Journal of the Operations Research
Society of Japan
2007, Vol. 50, No. 4, 463-487

# MULTI-PERIOD OPTIMIZATION MODEL FOR A HOUSEHOLD, AND OPTIMAL INSURANCE DESIGN

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(Received October 17, 2006; Revised May 7, 2007)

*Abstract* We discuss an optimization model to obtain an optimal investment and insurance strategy for a household. In this paper, we extend the studies in Hibiki and Komoribayashi (2006). We introduce the following points, and examine the model with numerical examples.

① We consider cash flow due to a serious disease and involve medical insurance.

② An optimization model is formulated with term life insurance which variable insurance money is received.

③ We propose a model to decide optimal life and medical insurance money received at each time.

④ Sampling error is examined with 100 kinds of 5,000 sample paths.

**Keywords**: Finance, multi-period optimization, financial planning, investment and insurance strategy, insurance design

## 1. Introduction

We discuss an optimization model to obtain an optimal investment and insurance strategy for a household. Recently, financial institutions have promoted giving a financial advice for individual investors. A household is exposed to risk associated with the decrease in real financial wealth due to inflation, loss of wage income due to the householder's death, loss of a house or non-financial wealth due to the fire, and the increase in medical cost due to a serious disease. Financial institutions need to recommend appropriate financial products in order to hedge risk against these accidents. We clarify how a set of asset mix, and life, fire and medical insurance affect asset and liability management for a household. We develop a multi-period optimization model which involves determining a set of financial products, hedging risk associated with a life cycle of a household and saving for the old age. The simulated path approach [3, 4] can be used to solve this problem.

There are some studies in the literature for individual optimal investment strategy; Bodie, Merton and Samuelson [1], Merton [7, 8], Samuelson [10]. Chen, Ibbotson, Milevsky and Zhu [2] advocate an optimization model with the inclusion of wage income, consumption expenditure, and both optimal asset allocation and life insurance. Yoshida, Yamada and Hibiki [12] solve an optimal asset allocation problem for a household using a multi-period optimization approach. Hibiki, Komoribayashi and Toyoda [6] describe a multi-period optimization model to determine an optimal set of asset mix, life insurance and fire insurance in conjunction with their life cycle and characteristics. The model is examined with numerical examples. In addition, some financial advices for three households are illustrated for practical use, and results which coincide with a practical feeling are obtained.

Hibiki and Komoribayashi [5] extend the studies in Hibiki, Komoribayashi and Toyoda [6] for practical use. Risk associated with the householder's death is hedged by life insurance. The model is proposed involving the associated three factors: receipt of a survivor's pension, exemption from mortgage loan payments, and change in the consumption level. Additional effects by three factors are examined with numerical examples. Moreover, the sensitivity of parameters associated with home buying is analyzed in order to examine the home buying strategy.

We obtain the following practical and interesting results in Hibiki, Komoribayashi and Toyoda [6], and Hibiki and Komoribayashi [5].

(1) The older a householder is, the less optimal life insurance money is.

(2) Optimal fire insurance money is nearly equal to the maximum loss of non-financial wealth.

(3) Expected terminal financial wealth does not affect optimal life and fire insurance money.

(4) If a household receives a survivor's pension and keeps the consumption level lower after a householder died, optimal life insurance money and investment units of a risky asset are reduced.

(5) If loan payments are forgiven due to the householder's death, optimal investment units of a risky asset are reduced, but optimal life and fire insurance money are not influenced.

(6) Home buying strategy affects an optimal asset mix and life insurance money.

In this paper, we extend the studies in Hibiki and Komoribayashi [5]. We introduce the following points in a multi-period optimization model, and examine the model with numerical examples.

① We consider cash flow due to a serious disease and involve medical insurance to cover the expensive medical cost.

② An optimization model is formulated with term life insurance which variable insurance money is received, and it is compared with the constant receipt of life insurance money by using numerical examples.

③ We propose a model to decide optimal life and medical insurance money received at each time.

④ Sampling error is examined with 100 kinds of 5,000 sample paths.

This paper is organized as follows. We describe a household, income, consumption expenditure, and four kinds of financial products, or securities, life insurance, fire insurance, and medical insurance to develop a model in Section 2. Section 3 shows the formulation of a multi-period ALM optimization model for a household. We analyze the sensitivity of parameters associated with a serious disease, and examine the effect of medical insurance. We solve the problem with decreasing life insurance money over time, and compare it with constant life insurance money over time by using numerical examples. In Section 4, we propose a model to decide optimal life and medical insurance money at each time, and numerical examples are shown. Sampling error is examined with 100 kinds of 5,000 sample paths in Section 5. Section 6 provides our concluding remarks.

## 2. Model Structure

We define a household, and describe income and consumption expenditure. We clarify the characteristics of financial instruments such as securities, life insurance, fire insurance, and medical insurance. We attach a superscript $(i)$ to a random and path dependent parameter in order to formulate a model in the simulated path approach.

### 2.1. Household

We define a household as a group composed of a householder and members of family as in the previous papers [5, 6]. Wealth at time $t$ held by a household can be divided into two kinds of wealth: financial wealth $W_{1,t}^{(i)}$ and non-financial wealth $W_{2,t}^{(i)}$. A household is exposed to risk associated with three kinds of accidents: a death and a serious disease of a householder, and a fire of a house. It is assumed that a death of a householder makes wage earnings stop, a serious disease of a householder decreases wage income and makes large payment, and a fire of a house damages a fraction $\alpha$ of non-financial wealth. A householder can purchase a life insurance policy, a fire insurance policy and a medical insurance policy to hedge risk in addition to the investment in securities such as stocks and bonds.

Cash flow streams are influenced by risk exposure associated with income and expenditure. We set the following parameters associated with the accidents to describe cash flow streams.

- $\tau_{1,t}^{(i)}$ : one if a householder dies on path $i$ at time $t$ and zero otherwise.
- $\tau_{2,t}^{(i)}$ : one if a fire of a house occurs on path $i$ at time $t$ and zero otherwise.
- $\tau_{3,t}^{(i)}$ : one if a householder is alive on path $i$ at time $t$ and zero otherwise.
- $\tau_{4,t}^{(i)}$ : one if a householder has a serious disease on path $i$ at time $t$ and zero otherwise.[1]
- $\lambda_{1,t}$ : mortality rate at time $t$, or the probability that a person who is alive at time 0 will die at time $t$, $\lambda_{1,t} = \Pr(\tau_{1,t} = 1) = \frac{1}{I}\sum_{i=1}^{I} \tau_{1,t}^{(i)}$ where $I$ is the number of simulated paths.
- $\lambda_2$ : rate of a fire (which is assumed to be time independent), or the probability that a fire occurs, $\lambda_2 = \Pr(\tau_{2,t} = 1) = \frac{1}{I}\sum_{i=1}^{I} \tau_{2,t}^{(i)}$.
- $\lambda_{4,t}$ : disease rate at time $t$, or the probability that a person who is alive at time 0 will have a serious disease at time $t$, $\lambda_{4,t} = \Pr(\tau_{4,t} = 1) = \frac{1}{I}\sum_{i=1}^{I} \tau_{4,t}^{(i)}$.

### 2.2. Income

Income at time $t$ is a householder's wage $m_t$ if a householder is alive and investment return from financial wealth $W_{1,t}^{(i)}$. If a householder dies, a household cannot get wages, but receive severance pay, and draw a survivor's pension. Amounts of severance pay and a survivor's pension are calculated based on the wage level. Let $a_{t_m}^{(i)}$ be the amount of a survivor's pension. The amount of a survivors' pension is dependent on the time of the householder's death $t_m$. An amount of severance pay $e_t^{(i)}$ is also dependent on years of continuous employment(age). When a householder has a serious disease, it is assumed that a fraction $\nu_3$ $(\nu_3 < 1)$ of wage income decreases because a householder has to take a rest from work. Amounts of wage income, severance pay and a survivor's pension, or cash inflow except investment return, borrowing, and insurance money, $M_t^{(i)}$ can be shown as follows :

$$M_t^{(i)} = \tau_{3,t}^{(i)} m_t^{(i)} + \left(1 - \tau_{3,t}^{(i)}\right) a_{t_m}^{(i)} + \tau_{1,t}^{(i)} e_t^{(i)} + \mathbf{1}_{\{t=T\}} \tau_{3,T}^{(i)} e_T^{(i)} - \nu_3 \tau_{4,t}^{(i)} m_t^{(i)} \ (t = 1, \ldots, T) \quad (1)$$

where $\mathbf{1}_{\{A\}}$ is an indicator function which shows one if the condition $A$ is satisfied, and zero otherwise.

[1]If $\tau_{3,t}^{(i)} = 0$, then $\tau_{4,t}^{(i)} = 0$.

### 2.3. Consumption expenses

There assumes to be two kinds of expenses: living expenses $C_{1,t}^{(i)}$ and payments associated with non-financial wealth $C_{2,t}^{(i)}$, such as a house, goods, and repair costs. Besides these costs, we need to pay the restoration cost if a fire of a house occurs.

**(1) Expenses for purchasing a house**

We assume that a household purchases a house by making a down payment and a debt loan at a bank ($H_t$). Let $t_e$ be the time when a house is purchased. The debt loan $H_{t_e}$ is the difference between the price of the house and the down payment. The expenditure for the house $C_{2,t_e}$ is the price of the house, and therefore non-financial wealth $W_{2,t_e}$ increases by the expenditures $C_{2,t_e}$ at time $t_e$. However, net cash outflow of purchasing the house at time $t_e$ is not the price of the house, but the down payment. The household pays the debt loan periodically under the determined mortgage interest rate and the loan period after the time $t_e+1$. We include periodic payments $C_{1,t}^2$ in the living expense for life ($C_{1,t}^{(i)}$) in this paper.

**(2) Restoration cost due to a fire**

It is assumed that a fraction $\alpha$ of non-financial wealth $W_{2,t-1}^{(i)}$ is damaged and the restoration cost $A_t^{(i)}$ is paid if a fire of a house occurs. Explicitly,

$$A_t^{(i)} = \tau_{2,t}^{(i)}\alpha(1-\gamma_t)W_{2,t-1}^{(i)} \tag{2}$$

where $\gamma_t$ is a depreciation ratio of non-financial wealth at time $t$. $A_t^{(i)}$ does not affect non-financial wealth.[2] Instead, it affects cash flow streams as shown in Equation (12) in Section 3.2.

**(3) Medical cost**

It is assumed that a household pays $\nu_2$ if a householder has a serious disease such as cancer, cardiac infarction, apoplexy. Payment is $\tau_{4,t}^{(i)}\nu_2$, and it is included in the living expense $C_{1,t}^{(i)}$.

**(4) Living expenses $C_{1,t}^{(i)}$**

The following four kinds of parameters are used to describe living expenses.

$C_{1,t}^{1(i)}$ : cost independent of the householder's death, such as education cost and rent.
$C_{1,t}^{2}$ : annual payment for a mortgage loan (when a householder is alive).
$\nu_2$ : medical cost due to a serious disease a householder has.
$C_{1,t}^{3(i)}$ : other living expense except $C_{1,t}^{1(i)}$, $C_{1,t}^{2}$, and $\nu_2$ (when a householder is alive).

Next, we explain how to compute the annual payment for the debt loan and other living costs dependent on the householder's death.

**① Mortgage loan**

If a household purchases a group credit insurance policy, the loan payments are forgiven after a householder died. This shows that an amount of annual payment can be $\tau_{3,t}^{(i)}C_{1,t}^2$. However, the loan payment is not forgiven if a household purchases a house after a householder died. By using the condition that $\tau_{3,t}^{(i)}=0$ for $t>t_e$ if $\tau_{3,t_e}^{(i)}=0$, the amount of annual payment for mortgage loan can be $\left(1-\tau_{3,t_e}^{(i)}+\tau_{3,t}^{(i)}\right)C_{1,t}^2$.

**② Change of the consumption level**

It is assumed that a household can keep a normal consumption level if a householder is alive, however a consumption level must be $\kappa$ times a normal level if a householder is dead,

[2]Non-financial wealth decreases by $A_t^{(i)}$ due to a fire, but the same money is spent to recover the loss, and non-financial wealth increases by $A_t^{(i)}$.

where $\kappa$ is a parameter associated with a consumption level. For example, we set $\kappa = 1$ when a household keeps a normal level, and we set $\kappa = 0.7$ when it has to allow for the 70% consumption level. Therefore, other living cost becomes

$$\left\{\tau_{3,t}^{(i)} + \left(1 - \tau_{3,t}^{(i)}\right)\kappa\right\} C_{1,t}^{3(i)} = \left\{\kappa + (1-\kappa)\tau_{3,t}^{(i)}\right\} C_{1,t}^{3(i)}. \tag{3}$$

The total living cost is

$$C_{1,t}^{(i)} = C_{1,t}^{1(i)} + \left(1 - \tau_{3,t_e}^{(i)} + \tau_{3,t}^{(i)}\right) C_{1,t}^{2} + \left\{\kappa + (1-\kappa)\tau_{3,t}^{(i)}\right\} C_{1,t}^{3(i)} + \tau_{4,t}^{(i)}\nu_2. \tag{4}$$

### 2.4. Securities

Investment in risky assets contributes to a hedge against inflation. We invest in $n$ risky assets and cash. Using a price $\rho_{jt}$, a rate of return of a risky asset $j$ at time $t$ is

$$R_{jt} = \frac{\rho_{jt}}{\rho_{j,t-1}} - 1 \ (j = 1,\dots,n;\ t = 1,\dots,T). \tag{5}$$

A risk-free rate $r_t$ at time $t(= 0, 1, \dots, T-1)$ is fixed in the period from time $t$ to $t+1$. We can assume any probability distributions of $R_{jt}$ and $r_t$ in the simulated path approach if we can sample random paths for $R_{jt}$ and $r_t$. However, it is assumed that $R$ is normally distributed with the mean vector $\mu$, and the covariance matrix $\Sigma$ ($R \sim N(\mu, \boldsymbol{\Sigma})$), and $r_t$ is constant for all $t$ in this paper. We calculate a price $\rho_{jt}$ by using $R_{jt}$.

### 2.5. Life insurance

We use term life insurance with maturity $T$ against the householder's death. If a householder purchases a term life insurance policy and dies by time $T$, a household can receive insurance money. In this model, we look upon life insurance as a financial product which can hedge risk associated with wage income earned by a householder.

We assume that a household makes level payment. Because only insured person who is alive pays a premium, a premium of level payment per unit is

$$y_l = \left(\sum_{t=0}^{T-1} \frac{1 - \sum_{k=0}^{t} \lambda_{1,k}}{(1+g_1)^t}\right)^{-1} \tag{6}$$

where $g_1$ is a guaranteed interest rate of life insurance with maturity $T$.

Using the principle of equalization of income and expenditure, insurance money is calculated for the corresponding present value of premium income. We explain how to compute variable life insurance money with various kinds of payment flow at each time.[3] Let $\theta_{1,t}$ be variable life insurance money per unit of present value of premium income at time $t$. We have

$$1 = \sum_{t=1}^{T} \frac{\theta_{1,t}\lambda_{1,t}}{(1+g_1)^t} = \sum_{t=1}^{T} \frac{\eta_{1,t}\theta_1\lambda_{1,t}}{(1+g_1)^t} \tag{7}$$

where $\theta_{1,t} = \eta_{1,t}\theta_1$. Equation (7) is transformed, and life insurance money per unit is

$$\theta_{1,t} = \eta_{1,t}\left\{\sum_{k=1}^{T} \frac{\eta_{1,k}\lambda_{1,k}}{(1+g_1)^k}\right\}^{-1}. \tag{8}$$

[3]Insurance money of well-known type of life insurance is constant over time, and the problem is solved with constant life insurance in Hibiki, Komoribayashi and Toyoda [6], and Hibiki and Komoribayashi [5].

If $\eta_{1,t}$ is constant, life insurance money is constant over time regardless of the time of the householder's death.[4]

### 2.6. Fire insurance

A household purchases one year fire insurance to hedge loss of non-financial wealth due to a fire. It can update the insurance contract every year, and purchase the fire insurance policy corresponding to the future non-financial wealth. Using the principle of equalization of income and expenditure, the relationship between one unit of present value of premium income and the corresponding insurance money $\theta_2$ is shown as:

$$1 = \frac{\theta_2 \lambda_2}{1+g_2}, \quad \text{or} \quad \theta_2 = \frac{1+g_2}{\lambda_2} \tag{9}$$

where $g_2$ is a guaranteed interest rate of one year fire insurance. It is independent of time $t$. We can only select a single payment because of one year fire insurance. A premium of single payment per unit $y_F$ is equal to a unit of the present value of future premium income ($y_F = 1$).

### 2.7. Medical insurance

We use term medical insurance with maturity $T$ against a householder's serious disease. If a householder purchases a term medical insurance policy and has a serious disease by time $T$, it can receive medical insurance money. In this model, we look upon medical insurance as a financial product which can hedge loss associated with the expensive medical cost and the decrease in wage income.

We assume that a household makes level payments, and its premium per unit is calculated as[5]

$$y_b = \left( \sum_{t=0}^{T-1} \frac{1 - \sum_{k=0}^{t} \lambda_{1,k}}{(1+g_1)^t} \right)^{-1}, \tag{10}$$

as well as life insurance. The same guaranteed interest rate $g_1$ as life insurance is used. We explain how to compute variable medical insurance money as well as life insurance. Let $\theta_{4,t}$ be variable medical insurance money per unit of the present value of premium income at time $t$. We have

$$\theta_{4,t} = \eta_{4,t} \left\{ \sum_{k=1}^{T} \frac{\eta_{4,k}\lambda_{4,k}}{(1+g_1)^k} \right\}^{-1}. \tag{11}$$

We define the function of medical insurance money with the $\eta_{4,t}$ values as well as the $\eta_{1,t}$ values for life insurance. If $\eta_{4,t}$ is constant, medical insurance money is constant over time.

[4]We show the following two kinds of functions of life insurance money besides a constant function, which have higher values as a householder dies earlier.

$$\text{Decreasing linear function} \;:\; \eta_{1,t} = T - t + 1$$
$$\text{Reciprocal of mortality rate} \;:\; \eta_{1,t} = \frac{1}{\lambda_{1,t}}$$

[5]A premium of level payment per unit of medical insurance is the same as that of life insurance (Equation (6)) with the same maturity because only insured person who is alive pays a premium. A disease rate influences insurance money.

## 3. Multi-period ALM Optimization Model for a Household

We formulate a multi-period optimization model in the simulated path approach. Conditional value at risk (CVaR) is used as a risk measure [9]. We assume that the current time is 0 ($t = 0$), and a householder retires at time $T$, which is a planning horizon. As mentioned in Section 2.4, a household invests in $n$ risky assets and cash, and it can rebalance positions at each time. It purchases a $T$-years life insurance and medical insurance policies at time 0, and makes level payments. It also purchases an one-year fire insurance policy which is updated every year in the planning period.

### 3.1. Notations

**(1) Subscript/Superscript**

- $j$ : asset ($j = 1, \ldots, n$).
- $t$ : time ($t = 1, \ldots, T$).
- $i$ : path ($i = 1, \ldots, I$).

**(2) Parameters**[6]

- $\rho_{j0}$ : price of risky asset $j$ at time 0 ($j = 1, \ldots, n$).
- $\rho_{jt}^{(i)}$ : price of risky asset $j$ on path $i$ at time $t$ ($j = 1, \ldots, n;\ t = 1, \ldots, T;\ i = 1, \ldots, I$),

$$\begin{aligned}\rho_{j1}^{(i)} &= \left(1 + R_{j1}^{(i)}\right) \rho_{j0}\ (j = 1, \ldots, n;\ i = 1, \ldots, I),\\ \rho_{jt}^{(i)} &= \left(1 + R_{jt}^{(i)}\right) \rho_{j,t-1}^{(i)}\ (j = 1, \ldots, n;\ t = 2, \ldots, T;\ i = 1, \ldots, I)\end{aligned}$$

where $R_{jt}^{(i)}$ is a rate of return of risky asset $j$ on path $i$ at time $t$.

- $r_0$ : interest rate in period 1 or at time 0.
- $r_{t-1}^{(i)}$ : interest rate on path $i$ in period $t$ or at time $t-1$ ($t = 2, \ldots, T;\ i = 1, \ldots, I$).
- $g_1$ : guaranteed interest rate on life insurance policies.
- $y_l$ : premium of level payment life insurance per unit, calculated in Equation (6).
- $y_{L,t}^{(i)}$ : premium of level payment life insurance per unit on path $i$ at time $t$, calculated as $y_{L,t}^{(i)} = \tau_{3,t}^{(i)} y_l$.
- $\theta_{1,t}$ : life insurance money per unit at time $t$, calculated in Equation (8).
- $L_t^{(i)}$ : life insurance money per unit on path $i$ at time $t$, calculated as $L_t^{(i)} = \tau_{1,t}^{(i)} \theta_{1,t}$.
- $y_b$ : premium of level payment medical insurance per unit, calculated in Equation (10).
- $y_{B,t}^{(i)}$ : premium of level payment medical insurance per unit on path $i$ at time $t$, calculated as $y_{B,t}^{(i)} = \tau_{3,t}^{(i)} y_b$.
- $\theta_{4,t}$ : medical insurance money per unit at time $t$, calculated in Equation (11).
- $B_t^{(i)}$ : medical insurance money per unit on path $i$ at time $t$, calculated as $B_t^{(i)} = \tau_{4,t}^{(i)} \theta_{4,t}$.
- $g_2$ : guaranteed interest rate on fire insurance policies.
- $y_F$ : premium of one year fire insurance per unit, set as $y_F = 1$.
- $\theta_2$ : one year fire insurance money per unit, calculated in Equation (9).
- $F_t^{(i)}$ : one year fire insurance money per unit on path $i$ at time $t$, calculated as $F_t^{(i)} = \tau_{2,t}^{(i)} \theta_2$.
- $\alpha$ : loss ratio of non-financial wealth due to a fire of a house.
- $\gamma_t$ : depreciation ratio of non-financial wealth at time $t$.

[6]The other parameters, $\tau_{1,t}^{(i)}, \tau_{2,t}^{(i)}, \tau_{3,t}^{(i)}, \tau_{4,t}^{(i)}, \lambda_{1,t}, \lambda_2$, and $\lambda_{4,t}$, can be referred in Section 2.1.

$A_t^{(i)}$ : loss of non-financial wealth on path $i$ at time $t$, calculated in Equation (2).

$M_t^{(i)}$ : cash income associated with wage, severance pay, and survivor's pension on path $i$ at time $t$, calculated in Equation (1).

$H_t^{(i)}$ : debt loan on path $i$ at time $t$.

$C_t^{(i)}$ : total consumption expenditures on path $i$ at time $t$, calculated as $C_t^{(i)} = C_{1,t}^{(i)} + C_{2,t}^{(i)}$.

$W_{1,t}^{(i)}$ : financial wealth on path $i$ at time $t$. $W_{1,0}$ is an initial financial wealth at time 0.

$W_{2,t}^{(i)}$ : non-financial wealth on path $i$ at time $t$, calculated as $W_{2,t}^{(i)} = (1-\gamma_t)W_{2,t-1}^{(i)} + C_{2,t}^{(i)}$. $W_{2,0}$ is an initial non-financial wealth at time 0.

$W_E$ : lower bound of expected terminal financial wealth.

$\beta$ : probability level used in the CVaR calculation.

$L_{v,t}$ : lower bound of cash at time $t$. When $L_{v,t} < 0$, the borrowing can be allowed.

**(3) Decision variables**

$z_{jt}$ : investment unit of risky asset $j$ at time $t$ ($j = 1, \ldots, n;\ t = 0, \ldots, T-1$).

$v_0$ : cash at time 0.

$v_t^{(i)}$ : cash on path $i$ at time $t$ ($t = 1, \ldots, T-1;\ i = 1, \ldots, I$).

$u_L$ : number of life insurance bought at time 0.

$u_{F,t}$ : number of one year fire insurance bought at time $t$ ($t = 0, \ldots, T-1$).

$u_B$ : number of medical insurance bought at time 0.

$V_\beta$ : $\beta$-VaR used in the CVaR calculation.

$q^{(i)}$ : shortfall below $\beta$-VaR ($V_\beta$) of terminal financial wealth($W_{1,T}^{(i)}$) on path $i$, $q^{(i)} \equiv \max\left(V_\beta - W_{1,T}^{(i)}, 0\right)$ ($i = 1, \ldots, I$).

## 3.2. Formulation

Cash flow constraints are important in a multi-period optimization approach. Cash flow except trading financial assets $D_t^{(i)}$ is associated with income, expenditures, and insurance. Premium payment is not required at time $T$. It is formulated as:

$$\begin{aligned} D_t^{(i)} &= M_t^{(i)} + H_t^{(i)} - C_t^{(i)} - \mathbf{1}_{\{t \neq T\}}\left(y_{L,t}^{(i)} u_L + y_F u_{F,t} + y_{B,t}^{(i)} u_B\right) + L_t^{(i)} u_L + F_t^{(i)} u_{F,t-1} \\ &\quad + B_t^{(i)} u_B - A_t^{(i)} \ (t = 1, \ldots, T-1;\ i = 1, \ldots, I). \end{aligned} \tag{12}$$

The objective is the maximization of the CVaR associated with terminal financial wealth subject to the minimum return requirement.[7] Namely,

$$\mathrm{CVaR}_\beta = \mathbf{Max}\left\{ V_\beta - \frac{1}{(1-\beta)I}\sum_{i=1}^{I} q^{(i)} \,\middle|\, W_{1,T}^{(i)} - V_\beta + q^{(i)} \geq 0 \ (i = 1, \ldots, I) \right\}$$

Expected terminal financial wealth $\mathrm{E}[W_{1,T}]$ is defined as a return measure, and therefore the minimum return requirement is formulated as

$$\frac{1}{I}\sum_{i=1}^{I} W_{1,T}^{(i)} \geq W_E.$$

[7]Even if the CVaR of $W_0 - W_{1,T}^{(i)}$ is used to minimize the objective, we have the same solution as the solution derived from the maximization of the CVaR of $W_{1,T}^{(i)}$.

The model is formulated as follows:

$$\textbf{Maximize} \qquad V_\beta - \frac{1}{(1-\beta)I}\sum_{i=1}^{I} q^{(i)}, \tag{13}$$

**subject to**

$$\sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 + y_{L,0}u_L + y_F u_{F,0} + y_{B,0}u_B = W_{1,0}, \tag{14}$$

$$(W_{1,1}^{(i)} \;=)\quad \sum_{j=1}^{n} \rho_{j1}^{(i)} z_{j0} + (1+r_0)v_0 + D_1^{(i)} = \sum_{j=1}^{n} \rho_{j1}^{(i)} z_{j1} + v_1^{(i)} \ (i=1,\ldots,I), \tag{15}$$

$$(W_{1,t}^{(i)} \;=)\quad \sum_{j=1}^{n} \rho_{jt}^{(i)} z_{j,t-1} + \left(1+r_{t-1}^{(i)}\right) v_{t-1}^{(i)} + D_t^{(i)} = \sum_{j=1}^{n} \rho_{jt}^{(i)} z_{jt} + v_t^{(i)}$$

$$(t=2,\ldots,T-1;\ i=1,\ldots,I), \tag{16}$$

$$W_{1,T}^{(i)} = \left\{\sum_{j=1}^{n} \rho_{jT}^{(i)} z_{j,T-1} + \left(1+r_{T-1}^{(i)}\right) v_{T-1}^{(i)}\right\} + D_T^{(i)} \ (i=1,\ldots,I), \tag{17}$$

$$\frac{1}{I}\sum_{i=1}^{I} W_{1,T}^{(i)} \geq W_E, \tag{18}$$

$$W_{1,T}^{(i)} - V_\beta + q^{(i)} \geq 0 \ (i=1,\ldots,I), \tag{19}$$

$$z_{jt} \geq 0 \ (j=1,\ldots,n;\ t=0,\ldots,T-1),$$
$$v_0 \geq 0,$$
$$v_t^{(i)} \geq L_{v,t} \ (t=1,\ldots,T-1;\ i=1,\ldots,I),$$
$$u_L \geq 0,$$
$$u_{F,t} \geq 0 \ (t=0,\ldots,T-1),$$
$$u_B \geq 0,$$
$$q^{(i)} \geq 0 \ (i=1,\ldots,I),$$
$$V_\beta \ : \ \text{free.}$$

## 3.3. Numerical analysis

### 3.3.1. Setting

We test numerical examples using the same setting of the family and the parameters as in Hibiki and Komoribayashi [5]. We show the sensitivity analysis associated with a serious disease because the model involves medical insurance.

All of the problems are solved using NUOPT (Ver. 7.1.5) – mathematical programming software package developed by Mathematical System, Inc. – on Windows XP personal computer which has 2.13 GHz CPU and 2GB memory.

The householder is thirty years old and the spouse is twenty-eight years old. The first child is an infant aged 0, and the second child will be born in three years.[8] The householder works at a financial institution, and the household plans that it will prepare twenty million yen as a down payment ten years later and buy an apartment in the center of Tokyo which costs fifty million yen. Twenty million yen is paid at the time ($t_e = 10$) when the house is bought. Thirty million yen is borrowed, and the mortgage loan is equally paid over twenty years. Equal yearly payment is calculated at a mortgage investment rate of 6%. The parents make an educational plan that the children will go to a private elementary school, a private

[8]It is assumed that the second child is not born if the householder dies in two years.

junior high school, a private high school, and a private university. The parameter values used in the examples are shown in Table 1.

Table 1: Parameter values

| Parameters | Values |
|---|---|
| number of risky assets | $n = 1$ |
| length of one period | one year |
| retirement age of a householder | 60 years old |
| number of periods | $T = 30$ |
| expected rate of return of a risky asset | $\mu = 0.1$ |
| standard deviation of rate of return of a risky asset | $\sigma = 0.2$ |
| risk-free rate | $r = 0.04$ |
| mortality rate | $\lambda_{1,t}$(*1) |
| rate of a fire | $\lambda_2 = 0.005$ |
| disease rate (*2) | $\lambda_{4,t} = \nu_1 \lambda_{1,t}$ |
| life insurance money per unit | $\theta_{1,t} = \theta_1$ (constant) |
| medical insurance money per unit | $\theta_{4,t} = \theta_4$ (constant) |
| guaranteed rate on life and medical insurance | $g_1 = 0.05$ |
| guaranteed rate on fire insurance | $g_2 = 0.05$ |
| maximum coefficient of severance pay (*3) | $\delta_U = 2$ |
| time of reaching maximum coefficient of severance pay | $T_\delta = 20$ |
| initial financial wealth | $W_{1,0} = 10$ (million yen) |
| initial non-financial wealth | $W_{2,0} = 10$ (million yen) |
| depreciation rate of non-financial wealth | $\gamma_t = 0.03$ |
| loss of non-financial wealth due to a fire | $\alpha = 1$ |
| lower bound of cash (million yen) | $L_{v,0} = 0,\ L_{v,t} = -10 (t \neq 0)$ |
| lower bound of expected terminal financial wealth | $W_E = 70$ (million yen) |
| probability level | $\beta = 0.8$ |
| number of paths | $I = 5,000$ |

*1 The rates are estimated by the life insurance standard life table 1996 for men [11].

*2 There must be a serious disease rate table for medical insurance, and a premium must be calculated by using the table. However, the table is not published outside. In this paper, we assume that the disease rate $\lambda_{4,t}$ is $\nu_1 (> 1)$ times the mortality rate $\lambda_{1,t}$. The reason is that a serious disease causes death, and the disease rate becomes higher with age.

*3 An amount of severance pay $e_t^{(i)}$ is calculated by multiplying an amount of wage when a householder retires or dies with the provision coefficient $\delta_t$ in Equation (20). It is assumed that the provision coefficient is a piecewise linear function with an upper bound $\delta_U$ at time $T_\delta$ in Equation (21). Explicitly,

$$e_t^{(i)} = \delta_t m_t^{(i)} \ (t = 1, \ldots, T), \tag{20}$$

$$\delta_t = \min\left[\left(\frac{t}{T_\delta}\right), 1\right] \delta_U \ (t = 1, \ldots, T;\ i = 1, \ldots, I;\ T_\delta \leq T). \tag{21}$$

Wage income depends on householder's age and occupation. We calculate wage income of a householder over time based on the Census of wage in 2003 by Ministry of Health, Labor and Welfare [15]. Consumption expenditure depends on wage income, family structure and school (education) plan. We calculate average consumption expenditures with respect to each number of family and each income level of family based on the national survey of family income and expenditure in 1999 by Statistic Bureau, Ministry of Internal Affairs and Communications [16]. We calculate average educational expenses based on the survey of household expenditure on education per student in 2001 [13] , the survey of student life by Ministry of Education, Culture, Sports, Science and Technology [14].

We clarify the effects of four parameters (factors) associated with a serious disease : ① a serious disease rate, ② medical cost, ③ the decrease in wage income, ④ the possibility of death after having a serious disease. We solve five kinds of problems for each parameter as in Table 2 to examine the sensitivity of four parameters. When one of the parameters is examined, the other parameter values are fixed at 'P3' values. We test 17 combinations in total.[9]

Table 2: Parameters associated with a serious disease

| Parameters | Notations | P1 | P2 | P3 | P4 | P5 |
|---|---|---|---|---|---|---|
| Coefficient of a disease rate | $\nu_1$ | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
| Medical cost | $\nu_2$ | 50 | 100 | 150 | 200 | 250 |
| Decreasing rate of wage income | $\nu_3$ | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 |
| Death probability after a serious disease(*4) | $\lambda_3$ | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 |

*4 A serious disease causes death, and therefore it is assumed that $\lambda_3(< 1 : \text{constant})$ is the probability that a householder having a serious disease dies after one year. For example, the probability is 60% if we set $\lambda_3 = 0.6$.

### 3.3.2. Result

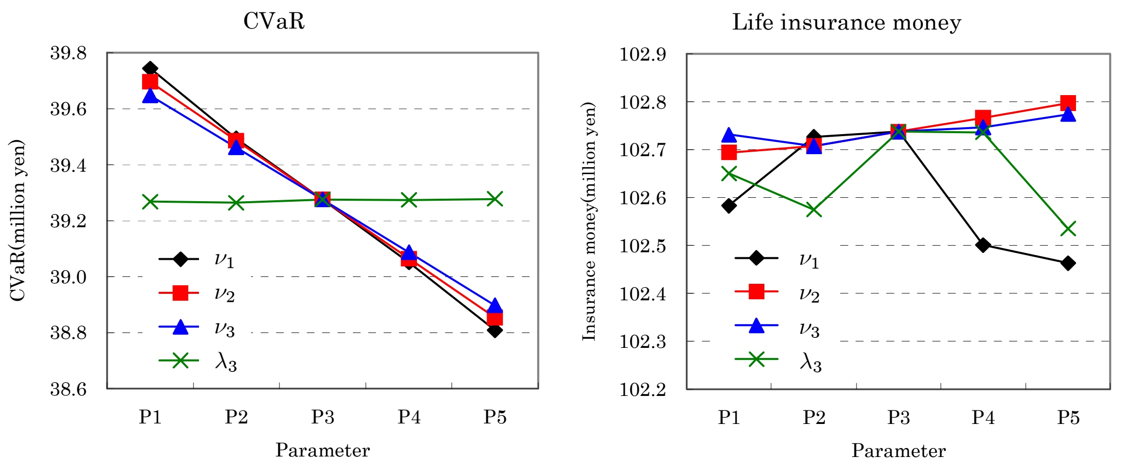


Figure 1: CVaR and optimal life insurance money

Figure 1 shows the CVaR on the left-hand side and life insurance money $(\theta_1 u_L^*)$ on the right-hand side for each combination of parameters. For example, a broken line of $\nu_1$ shows

[9]For example, when we examine the sensitivity of $\nu_1$ value, we solve the problems with one of five kinds of $\nu_1$ and 'P3' values of $\nu_2$, $\nu_3$ and $\lambda_3$ (i.e. $\nu_2 = 150$, $\nu_3 = 0.20$ and $\lambda_3 = 0.5$). Four combinations are overlapped, and therefore $17(= 5 \times 4 - 3)$ combinations are tested.

values of the CVaR for five kinds of $\nu_1$, i.e. P1= 2.0 through P5= 4.0 in Table 2.[10] When a $\nu_1$ value becomes larger, the CVaR value is smaller because the probabilities of the decrease in wage income and the increase in medical cost are higher due to a serious disease. When $\nu_2$ and $\nu_3$ values become larger, wage income decreases and medical cost increases, and therefore the CVaR value is smaller. However, the CVaR value is not influenced by the $\lambda_3$ value, or the probability of the householder's death after a householder has a serious disease. Life insurance money is not affected by four factors associated with a serious disease.

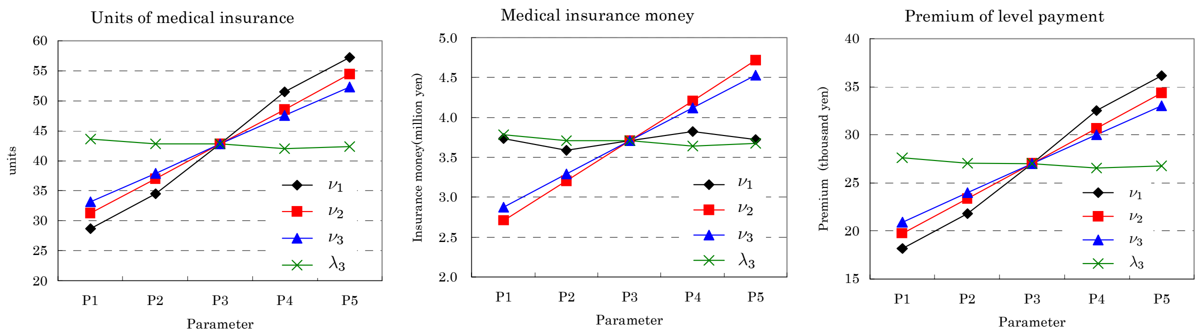


Figure 2: Optimal units, insurance money, and premium for medical insurance

Figure 2 shows units of medical insurance ($u_B^*$) on the left-hand side, medical insurance money ($\theta_4 u_B^*$) on the middle, and premium payments ($y_b u_B^*$) on the right-hand side. When $\nu_2$ and $\nu_3$ values become larger, the householder purchases more units of medical insurance policy to hedge against the decrease in wage income and the amount of medical cost. It means that the number of units of medical insurance and premium payments become larger. Medical insurance money is not influenced by the increase in a disease rate $\nu_1$ because wage income does not decrease and medical cost does not increase. However, medical insurance money per unit ($\theta_4$) becomes small, and the householder needs to purchase more units of medical insurance policy($u_B^*$) in order to receive medical insurance money which can cover cash outflow due to a serious disease. Therefore, premium payments ($y_b u_B^*$) become large. A $\lambda_3$ value does not influence the number of units of medical insurance, medical insurance money, and premium payments as well as the CVaR and life insurance money.

When a householder has a serious disease at time $t$, financial wealth decreases by payment for the expensive medical cost and the decrease in wage income at time $t$. We examine the relationship between the annual average or maximum decrease in wealth during thirty years and optimal medical insurance money in Figure 3. The decrease in wealth consists of the decrease in wage income and the increase in medical cost. The average and maximum decreases are almost equal to medical insurance money. It shows that medical insurance money is used to hedge against the decrease in cash inflow.

Figure 4 shows optimal investment units of a risky asset for each parameter. Investment units decrease gradually over time. When $\nu_1$, $\nu_2$, and $\nu_3$ values become large, investment units increase. The reason is that a household needs to invest in more amounts of a risky asset to cover the decrease in wage income and the increase in medical cost, and to increase expected terminal wealth. A $\lambda_3$ value does not influence optimal investment units.

[10]When we show values of the CVaR for five kinds of $\nu_1$, we can write five $\nu_1$ instead of the expression of P1 through P5 on the vertical axis so that readers understand the meaning of graphs easily. However, we employ the expression as in Figure 1 for lack of space. Readers can know parameter values of P1 through P5 by checking Table 2.

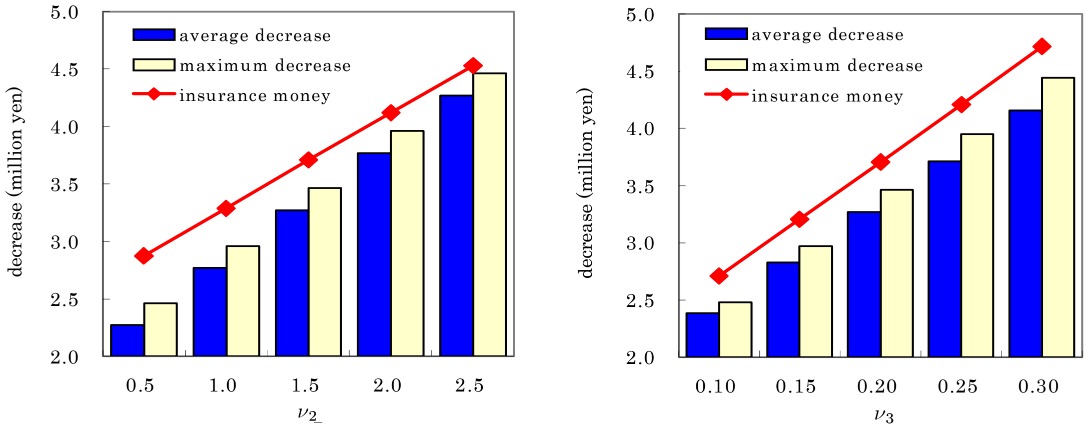


Figure 3: Relationship between the decrease in wealth and optimal medical insurance money

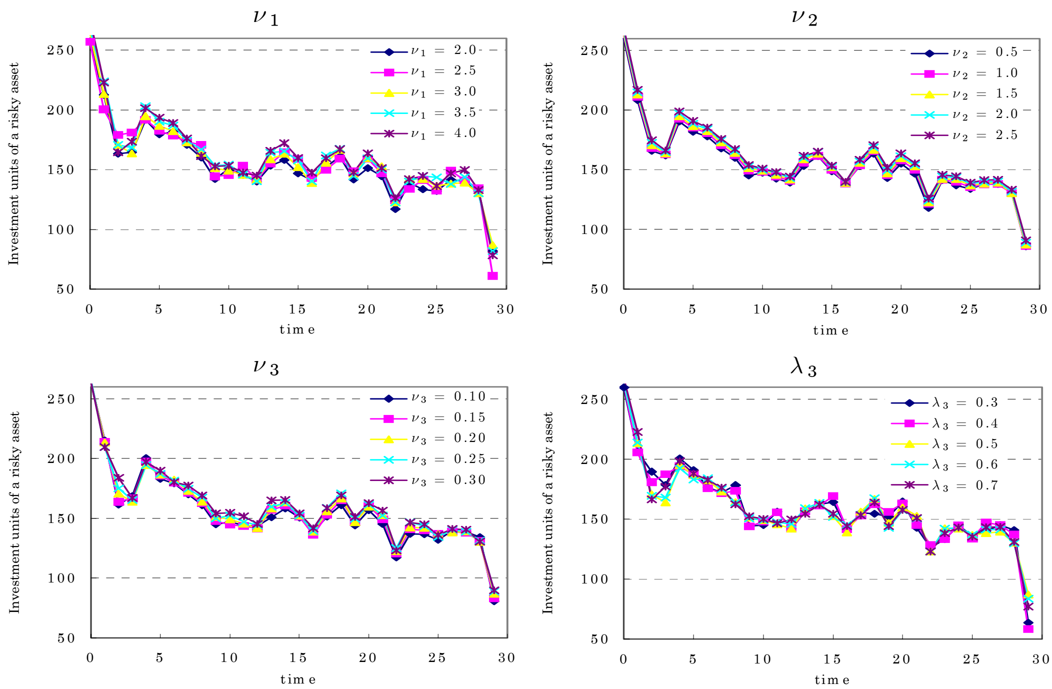


Figure 4: Optimal investment units of a risky asset

### 3.4. Life insurance with a decreasing linear function

Term life insurance of receiving constant insurance money is a very popular product. We solve the problem to examine the characteristics of the product. Four cases are the combinations of two kinds of $pf$ and two kinds of $np$ introduced in Hibiki and Komoribayashi [5], i.e. $pf = 0$ and $np = 0$, $pf = 0$ and $np = 1$, $pf = 1$ and $np = 0$, $pf = 1$ and $np = 1$. $pf$ is one if the problem is solved with receipt of a survivor's pension and zero without receipt, and $np$ is one if the problem is solved with exemption from a mortgage loan and zero without exemption.

Figure 5 shows conditional expected terminal financial wealth at the time of the householder's death.[11] A value at time 0 shows an expected value under the condition that a householder does not die in the planning period. Expected terminal financial wealth is increasing as the time of the householder's death becomes late.[12] The reason is that a household gets a wage in the longer period before a householder dies, and receives life insurance money when a householder dies. If a householder dies earlier, expected terminal financial wealth tends to be lower because a household receives a lower survivor's pension relative to an amount of wage.

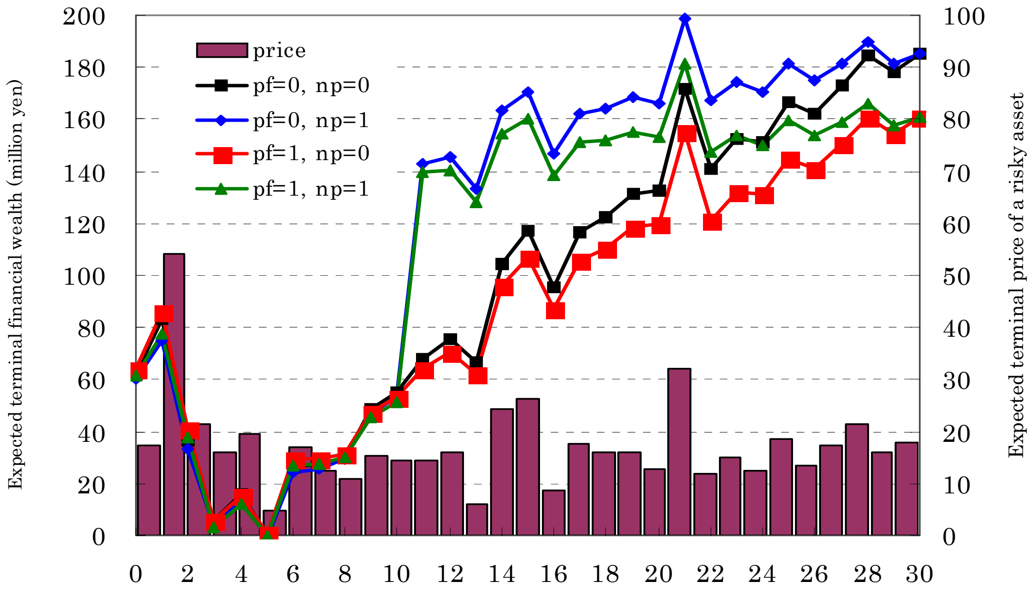


Figure 5: Conditional expected terminal financial wealth at each time

It is important for a household not to have a financial problem and to lead a stable life by receiving life insurance money even if a householder dies earlier. Therefore, we should design life insurance that a household can receive more insurance money as a householder dies earlier. We solve the problem with a decreasing life insurance policy that insurance money decreases in proportion to a householder's age. We calculate $\theta_{1,t}$ with $\eta_{1,t} = T - t + 1$ in Equation (8). We solve the problem, and compare a decreasing type of life insurance with a constant type.

[11]Conditional expected terminal financial wealth at each time $\overline{W}_t^{\tau_1}$ in Figure 5 can be calculated as follows:

$$\overline{W}_0^{\tau_1} = \frac{1}{|\tau_{3,T}|}\sum_{i=1}^{I}\tau_{3,T}^{(i)}W_T^{(i)}, \quad \overline{W}_t^{\tau_1} = \frac{1}{|\tau_{1,t}|}\sum_{i=1}^{I}\tau_{1,t}^{(i)}W_T^{(i)} \ (t=1,\ldots,T)$$

where $|\tau_{1,t}| = \sum_{i=1}^{I}\tau_{1,t}^{(i)}$, and $|\tau_{3,T}| = \sum_{i=1}^{I}\tau_{3,T}$ .

[12]If a householder dies after time 12, the loan payment is forgiven. Therefore, expected terminal financial wealth after time 12 for $np = 1$ are larger than those for $np = 0$ because the reduction in loan payment contributes to the increase in terminal financial wealth. Broken lines are not smooth, especially at time 1, 14, 15, and 21. The reason is that terminal financial wealth ($W_T^{(i)}$) is influenced by a terminal price of a risky asset ($\rho_{jT}^{(i)}$).

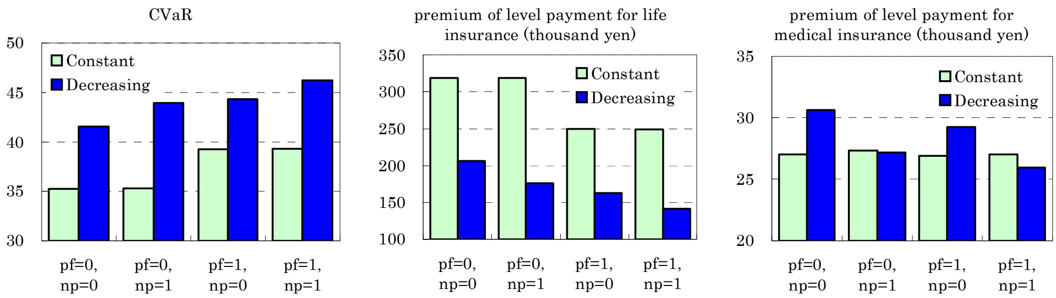


Figure 6: CVaR and premium payments for life insurance and medical insurance

Figure 6 shows the CVaR values, and premium payments for life insurance and medical insurance. The CVaR values of the decreasing type increases about 7 million yen or 18%, compared with the constant type for $pf = 1$ and $np = 1$. Premiums of the decreasing type decreases about 35% for $np = 0$, and about 45% for $np = 1$, compared with the constant type. We obtain dramatic effects by introducing the decreasing type of life insurance. The change in design of life insurance does not influence medical insurance money.

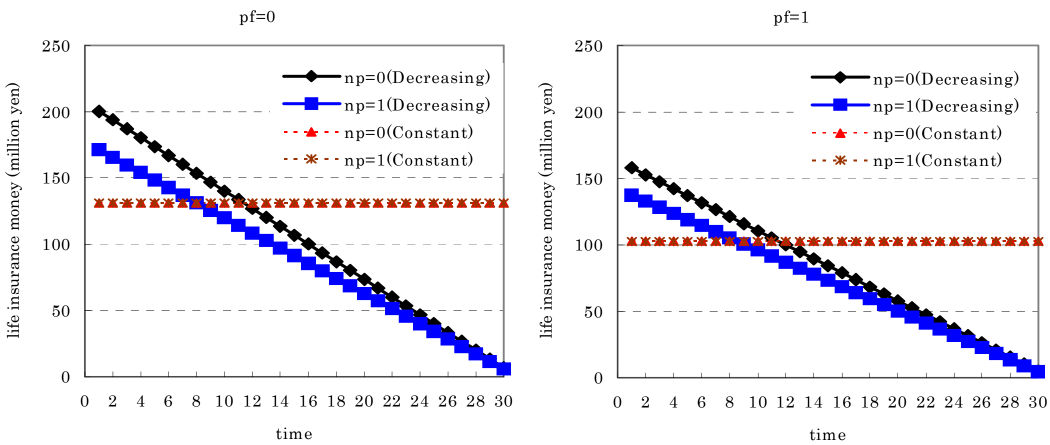


Figure 7: Optimal life insurance money

Figure 7 shows life insurance money at each time without receiving a survivor's pension ($pf = 0$) on the left-hand side, and with receiving ($pf = 1$) on the right-hand side. Life insurance money without receiving a survivor's pension is larger than insurance money with receiving for both types. Life insurance money of the decreasing type is larger until time 10, but smaller after 10 than that of the constant type. The reason premiums of constant type are larger than those of the decreasing type is that a mortality rate becomes higher as a householder gets older, and the period more life insurance money can be received for the constant type is longer than the period for the decreasing type.

Figure 8 shows conditional expected terminal financial wealth at each time of the householder's death. When a householder dies earlier, conditional expected terminal wealth becomes larger because a household can receive larger life insurance money.[13] Expected terminal financial wealth entirely becomes flatter than those in Figure 5. A decreasing type of life insurance reduces risk, and has similar expected terminal financial wealth regardless of the time of the householder's death, while a household saves premium payments.

[13]We might set up a decreasing linear function with higher life insurance money when a householder dies earlier.

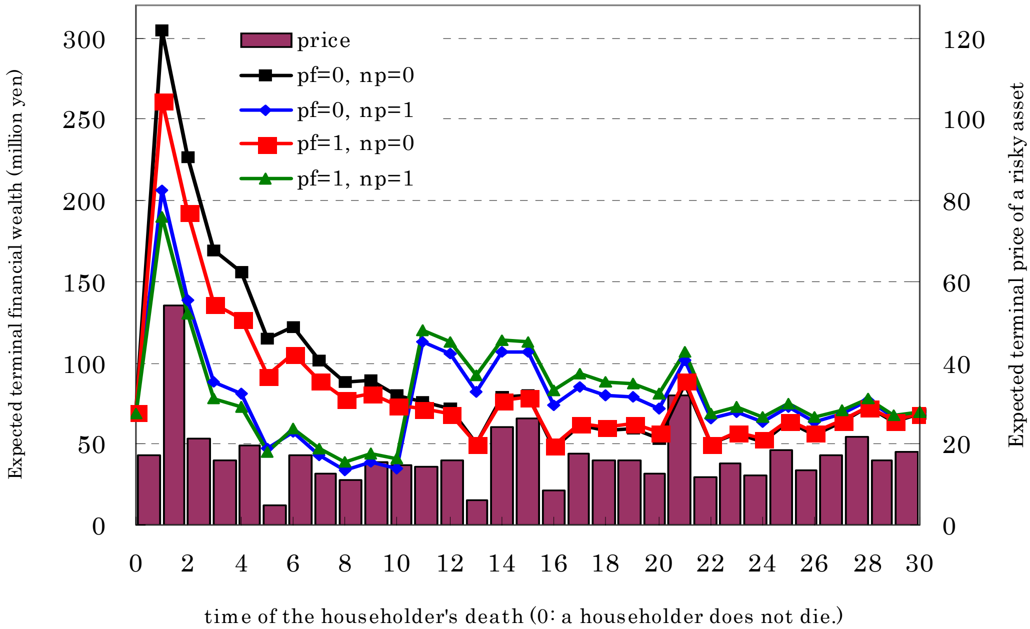


Figure 8: Conditional expected terminal financial wealth at each time

## 4. Optimal Insurance Design

The results derived in Section 3.4 show that we had better design life insurance so that a household can receive insurance money to fit cash flow needs. We propose a model to decide optimal life and medical insurance money received at each time, instead of the constant or decreasing insurance money.

### 4.1. Modification for optimal design

#### (1) Life insurance

Life insurance money at time $t$ ($x_t$) is calculated by multiplying life insurance money per unit ($\theta_{1,t}$) by the number of units ($u_L$), i.e. $x_t = \theta_{1,t}u_L$. Life insurance money per unit is not given as input parameters, and therefore we need to add a constraint which shows the principle of equalization of income and expenditure instead of Equation (7). We multiply the number of units of life insurance ($u_L$) by both sides in Equation (7), and we obtain

$$u_L = \sum_{t=1}^{T} \phi_t x_t \text{ where } \phi_t = \frac{\lambda_{1,t}}{(1+g_1)^t}. \tag{22}$$

$L_t^{(i)} u_L$ in Equation (12) is transformed as

$$L_t^{(i)} u_L = \tau_{1,t}^{(i)} \theta_{1,t} u_L = \tau_{1,t}^{(i)} x_t \ (t = 1, \dots, T; \ i = 1, \dots, I).$$

#### (2) Medical insurance

Medical insurance money at time $t$ ($w_t$) is calculated as $w_t = \theta_{4,t}u_B$. The principle of equalization of income and expenditure is

$$u_B = \sum_{t=1}^{T} \psi_t w_t \text{ where } \psi_t = \frac{\lambda_{4,t}}{(1+g_1)^t}. \tag{23}$$

$B_t^{(i)} u_B$ in Equation (12) is transformed as

$$B_t^{(i)} u_B = \tau_{4,t}^{(i)} \theta_{4,t} u_B = \tau_{4,t}^{(i)} w_t \ (t = 1, \dots, T; \ i = 1, \dots, I).$$

**(3) Other constraints**

In addition to the modification of the principle of equalization of income and expenditure, we need to add and modify the constraints in the formulation.

① **Cash flow except trading assets ($D_t^{(i)}$)** : modification of Equation (12)

$$D_t^{(i)} = M_t^{(i)} + H_t^{(i)} - C_t^{(i)} - \mathbf{1}_{\{t\neq T\}}\left(y_{L,t}^{(i)}u_L + y_F u_{F,t} + y_{B,t}^{(i)}u_B\right) + \tau_{1,t}^{(i)}x_t + \tau_{2,t}^{(i)}\theta_2 u_{F,t-1} + \tau_{4,t}^{(i)}w_t - \tau_{2,t}^{(i)}\alpha(1-\gamma_t)W_{2,t-1}^{(i)} \ (t=1,\ldots,T;\ i=1,\ldots,I). \quad (24)$$

② **Additional non-negativity constraints**

$$x_t \geq 0 \ (t=1,\ldots,T), \quad (25)$$

$$w_t \geq 0 \ (t=1,\ldots,T). \quad (26)$$

Except for the above-mentioned constraints, we do not have to change the formulation in Section 3.2.

### 4.2. Numerical analysis

We compare the combination (a) of a decreasing linear function for life insurance and a constant function for medical insurance titled 'Decreasing LI'(Life Insurance) with the combination (b) of optimal functions for life and medical insurance titled 'optimal' in Table 3. We solve the problems for four cases generated by the combination of $pf$ and $np$.

Table 3: Combination of functions for life and medical insurance

| | life insurance money | medical insurance money |
|---|---|---|
| (a) Decreasing LI | decreasing function ($\theta_{1,t}u_L$) | constant function ($\theta_4 u_B$) |
| (b) optimal | optimal function ($x_t$) | optimal function ($w_t$) |

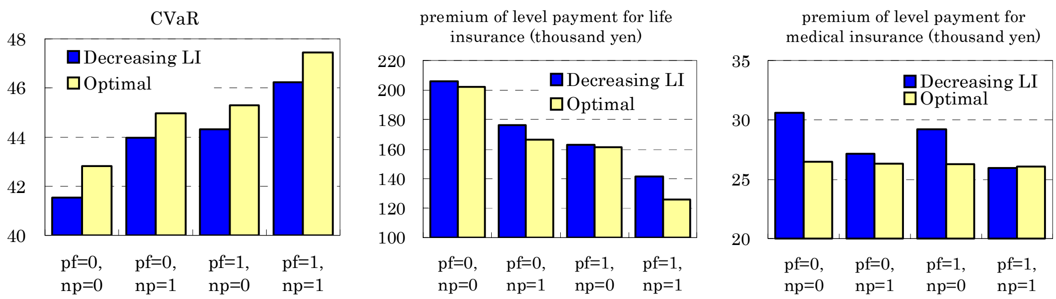


Figure 9: CVaR and premiums of level payment

Figure 9 shows the CVaR values on the left-hand side, premium payments for life insurance on the middle, and premium payments for medical insurance on the right-hand side. The CVaR values with optimal functions are larger than the CVaR values with a decreasing linear function for life insurance by 1.2 million yen or about 3% for $pf = 1$ and $np = 1$. Premium payments for life insurance with optimal functions can be reduced by about 10%. Premium payments for medical insurance can be reduced by 13% for $pf = 0$ and $np = 0$, and 10% for $pf = 1$ and $np = 0$.

We show optimal functions for life insurance in Figure 10, and describe the characteristics as follow.

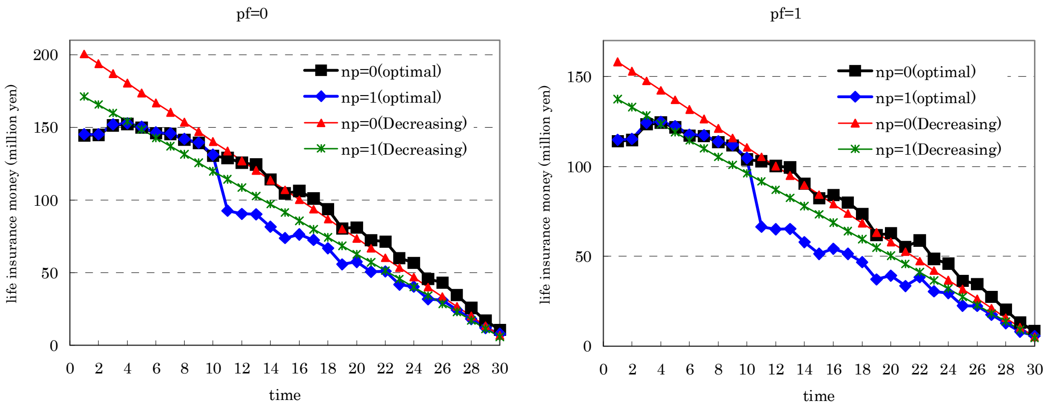


Figure 10: Optimal life insurance money

① The optimal functions also become decreasing functions entirely as we expect and examine the effect in Section 3.4.

② Amounts of optimal life insurance money rise sharply from time 2 to 3. The reason is that the second child will be born at time 3, and it costs for the household to raise the second child (e.g. educational cost[14]) if the householder dies after time 3.

③ Optimal life insurance money drops sharply from time 10 to time 11 for $np = 1$. The reason is that a house is bought at time 10, and loan payments are forgiven if a householder dies after time 11.[15]

When we solve the problem with a decreasing type of life insurance, the CVaR value becomes much larger, and premium payments become much lower drastically, compared with a constant type of life insurance. On the other hand, when we solve the problem with an optimal function of life insurance, the CVaR value becomes larger by 3%, and premium payments become lower by 10%, compared with a decreasing type of life insurance. The effect using the optimal function is not dramatic. The reason is that the decreasing linear function is similar to the optimal function as shown in Figure 10.

Figure 11 shows conditional expected terminal financial wealth at each time of the householder's death. They are flatter than those in Figure 8. This shows that conditional expected terminal financial wealth at each time has similar values, regardless of the time of the householder's death.[16] They have almost the same values, regardless of the values of $pf$ and $np$. The reason is that optimal life insurance money can be adjusted, depending on the values of $pf$ and $np$ as shown from the results in Figure 10. The model involving the optimal insurance design is solved usefully so that the above-mentioned feature can be reflected, and terminal financial wealth can get less affected by the time of the householder's death.

Figure 12 shows medical insurance money on the left-hand side, and fire insurance money on the right-hand side. The broken line with the legend 'decrease' on the left-hand side shows the decrease in wealth due to a serious disease, i.e. the decrease in wage income and the amount of medical cost. The broken line with the legend 'loss' on the right-hand side shows

[14]The increases in life insurance money from time 2 to 3 are about 6.7 million yen for $pf = 0$, and about 8.8 million yen for $pf = 1$. The value at time 3 of the educational cost is about 8.95 million yen with 4% discount rate.

[15]The decrease in life insurance money from time 10 to 11 is about 36.4 million yen for $np = 1$. Annual payment is 2.616 million yen for a mortgage loan of 30 million yen with 6% mortgage interest rate, and therefore the value at time 11 of the mortgage loan payment is 36.97 million yen with 4% discount rate.

[16]The reason the values at time 1 and 21 are higher is that average prices of a risky asset are higher, and there exists sampling errors.

the loss due to a fire. The left-hand side of Figure 12 shows the same results on average that a medical insurance policy is purchased to cover the decrease in wage income and the amount of medical cost in Section 3.3. However, the amounts of medical insurance money at each time are unstable, and the optimal function of medical insurance is easily affected by sampling error. Optimal fire insurance money is nearly equal to the maximum loss of non-financial wealth as well as the results derived by the previous models.

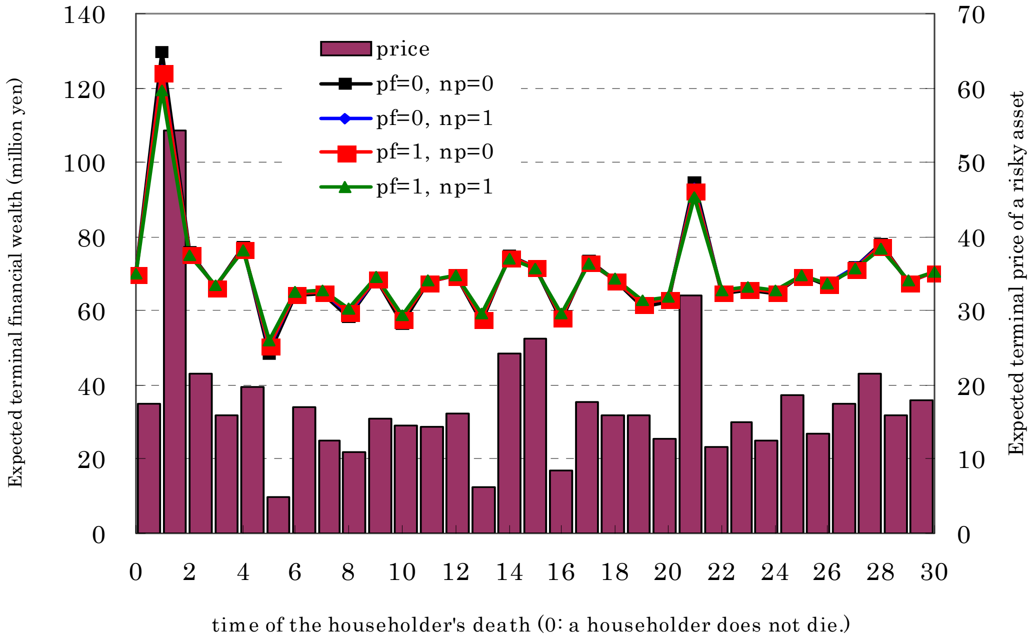


Figure 11: Conditional expected terminal financial wealth at each time

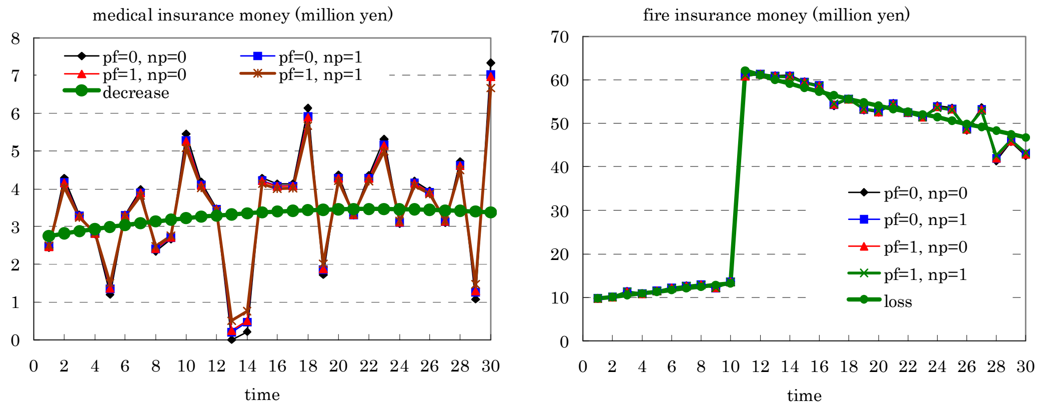


Figure 12: Optimal medical and fire insurance money

## 5. Examining sampling error

We find the characteristics of the model with numerical examples. However, sampling error occurs because a thirty-periods model is solved with 5,000 simulated paths. We solve 100 kinds of problems with different random seeds, and we examine sample distributions of optimal solutions. We need to provide 100 kinds of dataset associated with prices of a risky asset ($\rho_{j,t}^{(i)}$), 0-1 parameters for the householder's death ($\tau_{1,t}^{(i)}$, $\tau_{3,t}^{(i)}$), 0-1 parameter for a house of a fire ($\tau_{2,t}^{(i)}$), and 0-1 parameter for the householder's serious disease ($\tau_{4,t}^{(i)}$). We call the model with constant life insurance money in Section 3.2 'model A', and the model with optimal life insurance money in Section 4 'model B'.

### 5.1. Numerical Analysis : Model A

Figure 13 shows the change in the average of the CVaR on the left-hand side, life insurance money on the middle, and medical insurance money on the right-hand side as we increase the problems solved with different random seeds. There are five kinds of percentiles derived with different random seeds in Figure 13. The average values converge between forty percentile and sixty percentile by solving about thirty problems with different random seeds.

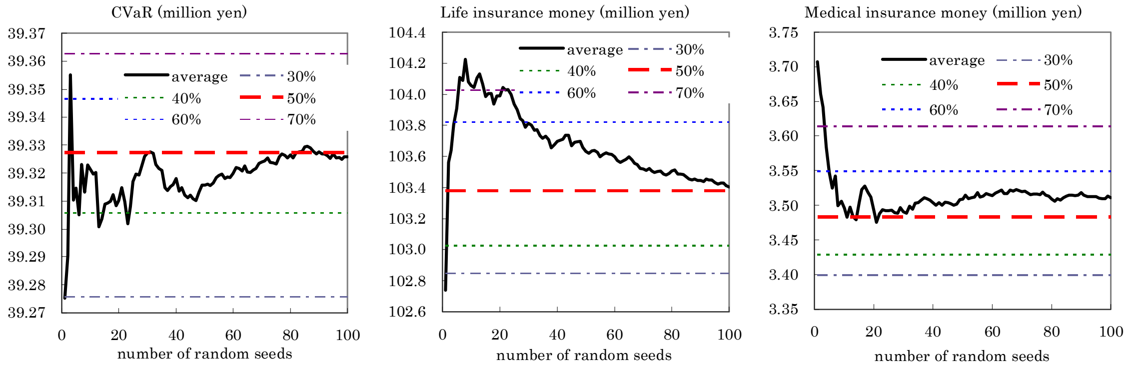


Figure 13: Convergence of the CVaR and life and medical insurance money for model A

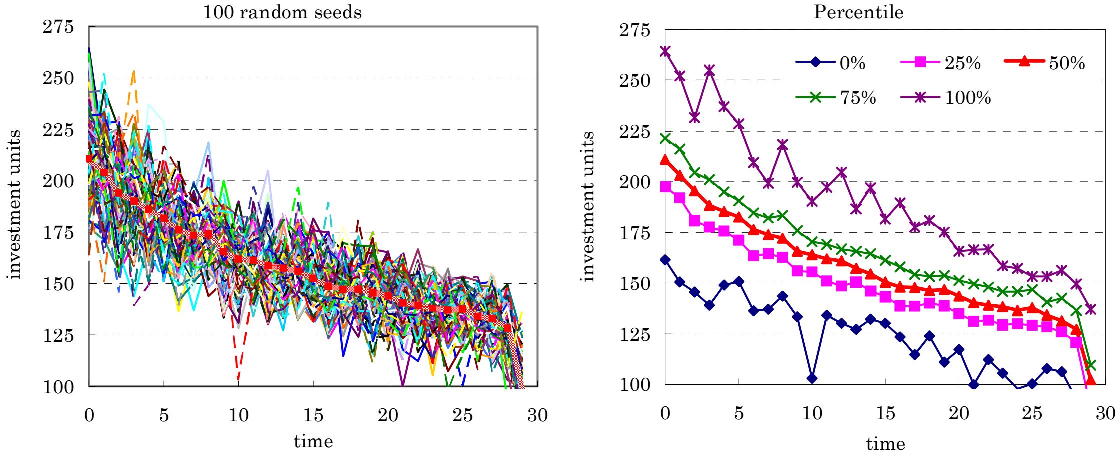


Figure 14: Optimal investment units of a risky asset

We show the number of investment units of a risky asset for 100 kinds of random seeds on the left-hand side, and for five kinds of percentiles on the right-hand side in Figure 14. We find that the number of investment units of a risky asset decreases through time as in Figure 4.

Percentiles change smoothly through time because percentiles are calculated separately at each time. The values over time are volatile as in Figure 4 when each problem is solved. However, the values are expected to fluctuate around the average values over time. Let $z_{1t}^{k*}$ be the optimal number of investment units for a risky asset when the problem is solved with the $k$-th random seed. The average of $z_{1t}^{k*}$ at time $t$ for 100 kinds of problems is as

$$\bar{z}_{1t}^{*} = \frac{1}{100}\sum_{k=1}^{100} z_{1t}^{k*} \ (t = 0, \ldots, T-1).$$

Let $D_z^{k*}$ be the average of deviation from the average $\overline{z}_{1t}^{*}$ for each random seed as

$$D_z^{k*} = \frac{1}{T}\sum_{t=0}^{T-1}\left(z_{1t}^{k*} - \overline{z}_{1t}^{*}\right) \ (k = 1, \ldots, 100).$$

A standard deviation of $D_z^{k*}$ is 1.69, a maximum value is 4.35, and a minimum value is $-3.72$. These values are much smaller than the number of investment units, and therefore it can be said that the values $z_{1t}^{k*}$ fluctuate around the average $\overline{z}_{1t}^{*}$.

We show conditional expected terminal financial wealth at each time of the householder's death for 100 kinds of random seeds on the left-hand side, and for seven kinds of percentiles on the right-hand side in Figure 15. We find the same characteristics as in Figure 5 even if we use different random seeds.

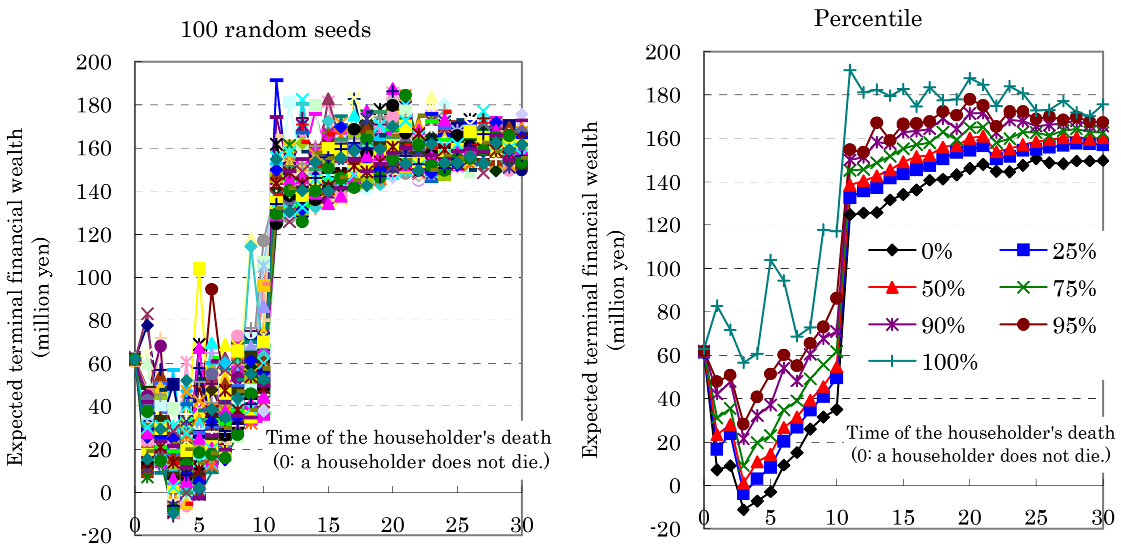


Figure 15: Conditional expected terminal financial wealth at each time for model A

## 5.2. Numerical analysis : Model B

We show life insurance money for 100 kinds of random seeds on the left-hand side, and for five kinds of percentiles on the right-hand side of Figure 16. We obtain optimal solutions stably even if we use different random seeds. The amounts of life insurance money rise until time 3, and decline afterward.

We show medical insurance money for 100 kinds of random seeds on the left-hand side, and for five kinds of percentile on the right-hand side of Figure 17. The fifty percentile (median) on the right-hand side of Figure 17 is almost equal to the sum of the decrease in wage income and the amount of medical cost, and therefore we find a medical insurance policy is purchased to cover the loss due to a serious disease. However, the amounts of optimal medical insurance money fluctuate around the average as shown on the left-hand side of Figure 17, and we cannot ignore the influence of sampling error.

We show conditional expected terminal financial wealth at each time of the householder's death for 100 kinds of random seeds on the left-hand side, and for seven kinds of percentiles on the right-hand side of Figure 18. We can find the characteristics shown in Figure 11 that terminal financial wealth can get less affected by the time of the householder's death. The fifty percentile (median) on the right-hand side of Figure 18 is almost flat. Some values

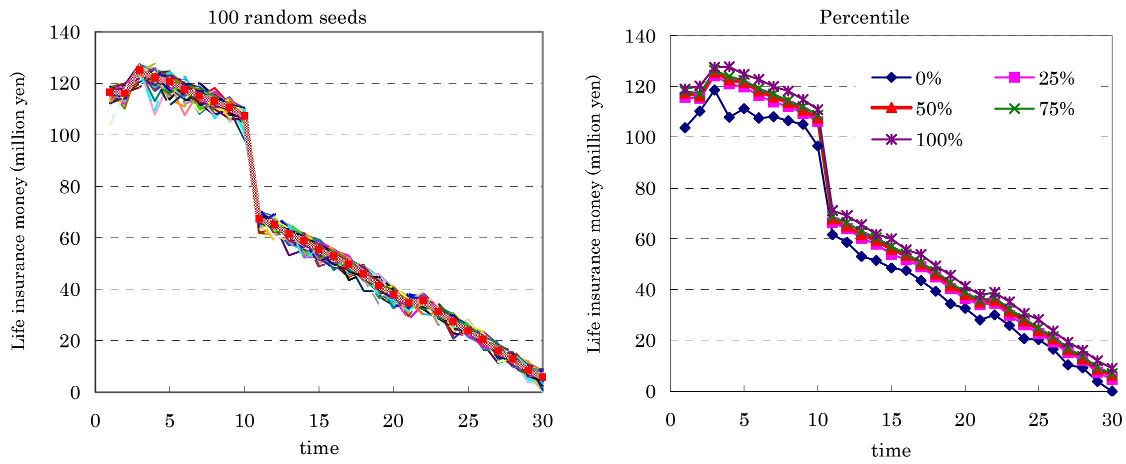


Figure 16: Optimal life insurance money for model B

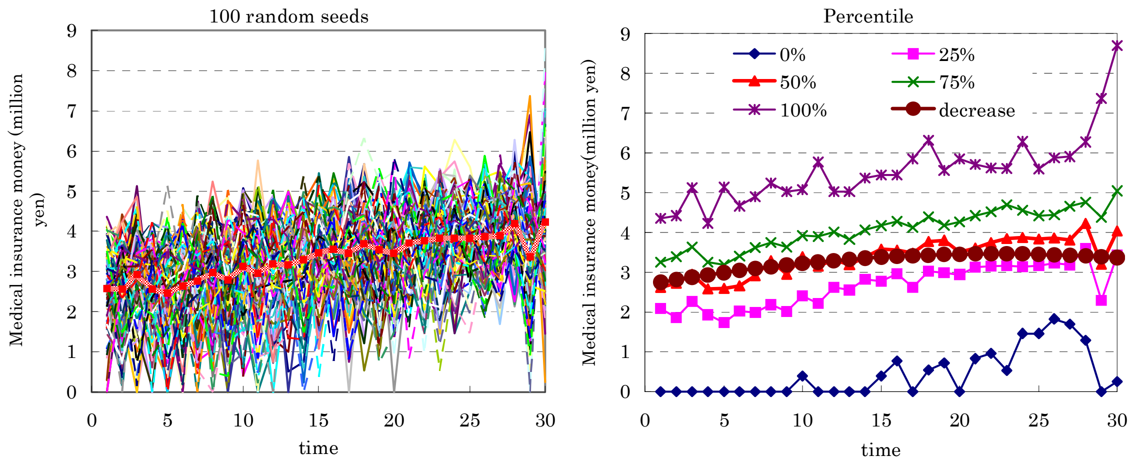


Figure 17: Optimal medical insurance money for model B

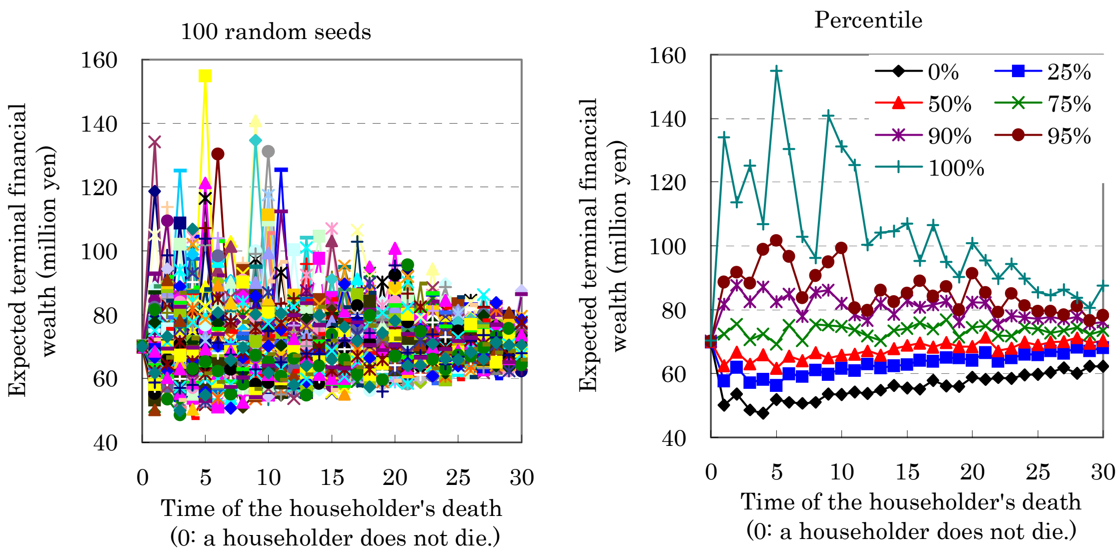


Figure 18: Conditional expected terminal financial wealth at each time for model B

fluctuate widely in the earlier periods because the number of paths that the householder dies is few,[17] and they are affected by prices of a risky asset. Figure 19 shows conditional expected terminal prices at each time of the householder's death. The fluctuation is due to sampling error because prices are not affected by the householder's death. Figure 19 is very similar to Figure 18. If we have enough paths at each time of the householder's death, and conditional prices become stable (flat) over time, time series of conditional wealth become also flat.

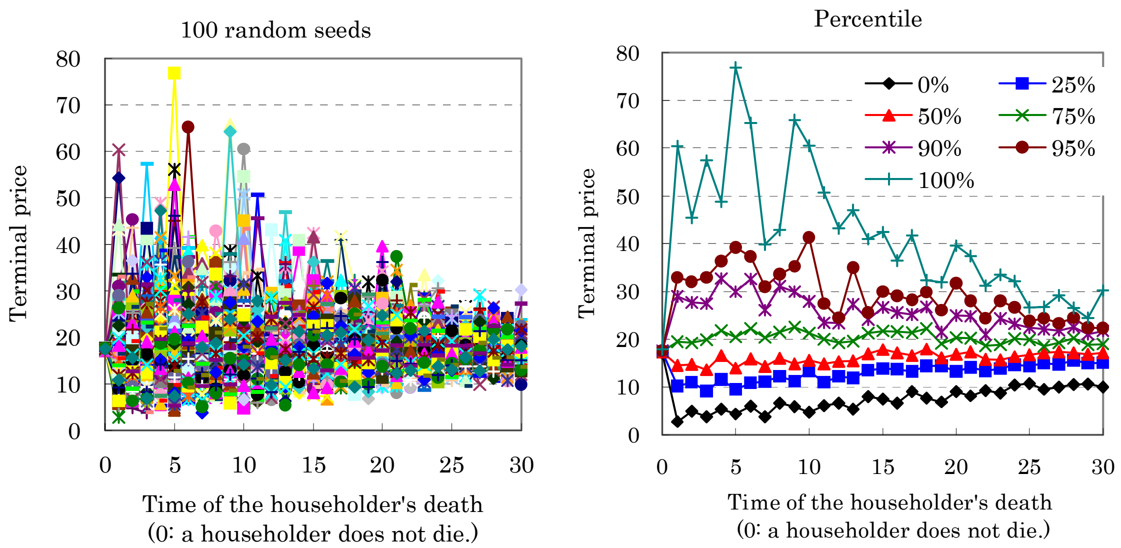


Figure 19: Conditional expected terminal prices at each time of the householder's death

## 6. Concluding Remarks

In this paper, we extend the optimization model for a household in Hibiki and Komoribayashi [5], and examine the models with numerical examples.

A household is exposed to risk associated with payments for the high cost due to the householder's disease in addition to the decrease in cash inflow due to the householder's death, and the decrease in non-financial wealth due to a fire. We consider the associated cash flow, and we propose the model involving medical insurance to hedge risk against payments for the high cost of medical care. We describe cash flow streams in consideration of four parameters and analyze the sensitivity of these parameters : ① coefficient of a disease rate, ② medical cost, ③ decreasing rate of wage income, and ④ death probability after a serious disease. When a disease rate and medical cost are higher, and the decrease in wage income is larger, respectively, the CVaR is lower, the number of medical insurance is larger, and a premium payment is higher. However, the death probability after a disease does not affect these values. Medical cost and the decrease in wage income are the increasing factors of medical insurance money. Even if a disease rate becomes higher, premium payments become higher, but medical insurance money does not increase. Optimal medical insurance money is almost equal to the sum of the decrease in wage income and the amount of medical cost, and therefore it can be said that a medical insurance policy is purchased to cover the loss due to a serious disease.

[17] For example, the number of paths is four at time 1.

The number of units of life insurance with the constant receipt is a decision variable in Hibiki, Komoribayashi and Toyoda [6], and Hibiki and Komoribayashi [5] because the amount of life insurance money is almost constant in practice. Expected terminal financial wealth becomes lower when a householder dies earlier, while it becomes higher than necessary when a householder dies later. As a result, a household has to pay relatively a high premium. It is important for a household not to have a financial problem and to lead a stable life by receiving life insurance money even if a householder dies earlier. We examine the effect of life insurance that a household can receive more insurance money as a householder dies earlier. By purchasing a decreasing type of life insurance, while premium payments are dramatically reduced, a household can receive large conditional terminal financial wealth on average even when a householder dies early, compared with purchasing the constant type.

We had better design life insurance that a household receives life insurance money so that it can fit cash flow needs, and therefore we propose a model to decide optimal life and medical insurance money at each time, instead of constant or decreasing insurance money. We can determine optimal time-dependent life insurance money to tailor cash flow needs of a household, and we find it is an useful model that expected terminal financial wealth are not influenced by the time of the householder's death.

We examine the model with numerical examples to find the characteristics. Sampling error occurs because a thirty-periods model is solved with 5,000 paths. We provide 100 kinds of dataset with different random seeds, and solve the problems. We obtain a sample distribution of optimal solutions. Sampling error occurs, but the average value and the values between 25 percentile and 75 percentile clearly show the feature of the optimal solutions.

The extended model can describe a detail cash flow of a household, compared with the previous models. It derives the optimal insurance and investment strategies appropriately, and we can use the model for giving a financial advice to individual investors.

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