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DUAL FORM OF MARKOV RENEWAL EQUATIONS AND AN APPLICATION TO ASYMPTOTIC ANALYSIS OF A SINGLE-SERVER QUEUE

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Abstract We propose dual form of Markov renewal equations. While the dual form is theoretically equivalent to the classical standard form of Markov renewal equation, it is shown to be a useful tool for analysis of recent stochastic models. To demonstrate the power of the dual form of Markov renewal equations, we apply one to tail asymptotic analysis of the stationary workload distribution for a single-server queue, where its arrival process is governed by a countable-state Markov chain. This application extends the existing result for the case with a finite-state Markov chain. We find that our approach with the dual form of Markov renewal equation gives a more straightforward proof than those in the previous works and makes the extension simple.

Keywords: Applied probability, Markov renewal equations, single-server queues, asymptotic analysis, Markovian arrival streams, stationary workload distribution.

1. Introduction

Theory of Markov renewal equations is well known to be a powerful tool for studying many stochastic models (see, e.g., Çinlar [2] and Asmussen [1]). A standard form of Markov renewal equation is given by $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{R} * \boldsymbol{f}$, which is the matrix expression of

$$f_i(x) = g_i(x) + \sum_{j\in E} \int_0^x \mathrm{d}R_{i,j}(y)\, f_j(x-y), \quad i \in E,\ x \ge 0, \tag{1}$$

where E is a finite or countable set called a state space, $\boldsymbol{f} = (f_i; i \in E)$ is an unknown vector-valued function on $\mathbb{R}_+ = [0,\infty)$, $\boldsymbol{g} = (g_i; i \in E)$ is a known vector-valued function on $\mathbb{R}_+$ and $\boldsymbol{R} = (R_{i,j}; i, j \in E)$ is a known matrix-valued function on $\mathbb{R}_+$ such that each $R_{i,j}$, $i, j \in E$, is nonnegative, nondecreasing and right-continuous with $R_{i,j}(\infty) = \lim_{x\to\infty} R_{i,j}(x) < \infty$. In this paper, we consider instead of (1) a dual form of Markov renewal equation $\boldsymbol{\phi} = \boldsymbol{\xi} + \boldsymbol{\phi} * \boldsymbol{R}$, which is the matrix expression of

$$\phi_j(x) = \xi_j(x) + \sum_{i\in E} \int_0^x \phi_i(x-y)\, \mathrm{d}R_{i,j}(y), \quad j \in E,\ x \ge 0, \tag{2}$$

where $\boldsymbol{\phi} = (\phi_j; j \in E)$ is an unknown vector-valued function on $\mathbb{R}_+$ and $\boldsymbol{\xi} = (\xi_j; j \in E)$ is a known vector-valued function on $\mathbb{R}_+$ (throughout the paper, we use boldface Latin lowercase letters for column vectors, boldface Greek lowercase letters for row vectors and boldface uppercase letters for matrices, except for cases where it is otherwise specified). One may assert that (2) is just a transposition of (1) and we have no argument against this claim. Indeed, in the next section, we can see without any difficulty that (1) and (2) are theoretically equivalent and the results for the dual form of Markov renewal equations follow from the corresponding ones for the standard form. The reason for proposing the dual form (2) does not lie in its own theoretical aspect but in its utility for applications. Namely, a motivation of this work arises from recent development of the study of stochastic models with Markovian environment, where a performance index is often provided as a row vector $\boldsymbol{\phi}(x)$, $x \geq 0$, such that its jth element $\phi_j(x)$ denotes the steady-state joint probability that some performance quantity is greater (or smaller) than $x \geq 0$ and the underlying Markov chain is in state $j \in E$. In such a case, the dual form (2) is sometimes more applicable than (1) and gives a straightforward approach.

To demonstrate the power of the dual form of Markov renewal equations, we apply the limit result for the solution of (2) to tail asymptotic analysis of the stationary workload distribution for a single-server queue, where its arrival process is governed by a countable-state Markov chain. In the case where the Markov chain governing the arrival process has a finite state space, the corresponding asymptotic result was provided by Takine [12] and Miyazawa [6]. Indeed, it would be possible to extend their proofs to the case with a countable state space. However, in the proof by Takine [12], we have to assume existence of the limits in advance since his proof relies on the Tauberian theorem (see, e.g., Feller [3, Chapter XIII]). On the other hand, Miyazawa [6] first obtained the asymptotic result for ruin probability of the corresponding risk process by applying a standard form of Markov renewal equation and then derived from it the asymptotics of the queue by introducing a time-reversed process. As other related works, Miyazawa [7] and Miyazawa and Zhao [8] provided tail asymptotics of the queue-length distributions for some classes of queues with countable underlying state spaces, where standard form of discrete-time Markov renewal equations and time-reversed processes played key roles in the analysis. While their results are very interesting, the procedures to handle the countable-state underlying Markov chains seem complicated. In the current paper, we show that a dual form of Markov renewal equation gives a simple and straightforward approach to the extension of the result in [6, 12], where we no longer resort to the Tauberian theorem or the time-reversed process.

The paper is organized as follows. In the next section, we describe some results for the dual form of Markov renewal equation (2), all of which have the corresponding results for the standard form (1). Both cases where $\boldsymbol{R}(\infty)$ is stochastic and where it is not necessarily stochastic are concerned there. In Section 3, we then apply the limit results for the solution of (2) to tail asymptotic analysis of the stationary workload distribution for a single-server queue, where its arrival process is governed by a countable-state Markov chain. We can see that the extension of the existing result is verified directly by applying a dual form of Markov renewal equation. Finally, Section 4 makes a concluding remark.

2. Dual Form of Markov Renewal Equations

Throughout this section, we impose the following assumption on matrix-valued function $\boldsymbol{R}$ in (2).

Assumption 2.1 (i) $\boldsymbol{R}(\infty)$ is irreducible and aperiodic; that is, there exists a positive integer n_0 such that, for any $n \geq n_0$, $\boldsymbol{R}(\infty)^n$ is strictly positive.

(ii) $\boldsymbol{R}$ is nonlattice in the sense that *every* $R_{i,j}$, $i, j \in E$ such that $R_{i,j}(\infty) > 0$, is *not* a step function with jumps only on the set $\{\delta_{i,j}, \delta_{i,j} + \delta, \delta_{i,j} + 2\,\delta, \ldots\}$ for some $\delta_{i,j} \geq 0$ and $\delta > 0$.

For convenience, we further suppose that $R_{i,j}(0) = 0$ for each pair $(i,j) \in E \times E$. When $\boldsymbol{R}(\infty)$ is stochastic, we say that the Markov renewal equation is proper. We consider the proper case first and then extend the results to the case where (2) is not necessarily proper.

2.1. Proper case

In the proper Markov renewal equation, $\mathrm{d}\boldsymbol{R}$ is interpreted as the semi-Markov kernel on $\mathbb{R}_+$ of a Markov renewal point process $\{(T_n, M_n)\}_{n\in\mathbb{Z}}$, where $\{M_n\}_{n\in\mathbb{Z}}$ is a discrete-time Markov chain on E driven by transition matrix $\boldsymbol{R}(\infty)$ and $\mathrm{P}(T_{n+1} - T_n \le x \mid M_n = i,\ M_{n+1} = j) = R_{i,j}(x)/R_{i,j}(\infty)$ for $n \in \mathbb{Z}$, $x \ge 0$ and $i,j \in E$ such that $R_{i,j}(\infty) > 0$. When state space E is finite, it is known that Markov chain $\{M_n\}_{n\in\mathbb{Z}}$ is positive recurrent under Assumption 2.1(i). When E is countable, we further impose the following.

Assumption 2.2 Each state of Markov chain $\{M_n\}_{n\in\mathbb{Z}}$ is recurrent.

Under Assumption 2.1(i), and Assumption 2.2 when E is countable, Markov chain $\{M_n\}_{n\in\mathbb{Z}}$ has a unique (up to a multiplicative factor) invariant measure, which is a positive solution $\boldsymbol{\nu} = (\nu_i; i \in E)$ of $\boldsymbol{\nu} = \boldsymbol{\nu}\,\boldsymbol{R}(\infty)$. In this case, return times to any fixed state of E in Markov renewal process $\{(T_n, M_n)\}_{n\in\mathbb{Z}}$ form a nonterminating renewal process, and by further imposing Assumption 2.1(ii), the distribution for inter-return times to any fixed state is nonlattice (see Proposition 2.27 in [2, Chapter 10]). Let $\boldsymbol{m} = \int_0^\infty x\,\mathrm{d}\boldsymbol{R}(x)\,\boldsymbol{e}$, where $\boldsymbol{e}$ denotes the column vector such that each element is equal to unity; that is, let m_i denote the mean sojourn time of each visit to state $i \in E$. Then, the mean inter-return time to state $i \in E$ is given by $\boldsymbol{\nu}\,\boldsymbol{m}/\nu_i$, which is possibly infinite when E is countable (see, e.g., [1, Chapter VII] or [2, Chapter 10]).

Remark 2.1 Under Assumption 2.1(i), and Assumption 2.2 when E is countable, let a matrix-valued function $\widetilde{\boldsymbol{R}} = (\widetilde{R}_{i,j}; i,j \in E)$ such that $\widetilde{\boldsymbol{R}}(x) = \mathrm{diag}(\boldsymbol{\nu})^{-1}\,\boldsymbol{R}^T(x)\,\mathrm{diag}(\boldsymbol{\nu})$ for $x \ge 0$, where $\mathrm{diag}(\boldsymbol{a})$ for $\boldsymbol{a} = (a_i; i \in E)$ denotes the diagonal matrix whose ith diagonal element is a_i and superscript "T" denotes the transposition; that is, $\widetilde{R}_{i,j}(x) = (\nu_j/\nu_i)\,R_{j,i}(x)$ for $i,j \in E$ and $x \ge 0$. Then, $\widetilde{\boldsymbol{R}}(\infty) = \lim_{x\to\infty} \widetilde{\boldsymbol{R}}(x)$ is also a stochastic matrix with invariant measure $\boldsymbol{\nu}$, and letting further $\widetilde{\boldsymbol{\phi}} = \mathrm{diag}(\boldsymbol{\nu})^{-1}\,\boldsymbol{\phi}^T$ and $\widetilde{\boldsymbol{\xi}} = \mathrm{diag}(\boldsymbol{\nu})^{-1}\,\boldsymbol{\xi}^T$, the dual form of Markov renewal equation (2) is converted to the standard form $\widetilde{\boldsymbol{\phi}} = \widetilde{\boldsymbol{\xi}} + \widetilde{\boldsymbol{R}} * \widetilde{\boldsymbol{\phi}}$. This conversion is nothing but considering the time-reversed process $\{(\widetilde{T}_n, \widetilde{M}_n)\}_{n\in\mathbb{Z}} = \{(-T_{-n}, M_{-n})\}_{n\in\mathbb{Z}}$ and indicates that (1) and (2) are theoretically equivalent; that is, all statements in this section follow from the corresponding ones for the standard form. Also, even without such a conversion, they can be confirmed directly as it is in the dual form by applying similar procedures.

First, we give the solution of (2). Let $\boldsymbol{U} = \sum_{n=0}^\infty \boldsymbol{R}^{*n}$, where $\boldsymbol{R}^{*n}$, $n = 0, 1, 2, \ldots$, are defined inductively by $\boldsymbol{R}^{*0} \equiv \boldsymbol{I}$ (identity matrix) and

$$R^{*n}_{i,j}(x) = \sum_{k\in E} \int_0^x \mathrm{d}R_{i,k}(y)\,R^{*(n-1)}_{k,j}(x-y), \quad i,j \in E,\ x \ge 0,\ n = 1, 2, \ldots. \tag{3}$$

Then, we have the following.

Lemma 2.1 *Let $\boldsymbol{R}(\infty)$ be stochastic and let ξ_i be nonnegative and bounded on finite intervals uniformly in $i \in E$. Then, under Assumption 2.1(i), and Assumption 2.2 when E is countable, the dual form of Markov renewal equation (2) has a unique solution $\boldsymbol{\phi} = \boldsymbol{\xi} * \boldsymbol{U}$, which is also bounded on finite intervals uniformly on E.*

This lemma corresponds to the result that the unique solution of (1) is given by $\boldsymbol{f} = \boldsymbol{U}*\boldsymbol{g}$ under the same condition (see, e.g., Proposition 4.4 in [1, Chapter VII]). In the case where state space E is finite, we can easily obtain the following result which also corresponds to Proposition 4.9 in [2, Chapter 10] for the standard form.

Proposition 2.1 *Suppose that E is finite and $\boldsymbol{R}(\infty)$ is stochastic. If each ξ_i, $i \in E$, is nonnegative and directly Riemann integrable (see, e.g., [1, 2] for its definition), then under Assumption 2.1,*

$$\lim_{x\to\infty} \phi_j(x) = \frac{\nu_j}{\boldsymbol{\nu}\,\boldsymbol{m}} \int_0^\infty \boldsymbol{\xi}(x)\,\boldsymbol{e}\,\mathrm{d}x, \quad j \in E. \tag{4}$$

Remark 2.2 In the standard Markov renewal equation (1), the limit of solution $\boldsymbol{f}$ is given by

$$\lim_{x\to\infty} f_i(x) = \frac{1}{\boldsymbol{\nu}\,\boldsymbol{m}} \int_0^\infty \boldsymbol{\nu}\,\boldsymbol{g}(x)\,\mathrm{d}x, \quad i \in E, \tag{5}$$

provided that each g_i, $i \in E$, is nonnegative and directly Riemann integrable (see, e.g., Proposition 4.9 in [2, Chapter 10]). In (5), we can see that the limit of $f_i(x)$ as $x \to \infty$ is invariant of $i \in E$. In contrast, (4) shows that the limit of ϕ_j is proportional to the steady-state probability that underlying Markov chain $\{M_n\}_{n\in\mathbb{Z}}$ is in state $j \in E$.

Next, we extend Proposition 2.1 to the case with a countable state space. For a nonnegative vector-valued function $\boldsymbol{\xi} = (\xi_j; j \in E)$ on $\mathbb{R}_+$, a nonnegative vector $\boldsymbol{h} = (h_j; j \in E)$ and any $b > 0$, let

$$\overline{\sigma}_b(\boldsymbol{\xi}, \boldsymbol{h}) = b \sum_{n\in\mathbb{Z}_+} \sum_{j\in E} h_j \sup_{x\in[nb,(n+1)b)} \xi_j(x),$$
$$\underline{\sigma}_b(\boldsymbol{\xi}, \boldsymbol{h}) = b \sum_{n\in\mathbb{Z}_+} \sum_{j\in E} h_j \inf_{x\in[nb,(n+1)b)} \xi_j(x).$$

Then, we say that $\boldsymbol{\xi}$ is *directly integrable associated with* $\boldsymbol{h}$ if $\overline{\sigma}_b(\boldsymbol{\xi}, \boldsymbol{h})$ is finite for any $b > 0$ and $\overline{\sigma}_b(\boldsymbol{\xi}, \boldsymbol{h}) - \underline{\sigma}_b(\boldsymbol{\xi}, \boldsymbol{h}) \to 0$ as $b \downarrow 0$.

Proposition 2.2 *Suppose that state space E is countable and $\boldsymbol{R}(\infty)$ is stochastic. If each ξ_i, $i \in E$, is nonnegative and $\boldsymbol{\xi}$ is directly integrable associated with $\boldsymbol{e}$, then (4) holds under Assumption 2.1 and 2.2, where $1/(\boldsymbol{\nu}\,\boldsymbol{m}) = 0$ if $\boldsymbol{\nu}\,\boldsymbol{m} = \infty$ conventionally.*

Remark 2.3 In the standard form (1) with a countable state space, a condition under which (5) holds is known that $\boldsymbol{g}$ is directly integrable with respect to invariant measure $\boldsymbol{\nu}$ (see, e.g., Proposition 4.17 in [2, Chapter 10]). Following the terminology in [2], the condition that $\boldsymbol{\xi}$ is directly integrable associated with $\boldsymbol{e}$ is rephrased by that $\boldsymbol{\xi}^T$ is directly integrable with respect to uniform measure $\boldsymbol{e}^T$, which is also equivalent to that $\widetilde{\boldsymbol{\xi}}$ in Remark 2.1 is directly integrable with respect to $\boldsymbol{\nu}$.

As seen in Remark 2.3, it is clear that Proposition 2.2 follows from the corresponding result for the standard form (Proposition 4.17 in [2, Chapter 10]) by considering the time-reversed process in Remark 2.1. However, we here give a direct proof for completeness without such a time-reverse conversion.

Proof: Suppose that $T_0 = 0$ almost surely; that is, there is a point of the Markov renewal process $\{(T_n, M_n)\}_{n\in\mathbb{Z}}$ at the origin with probability one. Note that the (i,j)th element $U_{i,j}(y)$, $i,j \in E$, of $\boldsymbol{U}(y)$, $y \ge 0$, represents the expected number of visits to state j during $[0,y]$ when Markov renewal process $\{(T_n, M_n)\}_{n\in\mathbb{Z}}$ is in state i at time 0. Let $H_{i,j}$, $i,j \in E$, denote the distribution function for the first hitting time from state i to state j; that is,

$$H_{i,j}(x) = \mathrm{P}\Bigl(\min_{n\ge 0}\{T_n : M_n = j\} \le x \bigm| M_0 = i\Bigr), \quad i,j \in E,\ x \ge 0,$$

where it should be noted that $H_{i,i}$, $i \in E$, is the degenerated distribution at the origin. Then, the usual renewal argument yields that

$$U_{i,j}(y) = \int_0^y \mathrm{d}H_{i,j}(w)\, U_{j,j}(y-w), \quad i,j \in E,\ y \ge 0.$$

Applying this to the solution $\boldsymbol{\phi} = \boldsymbol{\xi} * \boldsymbol{U}$ in Lemma 2.1, we have

$$\begin{aligned}
\phi_j(x) &= \sum_{i\in E} \int_0^x \xi_i(x-y) \int_0^y \mathrm{d}H_{i,j}(w)\,\mathrm{d}U_{j,j}(y-w) \\
&= \sum_{i\in E} \int_0^x \xi_i(x-y)\,\mathrm{d}H_{i,j}(w) \int_w^x \mathrm{d}U_{j,j}(y-w) \\
&= \sum_{i\in E} \int_0^x \xi_i(x-y-w)\,\mathrm{d}H_{i,j}(w) \int_0^{x-w} \mathrm{d}U_{j,j}(y) \\
&= \sum_{i\in E} \int_0^x \int_0^{x-y} \xi_i(x-y-w)\,\mathrm{d}H_{i,j}(w)\,\mathrm{d}U_{j,j}(y),
\end{aligned} \tag{6}$$

where the second and fourth equalities follow from Fubini's theorem. In order to apply the key renewal theorem to (6), we have to show that $\sum_{i\in E}\int_0^x \xi_i(x-w)\,\mathrm{d}H_{i,j}(w)$ is directly Riemann integrable. However, this follows from Proposition 4.15 in [2, Chapter 10] due to Remark 2.3, and we have

$$\begin{aligned}
\lim_{x\to\infty} \phi_j(x) &= \frac{\nu_j}{\boldsymbol{\nu}\,\boldsymbol{m}} \int_0^\infty \sum_{i\in E} \int_0^x \xi_i(x-w)\,\mathrm{d}H_{i,j}(w)\,\mathrm{d}x \\
&= \frac{\nu_j}{\boldsymbol{\nu}\,\boldsymbol{m}} \sum_{i\in E} \int_0^\infty \int_w^\infty \xi_i(x-w)\,\mathrm{d}x\,\mathrm{d}H_{i,j}(w) \\
&= \frac{\nu_j}{\boldsymbol{\nu}\,\boldsymbol{m}} \sum_{i\in E} \int_0^\infty \xi_i(x)\,\mathrm{d}x,
\end{aligned}$$

where the last equality follows since $H_{i,j}$ is a proper distribution under Assumptions 2.1(i) and 2.2. □

2.2. Non-proper case

This subsection does not necessarily suppose that Markov renewal equation (2) is proper; that is, we admit that $\boldsymbol{R}(\infty)$ is not stochastic. Let $\widehat{\boldsymbol{R}}(\theta)$ denote the moment-generating function of $\boldsymbol{R}$ for a real number θ; that is, $\widehat{\boldsymbol{R}}(\theta) = \int_0^\infty e^{\theta x}\,\mathrm{d}\boldsymbol{R}(x)$. Clearly, $\widehat{\boldsymbol{R}}(\theta)$ always exists for $\theta \le 0$. We can also consider the case where $\boldsymbol{R}$ has light tails; that is, there is a $\theta_0 > 0$ such that $\widehat{\boldsymbol{R}}(\theta)$ exists for any $\theta < \theta_0$, where θ_0 is possibly infinite.

First, we consider the case where state space E is finite. Since $\widehat{\boldsymbol{R}}(\theta)$ and $\boldsymbol{R}(\infty)$ have the same incidence matrix, $\widehat{\boldsymbol{R}}(\theta)$ is, if exists, also irreducible and aperiodic under Assumptions 2.1(i). Thus, by the Perron-Frobenius theory (see, e.g., Seneta [11, Chapter 1]), $\widehat{\boldsymbol{R}}(\theta)$ has a positive eigenvalue $\delta(\theta)$ that dominates the real parts of all other eigenvalues and the left and right eigenvectors associated with $\delta(\theta)$ are strictly positive. Denote these eigenvectors by $\boldsymbol{\eta}(\theta)$ and $\boldsymbol{h}(\theta)$ respectively. Now, we impose the following.

Assumption 2.3 There exists an $\alpha \in \mathbb{R}$ such that $\delta(\alpha) = 1$.

A condition under which α in Assumption 2.3 exists is found in Problem 4.3 in [1, Chapter VII] and, in the case where $\boldsymbol{R}(\infty)$ is strictly substochastic and $\boldsymbol{R}$ has light tails, a condition under which the positive α exists is discussed in [6]. Using α in Assumption 2.3, we define an $E \times E$-matrix-valued function $\boldsymbol{R}^\dagger$ on $\mathbb{R}_+$ by $\boldsymbol{R}^\dagger(x) = \mathrm{diag}(\boldsymbol{h}(\alpha))^{-1} \int_0^x e^{\alpha y}\, \mathrm{d}\boldsymbol{R}(y)\, \mathrm{diag}(\boldsymbol{h}(\alpha))$ for $x \geq 0$; that is,

$$R^\dagger_{i,j}(x) = \frac{h_j(\alpha)}{h_i(\alpha)} \int_0^x e^{\alpha y}\, \mathrm{d}R_{i,j}(y), \quad i,j \in E,\ x \geq 0.$$

Then, we can see that $\boldsymbol{R}^\dagger(\infty) = \lim_{x\to\infty} \boldsymbol{R}^\dagger(x)$ is stochastic and $\boldsymbol{R}^\dagger$ is also nonlattice under Assumption 2.1(ii). An invariant measure of $\boldsymbol{R}^\dagger(\infty)$ is given by $\boldsymbol{\nu}^\dagger = \boldsymbol{\eta}(\alpha)\, \mathrm{diag}(\boldsymbol{h}(\alpha))$ and the vector of mean sojourn times is given by $\boldsymbol{m}^\dagger = \mathrm{diag}(\boldsymbol{h}(\alpha))^{-1}\, \widehat{\boldsymbol{R}}^{(1)}(\alpha)\, \boldsymbol{h}(\alpha)$, where $\widehat{\boldsymbol{R}}^{(1)}(\alpha) = \mathrm{d}\widehat{\boldsymbol{R}}(\theta)/\mathrm{d}\theta\big|_{\theta=\alpha}$; that is,

$$m^\dagger_i = \frac{1}{h_i(\alpha)} \sum_{j\in E} \int_0^\infty x\, e^{\alpha x}\, \mathrm{d}R_{i,j}(x)\, h_j(\alpha), \quad i \in E.$$

Define vector-valued functions $\boldsymbol{\phi}^\dagger$ and $\boldsymbol{\xi}^\dagger$ on $\mathbb{R}_+$ respectively by $\boldsymbol{\phi}^\dagger(x) = e^{\alpha x}\, \boldsymbol{\phi}(x)\, \mathrm{diag}(\boldsymbol{h}(\alpha))$ and $\boldsymbol{\xi}^\dagger(x) = e^{\alpha x}\, \boldsymbol{\xi}(x)\, \mathrm{diag}(\boldsymbol{h}(\alpha))$ for $x \geq 0$. From (2), we then have a proper Markov renewal equation $\boldsymbol{\phi}^\dagger = \boldsymbol{\xi}^\dagger + \boldsymbol{\phi}^\dagger * \boldsymbol{R}^\dagger$ in the dual form, and by applying Proposition 2.1, we readily obtain the following result which is also a dual of Theorem 4.6 in [1, Chapter VII].

Proposition 2.3 *Suppose that state space E is finite. Under Assumptions 2.1 and 2.3, if each ξ_i, $i \in E$, is nonnegative and $e^{\alpha x}\, \xi_i(x)$ is directly Riemann integrable, then*

$$\lim_{x\to\infty} e^{\alpha x}\, \phi_j(x) = \frac{\eta_j(\alpha)}{\boldsymbol{\eta}(\alpha)\, \widehat{\boldsymbol{R}}^{(1)}(\alpha)\, \boldsymbol{h}(\alpha)} \int_0^\infty e^{\alpha x}\, \boldsymbol{\xi}(x)\, \mathrm{d}x\, \boldsymbol{h}(\alpha), \quad j \in E. \tag{7}$$

Remark 2.4 α in Assumption 2.3 is, of course, equal to zero in the proper case and (7) in Proposition 2.3 then reduces to (4) in Proposition 2.1.

Next, we consider the case where state space E is countable. We here suppose that $\boldsymbol{R}^{*n}$ defined in (3) exists and is elementwise finite on $\mathbb{R}_+$ for each finite $n = 0, 1, 2, \ldots$. This is clearly the case if $\boldsymbol{R}(\infty)$ is substochastic (including stochastic). Otherwise, this is the case, for instance, if row sums of $\boldsymbol{R}(\infty)$ are uniformly bounded on E; that is, there exists a constant $c > 0$ such that $\sum_{j\in E} R_{i,j}(\infty) \leq c$ for all $i \in E$. Indeed, in this case, we have $R^{*n}_{i,j}(x) \leq c^n$ for $i, j \in E$ and $x \geq 0$. Let $\widehat{\boldsymbol{R}^{*n}}(\theta)$ denote the moment-generating function of $\boldsymbol{R}^{*n}$ for $\theta \in \mathbb{R}$. If $\theta \leq 0$, $\widehat{\boldsymbol{R}^{*n}}(\theta)$ exists whenever $\boldsymbol{R}^{*n}$ does. If there is a $\theta_0 > 0$ such that $\widehat{\boldsymbol{R}^{*n}}(\theta)$ exists for any $\theta < \theta_0$, we say that $\boldsymbol{R}^{*n}$ has light tails, where

θ_0 is possibly infinite. Note here that $\widehat{\boldsymbol{R}^{*n}}(\theta) = \widehat{\boldsymbol{R}}(\theta)^n$, $n = 0, 1, 2, \ldots$, which can be checked inductively. Therefore, if $\widehat{\boldsymbol{R}^{*n}}(\theta)$ exists for each finite $n = 0, 1, 2, \ldots$, there exists the convergence parameter $\Delta(\theta) \geq 0$ such that $\sum_{n=0}^{\infty} \widehat{R_{i,j}^{*n}}(\theta)\, z^n < \infty$ for each pair $(i, j) \in E \times E$ and any $z \in [0, \Delta(\theta))$ (see, e.g., [11, Chapter 6] or [15]). Now, we impose the following assumption.

Assumption 2.4 There exists an $\alpha \in \mathbb{R}$ such that $\Delta(\alpha) = 1$. For this $\alpha \in \mathbb{R}$, $\widehat{\boldsymbol{R}}(\alpha)$ is 1-recurrent; that is, $\sum_{n=0}^{\infty} \widehat{R_{i,i}^{*n}}(\alpha) = +\infty$ for each $i \in E$ (see, e.g., [11, Chapter 6]).

For instance, in the case where $\boldsymbol{R}(\infty)$ is strictly substochastic, we have $\Delta(0) > 1$. Thus, in this case, if $\boldsymbol{R}$ has light tails with $\theta_0 > 0$, α in the first part of Assumption 2.4 exists and is positive since Δ is continuous and $\Delta(\theta) \to 0$ as $\theta \uparrow \theta_0$ (indeed, $1/\Delta(\theta) = \lim_{n\to\infty} \big[\widehat{R_{i,i}^{*n}}(\theta)\big]^{1/n}$ for any $i \in E$ by Theorem 6.1 in [11, Chapter 6] and this is log-convex by Kingman [4]). Otherwise, if $\Delta(0) < 1$, then the negative α in the first part of Assumption 2.4 exists since $\Delta(\theta) \to \infty$ as $\theta \to -\infty$. Under Assumptions 2.1(i) and 2.4, $\widehat{\boldsymbol{R}}(\alpha)$ has unique (up to a multiplicative factor) 1-invariant measure $\boldsymbol{\eta}(\alpha)$ and 1-invariant vector $\boldsymbol{h}(\alpha)$ satisfying $\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{R}}(\alpha) = \boldsymbol{\eta}(\alpha)$ and $\widehat{\boldsymbol{R}}(\alpha)\,\boldsymbol{h}(\alpha) = \boldsymbol{h}(\alpha)$ respectively, where all elements of $\boldsymbol{\eta}(\alpha)$ and $\boldsymbol{h}(\alpha)$ are strictly positive (see, e.g., [11, Chapter 6]). Hence, we can construct the "†-system" as in the case with a finite state space and obtain the following from Proposition 2.2.

Proposition 2.4 *Suppose that state space E is countable. Under Assumptions 2.1 and 2.4, if each ξ_i, $i \in E$ is nonnegative and $e^{\alpha x}\,\boldsymbol{\xi}(x)$ is directly integrable associated with $\boldsymbol{h}(\alpha)$, then (7) holds, where the right-hand side of (7) is equal to zero conventionally if the denominator is infinite.*

Remark 2.5 As in the case with a finite state space, α in Assumption 2.4 is equal to zero in the proper case and the second part of Assumption 2.4 is then equivalent to Assumption 2.2, so that Proposition 2.4 reduces to Proposition 2.2.

3. Application to Asymptotic Analysis of Single-server Queue

In this section, we apply Proposition 2.4 in the preceding section to tail asymptotic analysis of the stationary workload distribution for a single-server queue with a Markovian arrival stream. The result extends the existing one to the case where the Markov chain governing the arrival process has a countable state space. We will find that our approach with a dual form of Markov renewal equation gives a straightforward proof without considering the time-reversed process, while a few problems arise in the extension. Throughout this section, E is a countable set.

Consider a work-conserving single-server queue with an infinite-size buffer. Customer arrivals and their service times are supposed to follow a Markovian arrival stream with representation $(\boldsymbol{C}, \boldsymbol{D})$, where $\boldsymbol{C}$ denotes an $E \times E$-matrix with negative diagonal elements and nonnegative off-diagonal elements, and $\boldsymbol{D}$ denotes an $E \times E$-matrix-valued function on $\mathbb{R}_+$ such that its (i, j)th element $D_{i,j}$, $i, j \in E$, is nonnegative, nondecreasing and right-continuous with $D_{i,j}(\infty) = \lim_{x\to\infty} D_{i,j}(x) < \infty$. Matrix $\boldsymbol{C} + \boldsymbol{D}(\infty)$ is a rate matrix; that is, $(\boldsymbol{C} + \boldsymbol{D}(\infty))\,\boldsymbol{e} = \boldsymbol{0}$, where $\boldsymbol{0}$ denotes the column vector such that each element is equal to zero. We suppose that the diagonal elements of $\boldsymbol{C}$ are uniformly bounded below (so that rate matrix $\boldsymbol{C} + \boldsymbol{D}(\infty)$ is uniformizable). The continuous-time Markov chain driven by rate matrix $\boldsymbol{C} + \boldsymbol{D}(\infty)$ is assumed to be irreducible and positive recurrent and its stationary distribution is denoted by $\boldsymbol{\pi}$; that is, $\boldsymbol{\pi}\,(\boldsymbol{C} + \boldsymbol{D}(\infty)) = \boldsymbol{0}^T$ and $\boldsymbol{\pi}\,\boldsymbol{e} = 1$. We refer to this stationary Markov chain as the underlying Markov chain. When a state transition driven by $\boldsymbol{D}(\infty)$ (including one not changing the current state driven by $D_{i,i}(\infty)$, $i \in E$) occurs, a customer arrives to the queue, where we suppose that there exists at least one pair $(i,j) \in E \times E$ such that $D_{i,j}(\infty) > 0$, so that arrivals are certain. Service times of customers whose arrivals are driven by $D_{i,j}(\infty) > 0$ are independent and identically distributed according to distribution $D_{i,j}(x)/D_{i,j}(\infty)$, $x \geq 0$. For convenience, we suppose that $D_{i,j}(0) = 0$ for each pair $(i,j) \in E \times E$. Traffic intensity ρ of the queue is given by $\rho = \boldsymbol{\pi} \int_0^\infty x \, \mathrm{d}\boldsymbol{D}(x) \, \boldsymbol{e}$, which is assumed to be less than unity, and thus, the queue is stable (see, e.g., Loynes [5]).

Let V denote the workload in the system in the steady state and let M denote the associated state of the underlying Markov chain. We consider vector-valued function $\boldsymbol{\phi}$ on $\mathbb{R}_+$, whose jth element gives the stationary joint probability $\phi_j(x) = \mathrm{P}(V > x, M = j)$, $j \in E$, $x \geq 0$. Then, extending the result by Takine [13] to the case with a countable underlying state space, we obtain the following.

Lemma 3.1 *Vector-valued tail distribution $\boldsymbol{\phi}$ satisfies a dual form of Markov renewal equation;*

$$\boldsymbol{\phi}(x) = \boldsymbol{\pi} \, \overline{\boldsymbol{R}}(x) + \int_0^x \boldsymbol{\phi}(x-y) \, \mathrm{d}\boldsymbol{R}(y), \quad x \geq 0, \tag{8}$$

where

$$\boldsymbol{R}(x) = \int_0^x \mathrm{d}y \int_y^\infty \mathrm{d}\boldsymbol{D}(w) \, e^{\boldsymbol{Q}(w-y)}, \quad x \geq 0, \tag{9}$$

$\overline{\boldsymbol{R}}(x) = \boldsymbol{R}(\infty) - \boldsymbol{R}(x)$*, $x \geq 0$, and $\boldsymbol{Q}$ is given as an $E \times E$-matrix satisfying*

$$\boldsymbol{Q} = \boldsymbol{C} + \int_0^\infty \mathrm{d}\boldsymbol{D}(x) \, e^{\boldsymbol{Q}x}. \tag{10}$$

Following the argument by Takine and Hasegawa [14], where a finite-state underlying Markov chain is concerned, we can show that, if the diagonal elements of $\boldsymbol{C}$ are uniformly bounded below, then the diagonal elements of $\boldsymbol{Q}$ in (10) are also uniformly bounded below and matrix $e^{\boldsymbol{Q}x}$ is well defined even when E is countable. Furthermore, as in the case with finite E, we can show that, when $\boldsymbol{C} + \boldsymbol{D}(\infty)$ is an irreducible and positive recurrent rate matrix, so is matrix $\boldsymbol{Q}$ if $\rho < 1$, while $\boldsymbol{Q}\,\boldsymbol{e} < \boldsymbol{0}$ if $\rho > 1$. In the proof of Lemma 3.1 below and thereafter, $\boldsymbol{\kappa}$ denotes the stationary distribution for $\boldsymbol{Q}$; that is, $\boldsymbol{\kappa}\,\boldsymbol{Q} = \boldsymbol{0}^T$ and $\boldsymbol{\kappa}\,\boldsymbol{e} = 1$.

Proof: Let Y denote the random variable representing the queue length under the preemptive last-come, first-served discipline in the steady state. Define vector-valued distributions $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^{(n)}$, $n = 0, 1, 2, \ldots$, on $\mathbb{R}_+$ such that their jth elements give $\psi_j(x) = \pi_j - \phi_j(x) = \mathrm{P}(V \leq x, M = j)$ and $\psi_j^{(n)}(x) = \mathrm{P}(V \leq x, M = j, Y = n)$ for $j \in E$, $x \geq 0$. In the case with finite E, Takine [13] derives that

$$\boldsymbol{\psi}^{(0)}(x) = (1-\rho)\,\boldsymbol{\kappa}, \quad x \geq 0, \tag{11}$$

$$\boldsymbol{\psi}^{(n)}(x) = \int_0^x \boldsymbol{\psi}^{(n-1)}(x-y) \, \mathrm{d}\boldsymbol{R}(y) \quad x \geq 0, \ n = 1, 2, \ldots, \tag{12}$$

which are also available in the case with countable E (see Remark 3.1 below). Thus, summing up (11) and (12) over $n = 0, 1, 2, \ldots$, we have

$$\boldsymbol{\psi}(x) = (1-\rho)\,\boldsymbol{\kappa} + \int_0^x \boldsymbol{\psi}(x-y) \, \mathrm{d}\boldsymbol{R}(y), \quad x \geq 0. \tag{13}$$

Hence, we have $\boldsymbol{\pi} = (1-\rho)\,\boldsymbol{\kappa} + \boldsymbol{\pi}\,\boldsymbol{R}(\infty)$ by taking $x \to \infty$ in (13) and obtain (8) by $\boldsymbol{\phi}(x) = \boldsymbol{\pi} - \boldsymbol{\psi}(x)$. □

Remark 3.1 Although (11) and (12) are provided in the case with finite E in [13], they are available even in the case with countable E. In fact, the author [9] derives the corresponding formulas in more general setting with marked-point-process arrivals associated with stochastic intensity.

Here, let us provide the moment-generating function of $\boldsymbol{R}$ in (9).

Lemma 3.2 *If $\boldsymbol{D}$ has light tails and its moment-generating function $\widehat{\boldsymbol{D}}(\theta)$ exists for some $\theta > 0$, then the moment-generating function of $\boldsymbol{R}$ in (9) also exists for such $\theta > 0$ and is given by*

$$\widehat{\boldsymbol{R}}(\theta) = (\widehat{\boldsymbol{D}}(\theta) + \boldsymbol{C} - \boldsymbol{Q})\,(\theta\boldsymbol{I} - \boldsymbol{Q})^{-1}. \tag{14}$$

Proof: First, we show that $(\theta\boldsymbol{I}-\boldsymbol{Q})^{-1}$ is well defined and finite for $\theta > 0$. Let $-\mu < 0$ denote a bound of the diagonal elements of $\boldsymbol{C}$ such that $C_{i,i} \ge -\mu$ for all $i \in E$. Then, from (10), $-\mu$ also bounds the diagonal elements of $\boldsymbol{Q}$ from below, so that $\mu\,\boldsymbol{I} + \boldsymbol{Q}$ is a nonnegative, irreducible and $1/\mu$-positive matrix with a $1/\mu$-invariant measure $\boldsymbol{\kappa}$ and a $1/\mu$-invariant vector $\boldsymbol{e}$ (see Theorem 6.4 in [11]). This implies that $1/\mu$ is the convergence parameter of $\mu\,\boldsymbol{I}+\boldsymbol{Q}$; that is, $\sum_{n=0}^{\infty}(\mu\,\boldsymbol{I}+\boldsymbol{Q})^n\,z^n < \infty$ for $z \in [0, 1/\mu)$. Therefore, $\sum_{n=0}^{\infty}(\mu\,\boldsymbol{I}+\boldsymbol{Q})^n\,z^{n+1} = \big((1/z-\mu)\,\boldsymbol{I} - \boldsymbol{Q}\big)^{-1} < \infty$ for $z \in (0, 1/\mu)$; that is, $(\theta\,\boldsymbol{I} - \boldsymbol{Q})^{-1} < \infty$ for $\theta > 0$. Next, from (9), the moment-generating function of $\boldsymbol{R}$ is given by

$$\begin{aligned}
\widehat{\boldsymbol{R}}(\theta) &= \int_0^\infty e^{\theta x} \int_x^\infty \mathrm{d}\boldsymbol{D}(w)\, e^{\boldsymbol{Q}(w-x)}\,\mathrm{d}x \\
&= \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\, e^{\boldsymbol{Q}w} \int_0^w e^{(\theta\boldsymbol{I}-\boldsymbol{Q})x}\,\mathrm{d}x \\
&= \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\,(e^{\theta w}\,\boldsymbol{I} - e^{\boldsymbol{Q}w})\,(\theta\boldsymbol{I} - \boldsymbol{Q})^{-1},
\end{aligned}$$

where Fubini's theorem is used in the second equality. Hence, we obtain (14) by applying (10). □

Let $-\mu < 0$ denote a bound of the diagonal elements of $\boldsymbol{C}$ as in the proof of Lemma 3.2. Then, we can show that $\boldsymbol{D}(\infty)\,\boldsymbol{e} \le \mu\,\boldsymbol{e}$ and the n-fold convolution of $\boldsymbol{D}$ is well defined for each finite $n = 0, 1, 2, \ldots$. Now, define $\boldsymbol{K}(\theta) = \mu\,\boldsymbol{I} + \boldsymbol{C} + \widehat{\boldsymbol{D}}(\theta)$ for a real number θ. Note that $\boldsymbol{K}(\theta)$ is nonnegative whenever it exists. We here impose the following assumption which includes the condition in Lemma 3.2.

Assumption 3.1 There exists a $\theta_0 > 0$ such that $\boldsymbol{K}(\theta)^n$ exists and is elementwise finite for any $\theta < \theta_0$ and each $n = 0, 1, 2, \ldots$, where θ_0 is possibly infinite.

Since $\widehat{\boldsymbol{D}}(\theta)^n = \widehat{\boldsymbol{D}^{*n}}(\theta) = \int_0^\infty e^{\theta x}\,\mathrm{d}\boldsymbol{D}^{*n}(x)$, Assumption 3.1 implies that the convolutions $\boldsymbol{D}^{*n}$, $n = 0, 1, 2, \ldots$, have light tails uniformly in n; that is, $\widehat{\boldsymbol{D}^{*n}}(\theta)$ exists for $\theta < \theta_0$ and each $n = 0, 1, 2, \ldots$. Under Assumption 3.1, $\boldsymbol{K}(\theta)$, $\theta < \theta_0$, has the convergence parameter $\Delta(\theta) \ge 0$ such that $\sum_{n=0}^{\infty}\boldsymbol{K}(\theta)^n\,z^n < \infty$ for any $z \in [0, \Delta(\theta))$ (see, e.g., [11, Chapter 6] or [15]). Note that $\Delta(0) = 1/\mu$ and $\boldsymbol{K}(0) = \mu\,\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}(\infty)$ has a $1/\mu$-invariant measure $\boldsymbol{\pi}$ and a $1/\mu$-invariant vector $\boldsymbol{e}$; that is, $\boldsymbol{K}(0)$ is $1/\mu$-positive.

Lemma 3.3 *Under Assumption 3.1, equation $\Delta(\theta) = 1/(\theta+\mu)$ has at most one solution in $(0,\theta_0)$. In particular, if $\theta_0 < \infty$, it has exactly one solution in $(0,\theta_0)$. This solution does not depend on the choice of the value of μ.*

Proof: Let $K^{(n)}_{i,i}(\theta)$, $i \in E$, denote the (i,i)th element of $\boldsymbol{K}(\theta)^n$. Since $1/\Delta(\theta) = \lim_{n\to\infty}\big[K^{(n)}_{i,i}(\theta)\big]^{1/n}$ for any $i \in E$ (see Theorem 6.1 in [11, Chapter 6]), we can show that $1/\Delta(\theta)$ is nondecreasing and convex in $\theta < \theta_0$ (indeed, it is log-convex by [4]). Since $1/\Delta(0) = \mu$ and $1/\Delta(\theta) \to +\infty$ as $\theta \uparrow \theta_0$, it is sufficient to show that $\mathrm{d}\big(1/\Delta(\theta)\big)/\mathrm{d}\theta\big|_{\theta=0} < 1$. Let $\boldsymbol{\eta}(\theta)$ and $\boldsymbol{r}(\theta)$ respectively denote a $\Delta(\theta)$-subinvariant measure and a $\Delta(\theta)$-subinvariant vector of $\boldsymbol{K}(\theta)$; that is, $\boldsymbol{\eta}(\theta)$ and $\boldsymbol{r}(\theta)$ are both strictly positive and satisfy

$$\Delta(\theta)\,\boldsymbol{\eta}(\theta)\,\boldsymbol{K}(\theta) \le \boldsymbol{\eta}(\theta), \quad \Delta(\theta)\,\boldsymbol{K}(\theta)\,\boldsymbol{r}(\theta) \le \boldsymbol{r}(\theta), \quad \theta < \theta_0. \tag{15}$$

Note that the equalities in (15) hold at $\theta = 0$ with $\Delta(0) = 1/\mu$ and that $\boldsymbol{\eta}(0)$ and $\boldsymbol{r}(0)$ are respectively multiples of $\boldsymbol{\pi}$ and $\boldsymbol{e}$. Since $\boldsymbol{\eta}(\theta)\,\boldsymbol{K}(\theta)\,\boldsymbol{r}(\theta) \le \boldsymbol{\eta}(\theta)\,\boldsymbol{r}(\theta)/\Delta(\theta)$, $\theta < \theta_0$, and the equality holds at $\theta = 0$, their gradients on both sides are also equal at $\theta = 0$, so that we have $\mathrm{d}\big(1/\Delta(\theta)\big)/\mathrm{d}\theta\big|_{\theta=0} = \boldsymbol{\pi}\,\boldsymbol{K}^{(1)}(0)\,\boldsymbol{e} = \boldsymbol{\pi}\,\widehat{\boldsymbol{D}}^{(1)}(0)\,\boldsymbol{e} = \rho < 1$.

To confirm the last assertion, it suffices to show that, when μ is replaced by $\mu+\epsilon$, then $1/\Delta(\theta)$ is shifted to $1/\Delta(\theta)+\epsilon$. Note that $\sum_{n=0}^{\infty}\boldsymbol{K}(\theta)^n z^n = \big(\boldsymbol{I} - z\,\boldsymbol{K}(\theta)\big)^{-1} < \infty$ for $z \in [0,\Delta(\theta))$. Here, we have for $z \in [0,\Delta(\theta))$,

$$\begin{aligned}\big(\boldsymbol{I} - z\,\boldsymbol{K}(\theta)\big)^{-1} &= \frac{1}{1+z\,\epsilon}\left(\boldsymbol{I} - \frac{z}{1+z\,\epsilon}\left(\epsilon\,\boldsymbol{I} + \boldsymbol{K}(\theta)\right)\right)^{-1}\\ &= \frac{1}{1+z\,\epsilon}\sum_{n=0}^{\infty}\big(\epsilon\,\boldsymbol{I} + \boldsymbol{K}(\theta)\big)^n\left(\frac{z}{1+z\,\epsilon}\right)^n,\end{aligned}$$

and $z/(1+z\,\epsilon) \in \big[0,(1/\Delta(\theta)+\epsilon)^{-1}\big)$, which implies that the convergence parameter of $\epsilon\,\boldsymbol{I} + \boldsymbol{K}(\theta) = (\mu+\epsilon)\,\boldsymbol{I} + \boldsymbol{C} + \widehat{\boldsymbol{D}}(\theta)$ is $(1/\Delta(\theta)+\epsilon)^{-1}$. □

Since Δ is continuous on $(-\infty,\theta_0)$ with $\Delta(0) = 1/\mu$ and $\boldsymbol{K}(0)$ is $1/\mu$-positive, if $\boldsymbol{K}(\theta)$ is $\Delta(\theta)$-recurrent within a neighborhood of the origin, it is $\Delta(\theta)$-positive at least inside that neighborhood. We here impose the following.

Assumption 3.2 For $\alpha \in (0,\theta_0)$ satisfying $\Delta(\alpha) = 1/(\alpha+\mu)$, $\boldsymbol{K}(\alpha)$ is $1/(\alpha+\mu)$-positive.

Note that, by Lemma 3.3, α in Assumption 3.2 does not depend on the choice of the value of μ. Under Assumption 3.2, $\boldsymbol{\eta}(\theta)$ and $\boldsymbol{r}(\theta)$, which are introduced as the $\Delta(\theta)$-subinvariant measure and the $\Delta(\theta)$-subinvariant vector of $\boldsymbol{K}(\theta)$ in the proof of Lemma 3.3, become respectively a $1/(\alpha+\mu)$-invariant measure and a $1/(\alpha+\mu)$-invariant vector at $\theta = \alpha$; that is, the equalities in (15) hold at $\theta = \alpha$ with $\Delta(\alpha) = 1/(\alpha+\mu)$. Since $\boldsymbol{\eta}(\alpha)\,\boldsymbol{r}(\alpha) < \infty$ under Assumption 3.2, we normalize $\boldsymbol{\eta}(\alpha)$ and $\boldsymbol{r}(\alpha)$ such that $\boldsymbol{\eta}(\alpha)\,\boldsymbol{r}(\alpha) = 1$. From (14), we can easily check that $\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{R}}(\alpha) = \boldsymbol{\eta}(\alpha)$; that is, $\boldsymbol{\eta}(\alpha)$ is also a 1-invariant measure of $\widehat{\boldsymbol{R}}(\alpha)$. Concerning a 1-invariant vector of $\widehat{\boldsymbol{R}}(\alpha)$, we have the following.

Lemma 3.4 *Under Assumptions 3.1 and 3.2, $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$ gives a 1-invariant vector of $\widehat{\boldsymbol{R}}(\alpha)$.*

It is easy from (14) to show that $\widehat{\boldsymbol{R}}(\alpha)\,\boldsymbol{h}(\alpha) = \boldsymbol{h}(\alpha)$ and it remains to verify that $\boldsymbol{h}(\alpha)$ in Lemma 3.4 is strictly positive. This verification is somewhat technical though it is not so long, and it is placed after the proof of the main theorem below.

Theorem 3.1 *Under Assumptions 3.1 and 3.2, if $\boldsymbol{\kappa}\,\boldsymbol{r}(\alpha) < \infty$, we have*

$$\lim_{x\to\infty} e^{\alpha x}\,\phi_j(x) = \frac{(1-\rho)\,\boldsymbol{\kappa}\,\boldsymbol{r}(\alpha)}{\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{D}}^{(1)}(\alpha)\,\boldsymbol{r}(\alpha) - 1}\,\eta_j(\alpha). \tag{16}$$

Formula (16) has the same form as (30) in [12, Theorem 4.6] and (13) in [6, Theorem 4.1], both of which are derived in the case where the underlying Markov chain has a finite state space. Thus, Theorem 3.1 directly extends them and we here prove this extension by using another approach. Of course, in the case with a finite-state underlying Markov chain, the same formula can be obtained by applying Proposition 2.3, which also gives a more straightforward proof than [12] and [6].

Proof: First, it is easy to see from (9) that matrix-valued function $\boldsymbol{R}$ satisfies Assumption 2.1 in the preceding section. From (8), $\boldsymbol{\xi}$ in Proposition 2.4 is now given by $\boldsymbol{\xi}(x) = \boldsymbol{\pi}\,\overline{\boldsymbol{R}}(x)$, $x \ge 0$. In order to apply Proposition 2.4, we have to check that $e^{\alpha x}\,\boldsymbol{\xi}(x)$, $x \ge 0$, is directly integrable associated with $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$. To this end, with a slight extension of Lemma 6.1.4 in Rolski *et al.* [10, Chapter 6], it is sufficient to show that $e^{\alpha x}\,\boldsymbol{\xi}(x)\,\boldsymbol{h}(\alpha)$ is integrable on $\mathbb{R}_+$ since $\boldsymbol{\xi}$ is nonincreasing on $\mathbb{R}_+$ and $e^{\alpha x}$ is nondecreasing with $e^{\alpha x} \to 1$ as $x \downarrow 0$. Using (9) and applying Fubini's theorem, we have

$$\begin{aligned}\int_0^\infty e^{\alpha x}\,\boldsymbol{\xi}(x)\,\mathrm{d}x &= \boldsymbol{\pi}\int_0^\infty \mathrm{d}x\,e^{\alpha x}\int_x^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^{w-x} e^{\boldsymbol{Q}y}\,\mathrm{d}y\\ &= \boldsymbol{\pi}\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^w \mathrm{d}y\,e^{\boldsymbol{Q}y}\int_0^{w-y} e^{\alpha x}\,\mathrm{d}x\\ &= \frac{\boldsymbol{\pi}}{\alpha}\Big[\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\,e^{\alpha w}\int_0^w e^{(\boldsymbol{Q}-\alpha\boldsymbol{I})y}\,\mathrm{d}y - \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^w e^{\boldsymbol{Q}y}\,\mathrm{d}y\Big].\end{aligned} \tag{17}$$

Here, the first term in the brackets of (17) reduces to

$$\begin{aligned}\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\,e^{\alpha w}\,(e^{(\boldsymbol{Q}-\alpha\boldsymbol{I})w} - \boldsymbol{I})\,(\boldsymbol{Q}-\alpha\,\boldsymbol{I})^{-1} &= \big(\boldsymbol{Q} - \boldsymbol{C} - \widehat{\boldsymbol{D}}(\alpha)\big)\,(\boldsymbol{Q}-\alpha\,\boldsymbol{I})^{-1}\\ &= (\alpha\,\boldsymbol{I} - \boldsymbol{C} - \widehat{\boldsymbol{D}}(\alpha))\,(\boldsymbol{Q}-\alpha\,\boldsymbol{I})^{-1} + \boldsymbol{I},\end{aligned} \tag{18}$$

where (10) is used in the first equality.

To consider the second term in the brackets of (17), we verify existence of $(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}$. It is sufficient to show that $\boldsymbol{\chi}\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q}) = \boldsymbol{0}^T$ if and only if $\boldsymbol{\chi} = \boldsymbol{0}^T$. The if-part is immediate and we check the only-if-part. Suppose that $\boldsymbol{\chi}\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q}) = \boldsymbol{0}^T$. We then have $\boldsymbol{\chi}\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})\,\boldsymbol{e} = \boldsymbol{\chi}\,\boldsymbol{e} = 0$, which implies that $\boldsymbol{\chi}\,\boldsymbol{Q} = \boldsymbol{0}^T$ and also $|\boldsymbol{\chi}|\,\boldsymbol{e} < \infty$, where $|\boldsymbol{a}| = (|a_i|; i \in E)$ for $\boldsymbol{a} = (a_i; i \in E)$. Thus, it holds that $\boldsymbol{\chi} = \boldsymbol{\chi}\,e^{\boldsymbol{Q}y}$ for any $y \ge 0$. Note here that $e^{\boldsymbol{Q}y}$, $y > 0$, is the irreducible and positive recurrent stochastic matrix with stationary distribution $\boldsymbol{\kappa}$, and hence, $\lim_{y\to\infty} e^{\boldsymbol{Q}y} = \boldsymbol{e}\,\boldsymbol{\kappa}$. Therefore, the dominated convergence theorem leads to $\boldsymbol{\chi} = \boldsymbol{\chi}\,\boldsymbol{e}\,\boldsymbol{\kappa} = \boldsymbol{0}^T$.

Now, using the relation $e^{\boldsymbol{Q}y}\,\boldsymbol{e} = \boldsymbol{e}$, the second term in the brackets of (17) reduces to

$$\begin{aligned}&\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^w e^{\boldsymbol{Q}y}\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})\,\mathrm{d}y\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}\\ &= \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\,(\boldsymbol{e}\,\boldsymbol{\kappa}\,w + \boldsymbol{I} - e^{\boldsymbol{Q}w})\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}\\ &= \big(\widehat{\boldsymbol{D}}^{(1)}(0)\,\boldsymbol{e}\,\boldsymbol{\kappa} + \boldsymbol{D}(\infty) + \boldsymbol{C} - \boldsymbol{Q}\big)\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}\end{aligned}$$

$$= \big(\widehat{\boldsymbol{D}}^{(1)}(0) - \boldsymbol{I}\big)\, \boldsymbol{e}\,\boldsymbol{\kappa} + \big(\boldsymbol{C} + \boldsymbol{D}(\infty)\big)\, (\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1} + \boldsymbol{I}, \tag{19}$$

where we use (10) again in the second equality. Substituting (18) and (19) into (17) and then post-multiplying by $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$, we have

$$\begin{aligned}\int_0^\infty e^{\alpha x}\,\boldsymbol{\xi}(x)\,\boldsymbol{h}(\alpha)\,\mathrm{d}x &= \frac{\boldsymbol{\pi}}{\alpha}\left[(\boldsymbol{I} - \widehat{\boldsymbol{D}}^{(1)}(0))\,\boldsymbol{e}\,\boldsymbol{\kappa} - (\boldsymbol{C} + \boldsymbol{D}(\infty))\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}\right](\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha) \\ &= (1-\rho)\,\boldsymbol{\kappa}\,\boldsymbol{r}(\alpha) < \infty, \end{aligned}\tag{20}$$

where we use $(\alpha\,\boldsymbol{I} - \boldsymbol{C} - \widehat{\boldsymbol{D}}(\alpha))\,\boldsymbol{r}(\alpha) = \boldsymbol{0}$ in the first equality, and $\boldsymbol{\kappa}\,\boldsymbol{Q} = \boldsymbol{\pi}\,(\boldsymbol{C}+\boldsymbol{D}(\infty)) = \boldsymbol{0}^T$, $\boldsymbol{\pi}\,\boldsymbol{e} = 1$ and $\boldsymbol{\pi}\,\widehat{\boldsymbol{D}}^{(1)}(0)\,\boldsymbol{e} = \rho$ in the second equality. Hence, the direct integrability of $e^{\alpha x}\,\boldsymbol{\xi}(x)$ associated with $\boldsymbol{h}(\alpha)$ is verified.

Next, consider the term corresponding to the denominator of (7). Differentiating both sides of (14) in Lemma 3.2 and then applying (14) again, we have

$$\begin{aligned}\widehat{\boldsymbol{R}}^{(1)}(\alpha) &= \big[\widehat{\boldsymbol{D}}^{(1)}(\alpha) - (\widehat{\boldsymbol{D}}(\alpha) + \boldsymbol{C} - \boldsymbol{Q})\,(\alpha\boldsymbol{I} - \boldsymbol{Q})^{-1}\big]\,(\alpha\boldsymbol{I} - \boldsymbol{Q})^{-1} \\ &= \big[\widehat{\boldsymbol{D}}^{(1)}(\alpha) - \widehat{\boldsymbol{R}}(\alpha)\big]\,(\alpha\boldsymbol{I} - \boldsymbol{Q})^{-1}.\end{aligned}$$

Thus, multiplying by $\boldsymbol{\eta}(\alpha)$ and $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$ from both sides, we have

$$\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{R}}^{(1)}(\alpha)\,\boldsymbol{h}(\alpha) = \boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{D}}^{(1)}(\alpha)\,\boldsymbol{r}(\alpha) - 1, \tag{21}$$

where we use $\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{R}}(\alpha)\,\boldsymbol{r}(\alpha) = \boldsymbol{\eta}(\alpha)\,\boldsymbol{r}(\alpha) = 1$. Finally, substituting (20) and (21) into (7), we obtain (16). □

Proof of Lemma 3.4: As stated before, we can easily check that $\widehat{\boldsymbol{R}}(\alpha)\,\boldsymbol{h}(\alpha) = \boldsymbol{h}(\alpha)$ with $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$. Thus, we here verify $\boldsymbol{h}(\alpha) > \boldsymbol{0}$. It is done by considering a twisted version of Markovian arrival stream $(\boldsymbol{C}, \boldsymbol{D})$. Define an $E \times E$-matrix $\boldsymbol{C}^\dagger$ and an $E \times E$-matrix-valued function $\boldsymbol{D}^\dagger$ on $\mathbb{R}_+$ by

$$\begin{aligned}\boldsymbol{C}^\dagger &= \mathrm{diag}(\boldsymbol{r}(\alpha))^{-1}\,\boldsymbol{C}\,\mathrm{diag}(\boldsymbol{r}(\alpha)) - \alpha\,\boldsymbol{I}, \\ \boldsymbol{D}^\dagger(x) &= \mathrm{diag}(\boldsymbol{r}(\alpha))^{-1}\int_0^x e^{\alpha y}\,\mathrm{d}\boldsymbol{D}(y)\,\mathrm{diag}(\boldsymbol{r}(\alpha)), \quad x \ge 0,\end{aligned}$$

Note that $\boldsymbol{D}^\dagger(\infty) = \lim_{x\to\infty}\boldsymbol{D}^\dagger(x) = \mathrm{diag}(\boldsymbol{r}(\alpha))^{-1}\,\widehat{\boldsymbol{D}}(\alpha)\,\mathrm{diag}(\boldsymbol{r}(\alpha))$. We can see that $\boldsymbol{C}^\dagger + \boldsymbol{D}^\dagger(\infty)$ is an irreducible and positive recurrent rate matrix with stationary distribution $\boldsymbol{\pi}^\dagger = \boldsymbol{\eta}(\alpha)\,\mathrm{diag}(\boldsymbol{r}(\alpha))$; that is, $(\boldsymbol{C}^\dagger, \boldsymbol{D}^\dagger)$ gives a stationary Markovian arrival stream. The traffic intensity $\rho^\dagger$ of this Markovian arrival stream is given by

$$\rho^\dagger = \boldsymbol{\pi}^\dagger \int_0^\infty x\,\mathrm{d}\boldsymbol{D}^\dagger(x)\,\boldsymbol{e} = \boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{D}}^{(1)}(\alpha)\,\boldsymbol{r}(\alpha). \tag{22}$$

Now, we show that $\rho^\dagger > 1$; that is, the single-server queue with arrival stream $(\boldsymbol{C}^\dagger, \boldsymbol{D}^\dagger)$ is unstable. Recall that $1/\Delta(\theta)$ is nondecreasing and convex in $\theta < \theta_0$ and that $\alpha \in (0, \theta_0)$ in Assumption 3.2 satisfies $1/\Delta(\alpha) = \alpha + \mu$ with $1/\Delta(\theta) < \theta + \mu$ for $\theta \in (0, \alpha)$ and $1/\Delta(\theta) > \theta + \mu$ for $\theta \in (\alpha, \theta_0)$. This implies $\mathrm{d}\big(1/\Delta(\theta)\big)/\mathrm{d}\theta\big|_{\theta=\alpha} > 1$. On the other hand, since $\boldsymbol{\eta}(\theta)\,\boldsymbol{K}(\theta)\,\boldsymbol{r}(\theta) \le \boldsymbol{\eta}(\theta)\,\boldsymbol{r}(\theta)/\Delta(\theta)$, $\theta < \theta_0$, by (15) and the equality holds at $\theta = \alpha$, their gradients on both sides are also equal at $\theta = \alpha$. Thus, differentiating both sides, we have $\mathrm{d}\big(1/\Delta(\theta)\big)/\mathrm{d}\theta\big|_{\theta=\alpha} = \boldsymbol{\eta}(\alpha)\,\boldsymbol{K}^{(1)}(\alpha)\,\boldsymbol{r}(\alpha) = \boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{D}}^{(1)}(\alpha)\,\boldsymbol{r}(\alpha)$; that is, $\rho^\dagger > 1$ by (22).

Let $\boldsymbol{Q}^\dagger$ denote an $E \times E$-matrix satisfying

$$\boldsymbol{Q}^\dagger = \boldsymbol{C}^\dagger + \int_0^\infty \mathrm{d}\boldsymbol{D}^\dagger(x)\, e^{\boldsymbol{Q}^\dagger x}; \tag{23}$$

that is, $\boldsymbol{Q}^\dagger$ for $(\boldsymbol{C}^\dagger, \boldsymbol{D}^\dagger)$ corresponds to $\boldsymbol{Q}$ for $(\boldsymbol{C}, \boldsymbol{D})$ in (10). Comparing (23) with (10) and checking the definition of $\boldsymbol{C}^\dagger$ and $\boldsymbol{D}^\dagger$, we can see that $\boldsymbol{Q}^\dagger = \mathrm{diag}(\boldsymbol{r}(\alpha))^{-1}\,\boldsymbol{Q}\,\mathrm{diag}(\boldsymbol{r}(\alpha)) - \alpha\,\boldsymbol{I}$. Since $\rho^\dagger > 1$, we have $\boldsymbol{Q}^\dagger\,\boldsymbol{e} < \boldsymbol{0}$, which finally leads us to the goal; $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha) = -\mathrm{diag}(\boldsymbol{r}(\alpha))\,\boldsymbol{Q}^\dagger\,\boldsymbol{e} > \boldsymbol{0}$. □

4. Concluding Remark

In this paper, we have proposed the dual form of Markov renewal equations as a tool for analysis of stochastic models. We then have applied one to tail asymptotic analysis of the stationary workload distribution for a single-server queue with a Markovian arrival stream. The application extends the existing result to the case where the Markov chain governing the arrival process has a countable state space. We have found that our approach is more straightforward than the previous works and makes the extension simple. It would further be expected that the approach with the dual form of Markov renewal equations could help the analysis of other stochastic models with Markovian environments.

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References

[1] S. Asmussen: *Applied Probability and Queues, 2nd edition* (Springer-Verlag, 2003).

[2] E. Çinlar: *Introduction to Stochastic Processes* (Prentice-Hall, 1975).

[3] W. Feller: *An Introduction to Probability Theory and Its Applications, Volume II, 2nd edition* (John Wiley & Sons, 1971).

[4] J.F.C. Kingman: A convexity property of positive matrices. *The Quarterly Journal of Mathematics, Oxford Ser. 2*, **12** (1961), 283–284.

[5] R.M. Loynes: The stability of queues with non-independent inter-arrival and service times. *Proceedings of the Cambridge Philosophical Society: Mathematical and Physical Sciences*, **58** (1962), 497–520.

[6] M. Miyazawa: A Markov renewal approach to the asymptotic decay of the tail probabilities in risk and queuing processes. *Probability in the Engineering and Informational Sciences*, **16** (2002), 139–150.

[7] M. Miyazawa: A Markov renewal approach to $M/G/1$ type queues with countably many background states. *Queueing Systems: Theory and Applications*, **46** (2004), 177–196.

[8] M. Miyazawa and Y.Q. Zhao: The stationary tail asymptotics in the GI/G/1-type queue with countably many background states. *Advances in Applied Probability*, **36** (2004), 1231–1251.

[9] N. Miyoshi: On the stationary workload distribution of work-conserving single-server queues: A general formula via stochastic intensity. *Journal of Applied Probability*, **38** (2001), 793–798.

[10] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels: *Stochastic Processes for Insurance and Finance* (John Wiley & Sons, 1999).

[11] E. Seneta: *Non-negative Matrices and Markov Chains, 2nd edition* (Springer-Verlag, 1981).

[12] T. Takine: A recent progress in algorithmic analysis of FIFO queues with Markovian arrival streams. *Journal of the Korean Mathematical Society*, **38** (2001), 807–842.

[13] T. Takine: Matrix product-form solution for an LCFS-PR single-server queue with multiple arrival streams governed by a Markov chain. *Queueing Systems: Theory and Applications*, **42** (2002), 131–151.

[14] T. Takine and T. Hasegawa: The workload in the MAP/G/1 queue with state-dependent services: Its application to a queue with preemptive resume priority. *Communications in Statistics: Stochastic Models*, **10** (1994), 183–204.

[15] D. Vere-Jones: Ergodic properties of non-negative matrices—I. *Pacific Journal of Mathematics*, **22** (1967), 361–386.

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