tilde{W} \qquad (3.1)$$

where S is the underlying asset, $r(t)$ is the risk-free interest rate, $d\tilde{W}$ is Brownian motion under a risk-neutral measure, and σ is the deterministic function that depends upon the price of the underlying asset and time (the σ in which S and t are specified is the local volatility).

The advantage of the deterministic volatility model is that it extends the BS model without increasing the dimension of uncertainty, other than the diffusion uncertainty. For this reason, the option can be hedged only by the underlying asset. In an attempt to specify the diffusion part in equation (3.1) based on the option market prices, [23] verified that all of the risk-neutral density, $\varphi_T(K)$, when the price of the underlying asset is K at time T, is derived from the call option prices $C(K,T)$ corresponding to the strike K and maturity T, as follows:

$$\varphi_T(K) = \frac{\partial^2 C}{\partial K^2}(K,T) \tag{3.2}$$

In reality, equation (3.2) is really equation (2.5) in BL introduced in Section 2.1. The price of the European option is evaluated based on the risk-neutral density $\varphi_T(K)$, with T corresponding to the maturity of the option. In the risk-neutral measure, the valuation of the path-dependent option must specify the stochastic process of the underlying asset. Under the appropriate conditions, [23] recovers the unique stochastic process corresponding to the risk-neutral density $\varphi_T(K)$. By assuming that the risk-free interest rate is 0, and focusing on the martingale stochastic process $dx = b(x,t)\,dW$, he derived equation (3.3) based on the Fokker–Planc equation:

$$\frac{b^2(K,T)}{2}\frac{\partial^2 C}{\partial K^2} = \frac{\partial C}{\partial T}, \tag{3.3}$$

where C expresses the price of the European call option of maturity T and strike price$K$.

Because of the nature of the option price, the partial derivatives $\frac{\partial^2 C}{\partial K^2}, \frac{\partial C}{\partial T}$ in equation (3.3) are both positive.

They are obtained from information about the implied volatility surface observed in the option market. We can compute $b(K,T)$, which is implicit in the option market based on equation (3.3). Returning to the original stochastic process of the underlying asset under the conditions $dS/S = \sigma(S,t)\,dW$, we obtain the local volatility as equation (3.4):

$$\sigma(S,t) = \frac{b(S,t)}{S}. \tag{3.4}$$

[23] also provided a method to construct the implied tree of the derived stochastic process, by incorporating option market price information. The implied tree has the Arrow–Debreu price at each node.

[50] and [19] proposed a method that constructed implied trees by assuming a binomial tree from the beginning. We show Rubinstein's [50] method in detail. [50] adopted the binomial model as a prior distributional form of the risk-neutral distribution, and derived the risk-neutral distribution (in this case, it may be more appropriate to call it an "implied" distribution) at option maturity T as a posterior distribution that was obtained by fitting the European call option model prices to their market price counterparts. This implied distribution corresponded to the risk-neutral distribution $\varphi_T(K)$ in [23]. Using a backward induction based on the implied probability distribution at maturity T, [50] built an implied tree that expressed the dynamics of the implied stochastic process. This part of his work is essentially the same as the implied tree that [23] built after deriving the local volatility that was implicit in the option market prices.

Rubinstein's [50] method to estimate implied probability distribution

The risk-neutral probabilities and the implied probabilities are defined assuming a binomial model, as $P'_j, j = 1,\cdots,n,\ P_j, j = 1,\cdots,n$, respectively. The implied probability distribution is estimated by solving the following mathematical programming:

$$\min_{P'_j} \sum_j \left(P_j - P'_j\right)^2$$
$$\text{s.t} \begin{cases} \sum_j P_j = 1 \ , \ P_j \geq 0 \ , \ j = 1, \cdots, n \\ S^b \leq S \leq S^a, \ S = \dfrac{\left(q^n \sum_j P_j S_j\right)}{r^n} \\ C_i^b \leq C_i \leq C_i^a, \ C_i = \dfrac{\left(\sum_j P_j \max[0, S_j - K_i]\right)}{r^n} \ , \ i = 1, \cdots, m \end{cases} \tag{3.5}$$

where r, q, K_i, and m are the risk-free interest rate, dividend yield, strike price, and the number of traded options, respectively. Furthermore, S^b, S^a are the bid and the ask prices of the equity and C_i^b, C_i^a are the bid and the asking price of the call option. The objective function of the mathematical programming is the sum of the square errors (the lower the better), or the difference between the risk-neutral probability distribution and the implied probability distribution.

With these three papers as a starting point, the research field has grown rapidly. Buchen and Kelly [12] proposed that prior bias could be excluded by using the maximum entropy principle to estimate the implied distribution based on option market prices. Edelman [25] introduced a similar estimation method based on local cross-entropy. Jackwerh and Rubinstein [36] revamped Rubinstein's [50] method to obtain a smoother estimated implied distribution. Li [43] points out that the construction method of the implied tree by Derman and Kani [19] is unstable and proposes a simple method of constructing a "recombining" tree for a quite general class of local volatility functions. Kahalé [41] also proposed a new technique to recover a volatility surface that satisfies both the arbitrage-free condition and the smoothness condition. Hentschel [28] pointed out the large estimation error that occurs when the implied volatility is estimated by inverting the BS formula, and proposed using a generalized least-squares estimator that causes less noise and bias. Nomura and Miyazaki [48] examined the option valuation in Rubinstein's [50] approach when the normal distribution, used as the prior distribution in the estimation of the implied distribution, is replaced by a normal-inverse Gaussian distribution. Applying the above methods to investment strategies, Derman and Zou [20] proposed that the method could be used to derive the fair value of the skew.

Most of the models used to construct the implied tree are deterministic volatility models. These are often compared to the stochastic volatility models introduced in Section 4. In the deterministic model, the local volatility is described only by the time and the state of the underlying asset. No other uncertainties exist. To construct an implied tree within the deterministic model in a way that limits the stochastic process of the underlying asset, many researchers will look for a deterministic volatility model that best incorporates option market prices. Brown and Randall [11] parameterized the local volatility with the skew function, using the nonsymmetric tanh(x), and the smile function, adopting the symmetric sec h. Dumas, Fleming, and Whaley [22] prepared several deterministic volatility functions (DVF) and examined their performance in forecasting and hedging the price data from the Standard & Poor's 500 (S&P 500) index covering the period from June 1988 to December 1993. Their performance was almost identical to the performance of *ad hoc* models that smooth the BS implied volatility across maturities and strike prices. When exotic derivatives are valued, the model risk of the implied volatility function (IVF) is negligible for the compound option, but large in the barrier option (Hull and Suo, [34]).

Therefore, the results of empirical analyses do not necessarily support using the deterministic volatility model. However, the deterministic model proposed by Brigo and Mercurio [9] shows promising performance and is analytically tractable. They noted two problems in Dupire's [23] valuation: the stochastic process of the underlying asset is not explicitly parameterized and therefore cannot evaluate the path-dependent option by way of the Monte Carlo simulation, and the price and the Greeks of the simple European option are not attainable. To address these problems, they proposed a model in which the underlying process follows a mixture of several lognormal distributions and incorporates the smile and skew information from the option market prices.

4. Stochastic Volatility Model

This section discusses the development of the stochastic volatility model, focusing on the incorporation of the GARCH model into option pricing, the efficient parameter estimation method, and building the implied tree for the stochastic volatility model. Since the latter half of 1980s, ARCH and GARCH models have gained popularity as powerful econometric tools for forecasting volatility. Beginning in the mid-1990s, financial engineers began applying these models to their own field, leading to GARCH option pricing. Duan [21] and Heston and Nandi [30], among others, incorporated the GARCH model into option pricing.

Adopting the nonlinear GARCH (N-GARCH) model in Engle and Ng [26], Duan [21] proposed an option pricing model based on a Monte Carlo simulation. The parameters of the GARCH model are usually estimated using the econometric price data of the underlying asset. Duan calibrated the parameters of the model based on option market prices and found that the estimated volatility was a reasonable fit for the out-sample implied volatility.

N–GARCH model (under risk-neutral measure):

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{1}{2}\sigma_{t+1}^2 + \sigma_{t+1}\xi_{t+1} \tag{4.1}$$

$$\sigma_{t+1}^2 = \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2\left(\xi_t - \theta - \lambda\right)^2, \tag{4.2}$$

where r and λ are the risk-free rate and the market price of risk, respectively; the positive θ expresses the leverage effect (the increase in the index causes the decrease in the volatility); the parameters are restricted as $\beta_0 > 0, \beta_1 \geq 0, \beta_2 \geq 0$, to guarantee that the conditional volatility is positive; and $\xi_{t+1} = \varepsilon_{t+1} + \lambda$, conditional on the time t, follows a standard normal distribution under the risk-neutral measure. In the model, the option price is evaluated by equations (4.1) and (4.2).

Duan [21] found that the implied volatility attained by GARCH option pricing model fit out-sample implied volatility reasonably well. This inspired Heston and Nandi [30] to propose a GARCH option pricing model similar to the N–GARCH and V–GARCH models, and to empirically analyze the option market based on their model. The basic approach of their pricing model is as same as in Hes, introduced in Section 2.2. Because of the discrete stochastic process of the underlying asset, they do not solve the PDE in the GARCH option pricing model. Instead, they computed the characteristic function of the risk-neutral distribution by applying induction to the characteristic function. In an empirical analysis, they compare the ability of their GARCH option pricing model to forecast volatility that of a valuation model that adopts the deterministic volatility models of Dumas et al. [22], as described in Section 3. The empirical analysis shows that the deterministic volatility model fits quite well with the implied volatility, but the implied volatility of the deterministic volatility model does not explain the future volatility dynamics, and the model can not predict the out-sample option price. However, the GARCH option pricing model achieves a good fit with the implied volatility surface and can forecast the out-sample option price.

Fouque, Papanicolaou, and Sircar [27] proposed a stochastic volatility model in which the volatility process follows the Ornstein–Uhlenbeck (OU) process. They first empirically verified that the mean-reversion parameter κ of the volatility process is large in the actual market, thus the reverse of the parameter, $\varepsilon = 1/\kappa$, is very small. Then, after scale transforming the adopted stochastic volatility model and expressing it with the small parameter ε, they reformed the PDE satisfied by the option price (corresponding to equation (2.12) in Section 2.2) in the power order of ε. Defining the option model price as P^{ε}, they used the appropriate differential operators, L_0, L_1, L_2, to derive the PDE that P^{ε} should satisfy:

$$\left(\frac{1}{\varepsilon}L_0 + \frac{1}{\sqrt{\varepsilon}}L_1 + L_2\right) P^{\varepsilon} = 0. \tag{4.3}$$

Then, they formally expanded the option model price P^{ε} in the power order of ε, as equation (4.4):

$$P^{\varepsilon} = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + \cdots, \tag{4.4}$$

where $P_0, P_1, \cdots$ are the functions of time t, the stock's present value x and the present value y of the process deriving the volatility that satisfy the terminal conditions $P_0(T, x, y) = h(x)$ ($h(x)$ is nonnegative payoff function) and $P_1(T, x, y) = 0$. This was substituted into equation (4.3) and reformed with the power order ofε (so its coefficients are composed of both $P_0, P_1 \cdots$ and the differential operators L_0, L_1, L_2) to derive the identity, and multiplied by the small parameter ε, to exclude the divergent part. When we compared the values of the coefficients on the both sides of the derived identity and added the coefficients of the orders, ε to 0, we found that the number of PDEs that $P_0, P_1 \cdots$ should satisfy was the same as the number of orders of the small parameter. Using equation (4.4), the precision of the model can be improved with the use of higher order information. Fouque et al. (2000) derived the convenient analytical valuation formula (4.5) and found that it corresponds to the approximation $P^{\varepsilon} \approx P_0 + \sqrt{\varepsilon}P_1$, using the first two parts in equation (4.4) (Refer to the paper for details on the derivation.):

$$P_0 - (T-t)\left(V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right), \tag{4.5}$$

where P_0 is the BS formula under the constant volatility, $\bar{\sigma}$, and V_2, V_3 are constants related to the parameters of the stochastic volatility model. Especially, both V_2 and V_3 include $\sqrt{\varepsilon} = 1/\sqrt{\kappa}$, the square-root of the inverse of mean reversion parameter.

Fouque et al. [27] proposed that the method first derive the implied volatilities expressed by the parameters in the option model price (4.5), and then to calibrate the parameters, fit the implied volatilities of the model with the actual implied volatilities of the market, in a least-squares sense.

Piterbarg [48] presented a parameter estimation method of the stochastic volatility model having time-dependent coefficients. In the method, averaged constants are substituted for time-dependent parameters. The method does not use the exact solution for calibration, but instead resorts to approximation, resulting in less computational burden. In detail, a stochastic volatility model with time-dependent coefficients in some time period can produce an underlying asset that generates a distribution that approximates the distribution generated by a stochastic volatility model with constant coefficients.

Britten-Jones and Neuberger ([10]; hereafter, BN) developed an approach to construct the implied tree of the stochastic volatility model. Notably, the BN framework does not assume any stochastic process for the underlying asset. Thus, their approach is not limited to the deterministic volatility model and is applicable to the stochastic volatility model. Also, because of the convenient construction of the lattice, the approach restricts the underlying stochastic process model to a continuous one (the stochastic volatility model is also continuous). They adopted a trinomial tree, with state space $\mathbf{K}$ given by $\mathbf{K} = \{K : K = S_0 u^i, i = 0, \pm 1, \pm 2, \ldots, \pm M\}$, using an initial equity price S_0.

In general, the implied tree is constructed by first expressing the risk-neutral probability $\pi(K;t) = \Pr\{S_t = K\}$, such that the equity price S_t at time t takes K, by the linear combination of call option prices using the BL equation (2.2), because the risk-neutral probability is attained just dividing the price of the elementary claim by \$1, thus it is the same amount as the price of the elementary claim though the unit is different. Second, assuming that the equity price in the discrete model moves upward or downward one unit of state, in one unit of time, because of the continuity of the equity process, we expressed the combined stochastic density $\Pr\{S_t = K, S_{t+h} = K^*\}$. The equity price is K at time t, and K^* at time $t+h$, and the conditional probability is $\Pr\{S_{t+h} = Ku \,|S_t = K\}$, through a linear combination of call option prices. To simplify the notation, we introduce $\lambda(K;t)$ to represent the linear combination of the call option prices divided by h; the conditional probability is denoted as $\Pr\{S_{t+h} = Ku \,|S_t = K\} = \lambda(K;t)\,h$.

The extension of the implied tree for the stochastic volatility model

Regarding the volatilities, we assumed that volatility follows the time-homogeneous Markov chain ($\Pr[Z_{t+h} = j \,|Z_t = k] = p_{jk}$) that has N states, $Z \in \{1, 2, \ldots, N\}$; the transition probability from state k to state j is given by $P = \{p_{jk}\}$. We also defined the probability of the upward jump by $hv(Z_t) \equiv \Pr\{S_{t+h} = Ku \,|S_t = K, Z_t\}$ and chose the function $v(\cdot)$ and the transition probability p_{jk}, corresponding to the adopted stochastic volatility model. We introduced the adjustment $q(t,K)$, as in equation (4.6), to make the adopted stochastic volatility model fit exactly with the initial option prices,

$$\Pr\{S_{t+h} = Ku \,|S_t = K, Z_t = z\} = q(t,K)\,v(z)\,h. \tag{4.6}$$

When we define the combined stochastic probability of the states of the equity and the volatility $\Pr\{S_t = K, Z_t = z\}$ as $\pi(K,z;t)$, the adjustment $q(t,K)$ for $t \in \mathbf{T}, K \in \mathbf{K}$ is estimated to satisfy equation (4.7).

$$\lambda(K;t)\,\pi(K;t) = q(t,K)\sum_{z=1}^{N} v(z)\,\pi(K,z;t). \tag{4.7}$$

BN used a calibration procedure based on above approach. In the mean-reversion type of stochastic volatility model in Wiggins [55], the function $v(\cdot)$ and the transition probability p_{jk} were specified. They also provided some numerical examples that showed the magnitude of the barrier option valuation in the deterministic volatility model and the stochastic volatility model. In Section 6, we will discuss an empirical analysis by Hino and Miyazaki [31] that verified that the price difference discussed above is also observed in the NIKKEI225 option market.

5. Models Including Jump and the Method to Analyze the Option Market

5.1. Models including jump

In Merton [44], a model including jump was first introduced to option pricing. It is amazing that the model was introduced only 3 years after the BS model. As a stochastic process of the underlying asset, Merton [44] introduced a Poisson jump process in addition to the Brownian motion with the drift, as in equation (5.1):

$$\frac{dS}{S} = (\alpha - \lambda k)\, dt + \sigma dZ + dq, \tag{5.1}$$

where α, σ^2, and dZ are the instantaneous return of the equity, the variance of the instantaneous return, and the Brownian motion, respectively, $q\,(t)$ is the independent Poisson process (also independent from dZ), and λ and k are the average arrival rate and the expectation of a percentage change in the equity price when the Poisson arrival occurs, respectively.

The BS approach tries to construct the risk-free portfolio by combining the option, the underlying equity, and the risk-free asset. Merton's [44] approach is quite similar. In brief, the method first derives the SDE (of the same form as equation (5.1)) that the option price F should satisfy using the Ito's formula. Then, to obtain the PDE that the option price should satisfy, it constructs a risk-free portfolio in which the expected return rate is risk-free, using appropriate weighting of the option and the equity. Although the risk originating from the diffusion using the delta-hedging weights is removed, the delta-hedged portfolio is not risk-free because of the jump risk. Thus the expected return of the portfolio is not at the risk-free rate, but a return of the τ-remaining maturity option $g\,(S, \tau)$ at the equilibrium. When we adopt equation (5.1) as the underlying asset process, the expected equity return α and the expected option return $g\,(S, \tau)$, which we cannot know at the pricing date, are included in the PDE that should be satisfied by the option price. Merton [44] called a portfolio with jump risk but no diffusion risk a zero-beta portfolio with unsystematic risk. He insisted that under the CAPM framework, the expected return of this portfolio should be equal to that of a risk-free portfolio, i.e., $\alpha = g\,(S, \tau) = r$, and derived the PDE (5.2) that the option price F should satisfy:

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + (r - \lambda k)\, S F_S - F_\tau - rF + \lambda \varepsilon \left\{ F\,(SY, \tau) - F\,(S, \tau) \right\} = 0, \tag{5.2}$$

where Y is the size of the jump and ε is the expectation operator over the random variable Y.

By placing conditions on the number of the jump, he provided the solution formula of equation (5.2) as the sum of the BS formulas, as shown in equation (5.3), and then verified that the solution formula (5.3) actually satisfies PDE (5.2):

$$F\,(S, \tau) = \sum_{n=0}^{\infty} \frac{\exp\,(-\lambda\tau)\,(\lambda\tau)^n}{n!} \varepsilon_n \left\{ W \left[S X_n \exp\,(-\lambda k \tau)\,, \tau; E, \sigma^2, r \right] \right\}, \tag{5.3}$$

where W is usual BS call option formula what depends on the spot equity price, the remaining maturity, the strike price, the variance and the risk-free rate, in order. It should be noted that equation (5.3) is not an analytical solution but provides a reasonable approximation of the solution when the density function of $\{X_n\}$ is tractable.

Merton [44] is innovative and interesting, but it is a theoretical leap to adopt a capital asset pricing model (CAPM) argument by assuming that the jump component of the portfolio is an unsystematic risk that can be diversified out of the market portfolio. In this respect, Jarrow and Rosenfeld [37] provided a compelling argument. They argued that when the jump components of the equities are "unsystematic" and are diversified out of the market portfolio, the instantaneous CAPM is characteristic of the expected equilibrium return. Each individual equity was described by the jump–diffusion process, and the diffusion part of the individual equity is split between the diffusion part that is common to all of the individual equities, and the diffusion part that is specific to the individual equity. The instantaneous CAPM characterizes the equilibrium expected return only when the diffusion part that is specific to the individual equity and the jump component of the individual equity can be diversified out of the portfolio. Their empirical analysis tells us that the jump component of the portfolio does not disappear and the instantaneous CAPM is not justified.

To examine whether the jump component exists in the market portfolio, Jarrow and Rosenfeld [37] derived two likelihoods. One assumed that the market portfolio follows the jump–diffusion process, and the other assumed that it follows the diffusion process. Because the test statistics consist of the two likelihoods that follow a chi-square distribution, they tested whether the jump–diffusion process of the market portfolio significantly deviates from the diffusion process of the portfolio. Jorion [40] applied the same kind of statistical test to the log of four relative price processes, the diffusion process, the jump–diffusion process, the ARCH process, and the jump–ARCH process. The jump–ARCH process, a combination of the jump process and the ARCH process, already has been introduced.

In contrast to Merton [44] and Jarrow and Rosenfeld [37], who focused on whether the jump component is diversified in a portfolio, Jones [39] provided the option valuation model based on the arbitrage-free condition, after hedging the risk of the jump component. When the equity price process follows a jump–diffusion process with both diffusion risk and jump risk, the derivative that we need to evaluate is not replicated by combining the equity and the risk-free bond. So, to evaluate the derivative price F, Jones [39] built a risk-free portfolio composed of the underlying equity (S), the risk-free bond (with r as the risk-free rate), and two other externally given derivative securities (G,H), that follow the jump–diffusion processes. As in BS, Jones [39] obtained the PDE that the derivative price $F\,(S, G, H, \tau)$ should satisfy by setting the expected return of the risk-free portfolio equal to the risk-free rate.

The approaches that have been described evaluated option diversifying or hedging-out the jump risk in the underlying asset process. In contrast, Naik and Lee [46] allowed the risk premium of the jump risk to appear in the option valuation formula. They accepted that when the underlying asset follows the jump–diffusion process, the hedged portfolio cannot be constructed using only the underlying equity and the risk-free bond. Adopting the framework of the general equilibrium, and restricting the preference of the investor to constant relative risk aversion, they derived the option valuation formula that is appropriate when the underlying asset is in the market portfolio with the return, including the jump component, and insisted that the premium for the diffusion and the jump risks occupy a large portion of the option price. This type of approach is considered a prototype of a valuation framework used in incomplete markets, including the valuation of insurance derivatives that has developed rapidly since the beginning of the 21st century.

Most of the literature published in this area during the period from Merton's 1976 paper to the beginning of the 1990s involves models that include the jump component in a manner similar to what has been described, although the jump process is usually not introduced in the option valuation. One reason may be that the computation of the formula, for example, in equation (5.3), is intractable when a complicated stochastic process (including one in which the density function itself cannot be spelled out explicitly) is adopted as an underlying asset process. Hes provided the breakthrough that resolves this problem, as we described in Section 2.2. In Hes, an option valuation that assumes a complicated stochastic process as an underlying asset process is based on the characteristic function. Major stochastic processes are often used as underlying asset processes; the characteristic functions for the logarithm of the major stochastic processes are listed in Table 1. In table 1,

α is the characteristic functions for MJ and MJD models indicates the meen jump intensity and α is the characteristic function for FMLS meens the tail index that belongs to (0,2].

Table 1: Major stochastic processes and their characteristic functions ($s_T = lnS_T$)

Model (Continuous or Jump)	Characteristic function: $\mathrm{E}\left[\exp\left(ius_T\right)\right]$
BS (Continuous)	
$dS_t/S_t = (r-q)\,dt + \sigma dW_t$	$\exp\left(iu\left(r-q-\frac{\sigma^2}{2}\right)T - \frac{1}{2}u^2\sigma^2 T\right)$
MJ (Finite Jump)	
$dS_t/S_t = (r-q)\,dt$ $+\left(e^{\omega+\eta\varepsilon}-1\right)dN_t$	$\exp\left[iu\left\{r-q-\alpha\left(e^{\omega+\frac{\eta^2}{2}}-1\right)\right\}T +\alpha\left(e^{iu\omega-\frac{(u\eta)^2}{2}}-1\right)T\right]$
MJD (Compound Jump)	
$dS_t/S_{t-} = (r-q)\,dt + \sigma dW_t$ $+\left(e^{\omega+\eta\varepsilon}-1\right)dN_t$	$\exp\left[iu\left\{r-q-\alpha\left(e^{\omega+\frac{\eta^2}{2}}-1\right)-\frac{\sigma^2}{2}\right\}T -\frac{1}{2}u^2\sigma^2T+\alpha\left(e^{iu\omega-\frac{(u\eta)^2}{2}}-1\right)T\right]$
VG (Infinite Jump)	$\exp\left[iu\left(r+\omega\right)T\right]\left(1-i\theta vu+\frac{1}{2}\sigma^2u^2v\right)^{-\frac{T}{v}}$ $\omega = \frac{1}{v}\ln\left[1-\theta v-\frac{1}{2}\sigma^2 v\right]$
FMLS (Infinite Jump)	$\exp\left[iu\left(r-q+\sigma^\alpha\sec\frac{\alpha\pi}{2}\right)T - T\left(iu\sigma\right)^\alpha\sec\frac{\alpha\pi}{2}\right]$

Based on Heston's [29] approach, Bates [6] proposed the efficient valuation method for American options. He adopted a comprehensive stochastic process, including the stochastic volatility model and the jump–diffusion model, as the dynamics of a FX rate. He tested the consistency of the models by estimating the parameters of each sub-model, such as the stochastic volatility model and a comprehensive model using the DM option prices that included the jump model. Statistical testing suggested that the stochastic volatility model cannot explain the volatility smile without adopting unreasonable parameters from the time-series property of the implied volatility. This therefore supports the jump model.

Aside from Bates [6], most major proposals of option valuation methods have been based on Heston's [29] characteristic functional approach. While the BN model extended the DVF model introduced by Dupire [23] to include the stochastic volatility model, as noted in Section 3, Andersen and Andreasen ([2]; hereafter, AA) extended the DVF model to include jump. In more detail, in a situation when the underlying asset process followed the stochastic process including jump, they derived not only the backward PDE that the derivative price should satisfy, but also the forward PDE, using the insight of Breeden and Litzenberger [8] and the Tanaka's formula. They then calibrated the parameters of their model by fitting their option model prices to the corresponding option market prices. Unlike BN, which used the implied tree to compute the option price, AA adopted a numerical computation using FFT.

In Sasaki, Miyazaki, and Nomura [51], a numerical experiment on option pricing that adopted the jump–diffusion model examined the magnitude of the effect of the central limit theorem on option pricing that depends on the strike price and the maturity of the option based on the Edgeworth expansion up to the fourth cummulant.

The valuation formula of the European call option based on the characteristic function

The price $C_0(T,\kappa)$ of the European call option that has a maturity T and the strike price $K = \exp(\kappa)$ is given by

$$C_0(T,\kappa) \equiv c_0(T,\kappa)\exp(-\alpha\kappa) = \frac{\exp(-\alpha\kappa)}{2\pi}\int_{-\infty}^{\infty} e^{-iv\kappa}\psi_T(v)\,dv, \tag{5.4}$$

$$\psi_T(v) = \frac{e^{-rT}\phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}. \tag{5.5}$$

The adjusted call option price is defined as $c_0(T,\kappa) \equiv \exp(\alpha\kappa)C_0(T,\kappa)$ using the parameter $\alpha\,(>0)$, which expresses the deterioration speed in equation (5.4). The call option price is square-integrable. $\phi_T(\cdot)$ is the characteristic function of the log of the equity price $s_T = \ln(S_T)$, i.e., $\phi_T(u) = \mathrm{E}[\exp(ius_T)]$. We used FFT to compute the Fourier transform that appeared in equation (5.4). Refer to Carr and Madan [13] for the detailed method of applying FFT to option pricing.

5.2. Methods of option market analysis

Since the beginning of the 21st century, research on actual option market prices has gained in popularity. For example, Carr and Wu ([14], [15]) identified the stochastic process of the underlying asset that is implied by the option market price, using the characteristic function approach. The main topic in Carr and Wu [15] is the situation in which the central limit theorem fails to be established in the option market. In most cases, the sum of the daily return tends to be close to the normal distribution because of the central limit theorem, even when the daily return does not follow the normal distribution. With respect to European options, the distribution of the accumulated return until the maturity of the option may deviate substantially from the normal distribution for the short maturity option, but can approach to the normal distribution in the long maturity option. However, Carr and Wu [15] observed the S&P 500 option market prices carefully and found this is not the case. The risk-neutral distribution at the maturity of the option does not approach the normal distribution even when the maturity becomes longer and the shape of the distribution of the return to the maturity deviates far from normal. This behavior is characteristic of infinite divisible distributions.

Carr and Wu [15] proposed the use of the finite moment log-stable process as the stochastic process of the underlying asset in accordance with the option market prices. The distribution generated by the process belongs to the family of infinite divisible distributions, with the distribution of the sum of the daily return equal to the distribution of the daily distribution. Thus, the central limit theorem is not established for the distribution, and it is an appropriate distribution of the return, in accordance with S&P 500 option market prices. In addition to the finite moment log-stable distribution, the family of the infinite divisible distributions includes the hyperbolic distribution (Eberlein and Keller, [24]) and the normal inverse Gaussian (NIG) distribution (Barndorff-Nielsen, [5]), among others.

A valuable method to identify the stochastic process of the underlying asset implicit in the option market prices was introduced by Carr and Wu [14]. With deep theoretical insight on the relationship between the option price and the time to maturity, they found that the stochastic process of the underlying asset implicit in the option market prices is identified based on the relationship between the ATM and OTM options. Using a graph called "Term Decay Plots" that expresses $\ln T$ on the horizontal axis and $\ln(P/T)$ (with P the option market price) on the vertical axis, they visualized the convergence speed of the option market price and then identified the stochastic process of the underlying asset based on Table 2. They also empirically examined the S&P 500 option market using their method and found that the stochastic process of the S&P 500 index that is implicit in the option market prices includes not only the diffusion component but also the jump component. Nomura and Miyazaki [47] empirically examined the NIKKEI225 option market based on the method. The result is summarized in Section 6.

Table 2: The shape of the term decay plots specifies the type of stochastic process

Type of Stochastic Process	OTM	ATM
Continuous (PC)	Concave Curve	Straight Line (Slope = -0.5)
Finite Jump (PJ)	Straight Line (Slope = 0)	Straight Line (Slope = 0)
Infinite Jump (PJ)	Straight Line (Slope = 0)	Straight Line (Slope <0)
Continuous and Jump (CJ)	Straight Line (Slope = 0)	Straight Line (Slope <-0.5)

Bakshi and Kapadia [4] adopted the stochastic volatility model to examine the volatility risk premium implicit in option market prices. Their influential research was quite innovative. They quantified the volatility risk premium by the delta-hedging gain from the hedged portfolio, excluding all risks except the volatility risk. When the volatility is constant and the implied volatility is equal to the realized volatility, the delta-hedging gain from the hedged portfolio is definitely 0. However, in reality, when we activate the delta-hedging strategy (short the option and long the delta amount of the underlying asset), without selecting the options with a forecast model of the volatility, the delta-hedging gain from the hedged portfolio tends to be negative. To provide a rigorous empirical analysis of this based in theory, [4] focused on the delta-hedging gain from the delta-hedged portfolio, assuming the Hes model introduced in Section 2.2 (and also a model that includes the jump component) as the stochastic process of the underlying asset. In reality, the expected delta-hedging gain from the delta-hedged portfolio is derived as

$$E_t\left(\Pi_{t,t+\tau}\right) = \int_t^{t+\tau} E_t\left(\lambda_u\left[\sigma_u\right]\frac{\partial C_u}{\partial \sigma_u}\right)du, \tag{5.6}$$

where $\prod_{t,t+\tau}$, $\lambda_u[\cdot]$ and $\frac{\partial C_t}{\partial \sigma_t}$ are the gain or loss on a delta-hedged option position, the volatility risk premium and the vega of the call option, respectively and E_t is the statistical measure's expectation operator. Thus, the volatility risk premium is related to amounts that are observable and computable, such as the expected delta-hedging gain from the hedged portfolio and the vega $\frac{\partial C_t}{\partial \sigma_t}$ of the option, allowing the volatility risk premium to be determined. Their empirical analysis of S&P 500 option market prices revealed three points. First, the more OTM the option becomes, the less poorly the delta-hedging gain performs. Second, the delta-hedging gain's underperformance becomes large in the high volatility period. Third, even when the impact of the jump risk is removed, the volatility risk premium largely influences the delta-hedging gain from the hedged portfolio.

6. Empirical Analysis of the NIKKEI225 Option Market

As I described in Introduction, most of the advanced empirical researches on the implied volatility examine the case of US equity market and such kind of researches on other equity markets are quite scarce. Thus, in this section, focusing on the Japanese equity market, which has the biggest outstanding amount among the Asian equity markets and also some linkage to the US equity market, I apply the models and the methods introduced up to the previous section to the market for the purpose to enrich the empirical analysis in this area. In more detail, main results and implications of 6 kinds of empirical researches on NIKKEI225 option market conducted by the author's group are introduced. Then, the general view attained from these researches and the comment on the similarity and the difference between the US and the Japanese equity option markets are added.

6.1. Diffusion component and jump component in the equity process implied by option prices in U.S. and Japanese markets

Nomura and Miyazaki [47] applied methods of option market analysis, such as the Term Decay Plots and the sum of least-squares error between the option model price and the option market price (Table 3) introduced in Carr and Wu [14] to the NIKKEI 225 option market. They identified differences between the U.S. market and the Japanese market in the components of the equity process that are implied by option price, as follows:
U.S. (S&P 500 Option)

- The diffusion component always exists, but the jump component does not always exist.
- The option model prices in the finite moment log stable (FMLS) process model provide the best fit to the option market prices.
- In the jump–diffusion model, the jump component is more prominent than the diffusion component.

Japan (NIKKEI225 Option)

- The jump component seldom appears and the diffusion component is dominant.
- The option model prices of the Merton Jump–Diffusion (MJD) (Merton, [44]) model provide the best fit to the option market prices.
- In the jump–diffusion model, the diffusion component is more prominent than the jump component.

The results suggest two implications: First, the jump–diffusion model should be adopted in the valuation of NIKKEI225 options, rather than a simple diffusion model. Second, although the jump component is present in the NIKKEI225 option market, it is relatively small compared to the U.S. market.

Table 3: The average of the estimated parameters

	BS	MJ	MJD	VG	FMLS
Parameters of Model	σ **0.2046**	α **7.8801**	α **0.1704**	α **0.2414**	α **1.7780**
			(1.8145)	(0.7681)	(1.5597)
		ϖ **-0.0307**	ϖ **-0.6841**	ϖ **-0.2410**	σ **0.1476**
			(-0.1045)	(-0.2013)	(0.1486)
			η **0.0816**	η **0.2961**	σ **0.2119**
			(0.1671)	(0.2295)	
			σ **0.1851**		
			(0.0638)		
Mse	$\mathbf{2.1856\times10^4}$	$\mathbf{2.9946\times10^4}$	$\mathbf{1.1872\times10^4}$	$\mathbf{2.1155\times10^4}$	$\mathbf{1.3475\times10^4}$
			1.4473×10^5	0.8042×10^5	0.3630×10^5
Vse	$\mathbf{1.2698\times10^8}$	$\mathbf{1.9073\times10^8}$	$\mathbf{1.0513\times10^8}$	$\mathbf{1.4452\times10^8}$	$\mathbf{1.0444\times10^8}$

6.2. Normal distribution and NIG distribution in the valuation of the NIKKEI225 option

Miyazaki and Nakao [45] studied the valuation of Japanese equity options assuming that the return of the equity process follows a NIG distribution. They empirically verified the advantage of the NIG distribution advocated by Barndorff-Nielsen [5] over other models, using a comparison analysis on the return of the underlying asset and the risk-neutral valuation of the derivative products. In detail, they performed an empirical analysis on the description of the return of the underlying asset in 29 Japanese equity industry sector indexes, comparing several percentile tail probabilities between the normal distribution and the NIG distribution. They reported that the latter was superior. However, [45] provide no analysis of the option market data.

To supplement the analysis in [45], Nomura and Miyazaki [48] focused on the analysis of the option market data. They compared a normal distribution and a NIG distribution from the standpoint of the valuation of a further OTM (FOTM) option consistent with the option market price information. In more detail, they examined whether a normal distribution or a NIG distribution captures the time-series FOTM option market price, based on price information of options close to ATM that have large trading volume. The analysis first recovered the implied probability distribution using Rubinstein's [50] approach, introduced in Section 3, based on the price information of the options close to ATM. Second, they derived the FOTM option model prices using the implied distribution. Third, they compared the fit of the option market prices in a normal distribution versus a NIG distribution. Their results support the superiority of a NIG distribution over a normal distribution.

The deviations of FOTM (Yen, 2000) option model prices, based on implied normal distribution and implied NIG distribution from the FOTM option market price, are shown in time series in Figure 4. Figure 4 lists only the time period in which the FOTM option is traded. In Figure 4, we see that the option model price based on the implied NIG distribution is closer to the actual option market price than the option model price based on the implied normal distribution. This tendency is more prominent when the time to maturity is short. The reason is that the probability around the FOTM area in the implied normal distribution is too large to satisfy the price information constraints close to ATM because of the low kurtosis of the normal distribution and because the implied normal distribution overestimates the value of the FOTM option. Because the kurtosis of the implied NIG distribution becomes higher when the time to maturity becomes shorter, the price information close to ATM effectively recovers the implied NIG distribution up to the FOTM area, and the implied NIG distribution correctly evaluates the FOTM option.

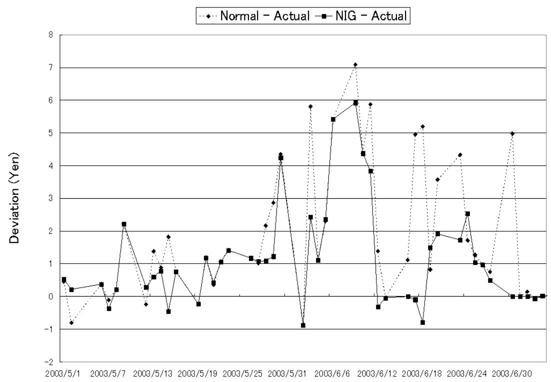

Figure 4: Deviation from the actual price (2000Yen FOTM)

6.3. Model risk in the valuation of barrier options

Model risk means that an option price will be different in each model depending upon the underlying stochastic processes that are assumed in that model. It would be hasty to conclude that the option price always differs depending upon the assumed underlying stochastic process. Two different underlying stochastic processes sometimes produce the same probability distribution at a given time, and the same European option price could arise from two different stochastic process models of the underlying asset. Where does the model risk come from? Listed options are usually European, and we calibrate our model parameters so that our option model prices fit the listed option market prices, as in Section 6.1. Thus, even though we observe the same listed option market prices (the same European option model prices), exotic option model prices will differ from model to model because most of the exotic options are path-dependent.

Hino and Miyazaki [31] adopted a deterministic volatility model and a stochastic volatility model (Wiggins, [55]) to examine the model risk in the valuation of barrier options based on BN, which was introduced in Section 4. Figure 5 shows that when the mean-reversion parameter of the volatility process gets smaller, and the volatility of the volatility process gets larger, the price difference resulting from the difference between the deterministic model and the stochastic volatility model will grow larger. In this situation, we observe a relatively large model risk.

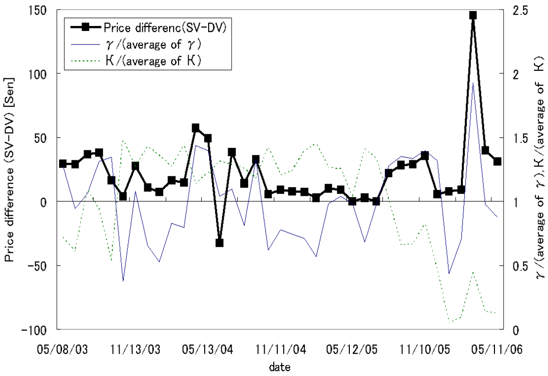

Figure 5: Price difference in barrier option

6.4. Phase of the Japanese equity market and implied volatility dynamics

Recently, an option describing the evaluation of the implied volatility surface in exotic equity options has been proposed. Thus, it becomes very important to discover the relationship between the phase of the equity market and the implied volatility dynamics. Derman [18] offered an interesting approach to implied volatility dynamics. [18] focused predominantly on the S&P 500 option market, and proposed three models to describe the dynamics of the implied volatility. In the first model, the implied volatility that corresponds to each strike price is almost constant until the option reaches maturity. This is called the sticky strike model. In the second, the implied volatility, corresponding to the difference between the strike price and the underlying asset price, is almost constant until the maturity of the option. This is the sticky delta model. In the third model, the implied volatility in the future is almost identical to the initial volatility indicated by the volatility tree. This is the sticky implied-tree model. Furthermore, he described the model most likely to be selected at each phase of the equity market. It is unfortunate that he proposes the three models in a qualitative way, and does not provide a quantitative method for selecting the appropriate model for each phase of the equity market.

Kato and Miyazaki [42] quantified the fit of each model to the actual implied volatility dynamics, and proposed a quantitative model that relates the phase of the equity market and the implied volatility dynamics. In the quantitative model, they used the average and the standard deviation of the daily equity return in each month as a proxy of the phase of the equity market, and related the three models to the proxy of the phases of the equity market using a trinomial logit model. The results of their empirical analysis of the NIKKEI225 index and NIKKEI225 option market data, spanning from 2003/5/1 to 2006/6/30, is shown in Figures 6A and 6B. These results demonstrate that the sticky delta model is selected when the equity market is in a trending phase (with large drift), while the sticky strike model is selected when the equity market is in the stable phase (with small drift).

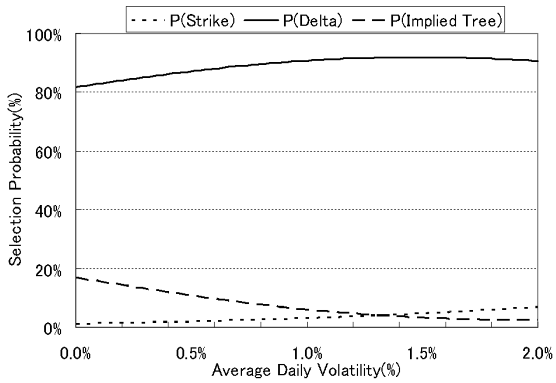

Figure 6A: Selection probability (Case of large drift)

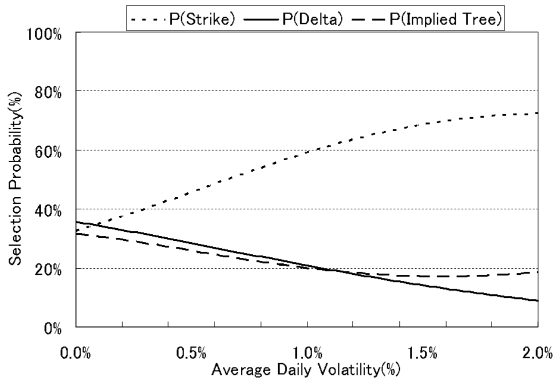

Figure 6B: Selection probability (Case of 0 drift)

6.5. Delta-hedging strategies of NIKKEI225 options using baskets of individual equities

In the option trading strategy when we sell an option and adopt a delta-hedging strategy, we profit when the implied volatility of the sold option is higher than the actual volatility realized at the maturity of the option, and we lose money when the opposite occurs. When we sell the option on the NIKKEI225 index, we dynamically hedge the short option position using the NIKKEI225 index in the usual delta-hedging strategy.

Relaxing the conditions of the delta-hedging strategy, Hoshika and Miyazaki [32] proposed a dynamic hedging strategy for the option position, using the baskets of individual equities instead of the NIKKEI225 index. The individual equities in the basket were selected to optimize the first moment of the basket, while the second moment was constrained to that of the NIKKEI225 index. In the proposed method, a limited increase in risk exists. Profits arise from the difference between the drift of NIKKEI225 index and that of the attained basket; in addition, profit is derived from the difference between the implied volatility of the sold option and the actual volatility realized until the maturity of the option.

In an empirical analysis, we picked up the NIKKEI225 ATM call option that had a remaining maturity of 1 month at the beginning of each month and daily activated both the usual and the proposed delta-hedging strategies in the corresponding month covering the period from 2002/5/1 to 2005/12/30. The accumulated profits attained by the two delta-hedging strategies are provided in Figure 7. Figure 7 indicates that the profit from the difference between the drift of the NIKKEI225 index and the basket is much larger than the usual profit from the difference between the implied volatility of the sold option and the actual volatility realized when the option matures.

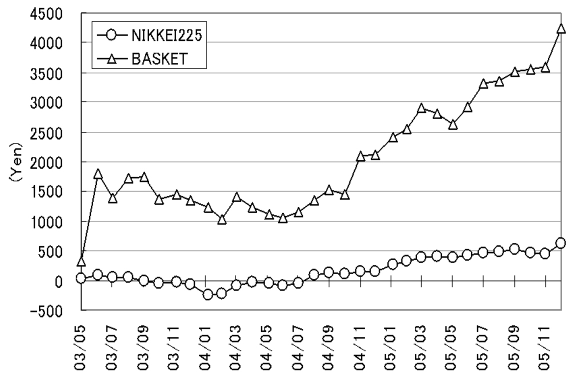

Figure 7: Performance of the delta-hedging strategy

6.6. Volatility risk premium in the NIKKEI225 option market

Uchida and Miyazaki [54] empirically examined the volatility risk premium in the NIKKEI225 option based on the approach introduced by Bakshi and Kapadia [4]. Using the daily NIKKEI225 option market price data during the period from 2003/5/01 to 2005/12/30, they derived the profit from the daily delta-hedging strategy, and discussed the volatility risk premium. In Table 4, the profits (π(Yen), π/S) from the daily delta-hedging strategy are categorized by the moneyness ($y - 1 = \ln(S_\tau/K) - 1$) and the remaining maturity (14-30 days, 31-60 days, all periods included)of the call options. Table 4 tells us that the underperformance of the delta-hedging strategy becomes salient in the ATM area rather than in the OTM area, and also in the longer maturity option as opposed to the short maturity option.

In terms of the negative delta-hedging strategy, our results are the same as those in Bakshi and Kapadia's [4] analysis of the U.S. market. The underperformance in the Japanese case is much larger than seen in the U.S. case. One reason for the phenomenon may be that the Japanese investor buys options for hedging purposes, even if they are expensive and the Japanese option market has not matured. In the U.S. market, the relationship between the delta-hedge gain and the level of the volatility is linear regardless of the moneyness of the option. However, in the Japanese case, the delta-hedge gain depends on the moneyness of the option; the relationship between the delta-hedge gain and the level of the volatility is not linear. The equity price and the volatility are negatively correlated.

Table 4: Delta-hedge gain

moneyness y-1			π(Yen)			π / S_t			
			14-30	31-60	ALL	14-30	31-60	ALL	π<=0
-10.0%	~	-7.5%	-5.18	-14.05	-10.79	-0.06%	-0.13%	-0.10%	54%
-7.50%	~	-5.0%	-13.94	-43.27	-32.03	-0.13%	-0.39%	-0.29%	70%
-5.0%	~	-2.5%	-17.24	-49.53	-37.39	-0.16%	-0.45%	-0.34%	71%
-2.5%	~	0.0%	-40.24	-54.86	-49.31	-0.36%	-0.50%	-0.45%	79%
0.0%	~	2.5%	-28.98	-30.31	-29.77	-0.26%	-0.27%	-0.26%	74%
2.5%	~	5.0%	-17.17	-23.74	-20.99	-0.16%	-0.21%	-0.19%	70%
5.0%	~	7.5%	-13.13	-12.58	-12.80	-0.12%	-0.12%	-0.12%	67%
7.5%	~	10.0%	-0.95	2.01	0.75	-0.02%	0.01%	-0.01%	56%

6.7. The general view attained from our researches and the comment on the similarity and the difference between the US and the Japanese equity option markets

In general, one of the consensuses of the opinions from the practitioners who have some interest on the model-based market analysis is that the elaborate model and the method are not useful when the market is immature because the richness and the cheapness clarified by the model are not corrected. In this context, is the NIKKEI225 option market matured or immature market? Judging from our 6 empirical analyses, my opinion is that the NIKKEI225 option market is matured enough to be analyzed by the elaborate models and methods introduced in this article.

Even though the NIKKEI225 option market is the matured market, but, of course, it does not mean that all of the features of the market are like those of the US equity option market. As we see in section6.1, the stochastic process of the underlying asset implicit in the NIKKEI225 option market has not only the diffusion component but also the jump component as in the case of the US. However, the magnitude of the diffusion component implicit in the NIKKEI225 option market is larger than that in the US option market. Further more, regarding the analysis on the volatility risk premium provided in section 6.6, even though in both Japanese and US option markets, the volatility risk premiums are observed, the risk premium implicit in the Japanese equity option market is larger than that in the US equity option market.

To capture the nature of the option markets even outside US, it is indispensable to understand the advanced approaches introduced in this article and to develop or customize them to each own domestic equity option market.

7. Summary and Concluding Remarks

This article focuses on the evaluation of exotic options in line with observable listed-option market prices. The development of valuation models is reviewed, and option market analyses that are based on the models are surveyed. I aim to explain the fundamental ideas, methods, and research directions in the literature. Two significant approaches (Breeden and Litzenberger, [8]; Heston, [29]) that serve as the foundations for market-based option pricing are described. The literature in the field is grouped into three categories: studies that focus on the deterministic volatility and the implied tree, stochastic volatility models, and models including a jump component. The application of these models to option market analysis and an overview of the developing process in each category are presented. Results from studies on the NIKKEI225 option market using these methods are also provided.

Aside from the U.S. option market, these types of empirical analyses are currently quite scarce. Moving forward, they should be more extensively used to help the growth of option markets. In the process, existing methods should be revamped, or new valuation models should be invented. It would be a pleasure if this article could serve as a guide to young newcomers in the field, and as a summary tool for senior researchers.

Acknowledgements:
I heartily thank two anonymous reviewers for their useful suggestions which improve the quality of this article significantly.

References

[1] C. Albanese and A. Kuznetsov: Unifying volatility models. *Risk*, March 2004, 94–98.

[2] L. Andersen and J. Andreasen: Jumping smiles. *Risk*, November 1999, 65–68.

[3] K.J. Arrow: The role of securities in the optimal allocation of risk-bearing. *Review of Economic Studies*, **31** (1964), 91–96.

[4] G. Bakshi and N. Kapadia: Delta-hedged gains and the negative market volatility risk premium. *Review of Financial and Studies*, **16** (2003), 527–566.

[5] O.E. Barndorff-Nielsen: Processes of normal inverse Gaussian type. *Finance & Stochastics*, **2** (1998), 41–68.

[6] D.S. Bates: Jumps and stochastic volatility: exchange rate processes implicit in Deutsche mark options. *Review of Financial and Studies*, **9** (1996), 69–107.

[7] F. Black and M. Scholes: The pricing of options and corporate liabilities, *Journal of Political Economy*, **81** (1973), 637–654.

[8] D.T. Breeden and R.H. Litzenberger: Prices of state-contingent claims implicit in option prices. *Journal of Business*, **51** (1978), 621–651.

[9] D. Brigo and F. Mercurio: A mixed-up smile. *Risk*, September 2000, 123–126.

[10] M. Britten-Jones and A. Neuberger: Option prices, implied price processes, and stochastic volatility. *The Journal of Finance*, **55** (2000), 839–866.

[11] G. Brown and C. Randall: If the skew fits. *Risk*, April 1999, 62–65.

[12] P.W. Buchen and M. Kelly: The maximum entropy distribution of an asset inferred from option prices. *Journal of Financial and Quantitative Analysis*, **31** (1996), 143–159.

[13] P. Carr and D.B. Madan: Option valuation using the fast Fourier transform. *Journal of Computational Finance*, **2** (1999), 61–73.

[14] P. Carr and L. Wu: What type of process underlies options? A simple robust test. *Journal of Finance*, **58** (2003a), 2581–2610.

[15] P. Carr and L. Wu: Finite moment log stable process and option pricing. *Journal of Finance*, **58** (2003b), 753–777.

[16] P. Carr and L. Wu: Time-changed Levy process and option pricing. *Journal of Financial Economics*, **71** (2004), 113–141.

[17] G. Debreu: *Theory of Value* (Wiley, New York, 1959).

[18] E. Derman: Regimes of volatility. *Risk*, April 1999, 55–59.

[19] E. Derman and I. Kani: Riding on a smile. *Risk*, February 1994, 32–39.

[20] E. Derman and J. Zou: A fair value for the skew. *Risk*, January 2001, 111–113.

[21] J-C. Duan: Cracking the smile. *Risk*, December 1996, 55–59.

[22] B. Dumas, J. Fleming, and R. Whaley: Implied volatility functions: empirical tests. *The Journal of Finance*, **53** (1998), 2059–2106.

[23] B. Dupire: Pricing with a smile. *Risk*, January 1994, 18–20.

[24] E. Eberlein and U. Keller: Hyperbolic distributions in finance. *Bernoulli*, **1** (1995), 281–299.

[25] D. Edelman: Local cross-entropy. *Risk*, July 2004, 73–76.

[26] R. Engle and V. Ng: Measuring and testing the impact of news on volatility. *Journal of Finance*, **43** (1993), 1749–1778.

[27] J-P. Fouque, G. Papanicolaou, and R. Sircar: Calibrating random volatility. *Risk*, February 2000, 89–92.

[28] L. Hentschel: Errors in implied volatility estimation. *Journal of Financial and Quantitative Analysis*, **38** (2003), 779–810.

[29] S.L. Heston: A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial and Studies*, **6** (1993), 327–343.

[30] S.L. Heston and S. Nandi: A closed-form GARCH option valuation model. *Review of Financial and Studies*, **13** (2000), 585–625.

[31] M. Hino and K. Miyazaki: Model risk in the valuation of barrier option. *RIMS Lecture Note*, **1548** (2007), 226–233. (in Japanese)

[32] H. Hoshika and K. Miyazaki: A thought on a delta hedging of Nikkei 225 Option. *RIMS Lecture Note*, **1548** (2007), 218–225. (in Japanese)

[33] J-Z. Huang and L. Wu: Specification analysis of option pricing models based on time-changed Levy processes. *The Journal of Finance*, **59** (2004), 1405–1439.

[34] J. Hull and W. Suo: A methodology for assessing model risk and its application to the implied volatility function model. *Journal of Financial and Quantitative Analysis*, **37** (2002), 297–318.

[35] J. Hull and A. White: The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, **42** (1987), 281–300.

[36] J.C. Jackwerth and M. Rubinstein: Recovering probability distributions from option prices. *The Journal of Finance*, **51** (1996), 1611–1631.

[37] R.A. Jarrow and E. Rosenfeld: Jump risks and the intertemporal capital asset pricing model. *Journal of Business*, **57** (1984), 337–352.

[38] G.J. Jiang and Y.S. Tian: The model-free implied volatility and its information content. *Review of Financial and Studies*, **18** (2005), 1305–1342.

[39] E.P. Jones: Option arbitrage and strategy with large price changes. *Journal of Financial Economics*, **13** (1984), 91–113.

[40] P. Jorion: On jump processes in the foreign exchange and stock markets. *Review of Financial Studies*, **1** (1988), 427–445.

[41] N. Kahalé: An arbitrage-free interpolation of volatilities. *Risk*, May 2004, 102–106.

[42] A. Kato and K. Miyazaki: Behavior of implied volatility in several phases of Japanese equity markets. *RIMS Lecture Note*, **1548** (2007), 210–217. (in Japanese)

[43] Y. Li: A new algorithm for constructing implied binomial trees. *Journal of Computational Finance*, **4** (2000), 69–95.

[44] R.C. Merton: Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, **3** (1976), 125–144.

[45] K. Miyazaki and T. Nakao: "Normal distribution or NIG distribution", "daily basis or weekly basis" risk management and option pricing in Japanese equity market. *JAFEE Journal* (2003), 3–31. (in Japanese)

[46] V. Naik and M. Lee: General equilibrium pricing of options on the market portfolio with discontinuous returns. *Review of Financial Studies*, **3** (1990), 493–521.

[47] S. Nomura and K. Miyazaki: Diffusion and jump components of equity process implicit in NIKKEI 225 option market. *Gendai Finance*, **18** (2005), 47–67. (in Japanese)

[48] S. Nomura and K. Miyazaki: A valuation of far out-of-the-money options based on implied Normal/NIG distributions. *JAFEE Journal* (2006), 3–31. (in Japanese)

[49] V. Piterbarg: Time to smile. *Risk*, May 2005, 71–75.

[50] M. Rubinstein: Implied binomial trees. *The Journal of Finance*, **49** (1994), 771–818.

[51] T. Sasaki, K. Miyazaki, and S. Nomura: An option valuation model based on Edgeworth expansion ∼a case of Merton's jump-diffusion model∼. *Japanese Journal of Applied Statistics*, **35** (2006), 113–128. (in Japanese)

[52] L. Scott: Option pricing when the variance changes randomly: theory, estimation and an application. *Journal of Financial and Quantitative Analysis*, **22** (1996), 419–438.

[53] E. Stein and J. Stein: Stock price distributions with stochastic volatility: an analytic approach. *Review of Financial and Studies*, **4** (1991), 727–752.

[54] Y. Uchida and K. Miyazaki: Volatility risk premium in Nikkei225 option market. *RIMS Lecture Note*, **1548** (2007), 202–209. (in Japanese)

[55] J.B. Wiggins: Option values under stochastic volatility; theory and empirical estimates. *Journal of Financial Economics*, **19** (1987), 351–372.

Koichi Miyazaki
The University of Electro-Communications
Department of Systems Engineering
1-5-1 Chofugaoka, Chofu-shi
Tokyo 182-8585, Japan
E-mail: miyazaki@se.uec.ac.jp