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APPROXIMATING MINIMUM COST MULTIGRAPHS OF SPECIFIED EDGE-CONNECTIVITY UNDER DEGREE BOUNDS

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Abstract In this paper, we consider the problem of constructing a minimum cost graph with a specified edge-connectivity under a degree constraint. For a set V of vertices, let $r : \binom{V}{2} \to \mathbb{Z}_+$ be a connectivity demand, $a : V \to \mathbb{Z}_+$ be a lower degree bound, $b : V \to \mathbb{Z}_+$ be an upper degree bound, and $c : \binom{V}{2} \to \mathbb{Q}_+$ be a metric edge cost. The problem (V, r, a, b, c) asks to find a minimum cost multigraph $G = (V, E)$ with no self-loops such that $\lambda(u, v) \geq r(u, v)$ holds for each vertex pair $u, v \in V$ and $a(v) \leq d(v) \leq b(v)$ holds for each vertex $v \in V$, where $\lambda(u, v)$ (resp., $d(v)$) denotes the local-edge-connectivity between u and v (resp., the degree of v) in G. We reveal several conditions on functions r, a, b and c for which the above problem admits a constant-factor approximation algorithm. For example, we give a $(2+2/k)$-approximation algorithm to (V, r, a, b, c) with $r(u, v) \geq 2$, $u, v \in V$ and a uniform $b(v)$, $v \in V$, where $k = \min_{u,v \in V} r(u, v)$. To design the algorithms in this paper, we derive new results on splitting and detachment, which are graph transformations to split vertices into several copies of them while preserving edge-connectivity.

Keywords: Combinatorial optimization, approximation algorithm, degree bound, detachment, edge-connectivity, network design, splitting

1. Introduction

We let $\mathbb{Z}_+$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the set of nonnegative integers, rational numbers and real numbers, respectively. Throughout the paper, we mean by a graph an undirected multigraph with no self-loop unless stated otherwise. Let $G = (V, E)$ be a graph. For $X \subset V$, let $\delta(X; G)$ (or $\delta(X)$) denote the set of edges which have one end vertex in X and the other in $V - X$, and let $d(X; G)$ (or $d(X)$) denote $|\delta(X; G)|$. We may represent a set $\{v\}$ of a single element by v. Hence $d(v; G)$ denote the degree of v in G. Moreover we let $\lambda(u, v; G)$ (or $\lambda(u, v)$) denote the local-edge-connectivity between two vertices u and v in G. For a function $r : \binom{V}{2} \to \mathbb{Z}_+$, a graph G with a vertex set V is called *r-edge-connected* when $\lambda(u, v; G) \geq r(u, v)$ holds for each pair $u, v \in V$. If $r(u, v) = k$ for all $u, v \in V$, then an r-edge-connected graph may be called k-edge-connected. In this paper, we consider the problem of constructing a minimum cost graph with a specified edge-connectivity under a degree constraint, which is formulated as follows: Given a set V of vertices, a connectivity demand $r : \binom{V}{2} \to \mathbb{Z}_+$, a lower degree bound $a : V \to \mathbb{Z}_+$, an upper degree bound $b : V \to \mathbb{Z}_+$, and an edge cost $c : \binom{V}{2} \to \mathbb{Q}_+$, find a minimum cost multigraph $G = (V, E)$ with no self-loops such that

$$\lambda(u, v; G) \geq r(u, v) \quad \text{for each pair } u, v \in V$$

and

$$a(v) \leq d(v; G) \leq b(v) \quad \text{for each } v \in V.$$

We denote a problem instance consisting of the above inputs by (V, r, a, b, c). For a function h and a constant $\ell \in \mathbb{Q}_+$, we denote $h \leq \ell$ (resp., $h \geq \ell$) to mean that the value of h is

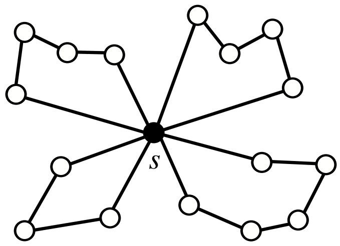

Figure 1: A solution for the 4-traveling salesperson problem

always less than or equal to ℓ (resp., greater than or equal to ℓ), where $h = \ell$ means $h \le \ell$ and $h \ge \ell$.

This problem includes a wide range of classic fundamental problems. For example, let $a = 0$ and $b = +\infty$, i.e., no degree constraints are imposed. Then the problem is equivalent to the minimum spanning tree problem if $r = 1$ (i.e., $(V, 1, 0, +\infty, c)$). If $r(x, y) = 1$ for $x, y \in \mathcal{T} \subseteq V$ and $r(x, y) = 0$ otherwise, then the problem is the Steiner tree problem with some terminal set $\mathcal{T}$. This is an NP-hard problem and a 1.55-approximation algorithm is given by G. Robins and A. Zelikovsky [13]. For a general r, $(V, r, 0, +\infty, c)$ is the Steiner network problem without edge capacity constraints. K. Jain [9] proved that the Steiner network problem $(V, r, 0, +\infty, c)$ is approximable within a factor of 2 by using the iterated rounding technique (his version can deal with the capacities on the number of multiple edges although we do not consider them in this paper). On the other hand, let $r = 0$, i.e., no connectivity demand is required. If $b = +\infty$, then the problem $(V, 0, a, +\infty, c)$ is called the a-edge-cover problem. If $a = b = 1$, i.e., the degree of each vertex is specified as 1, then the problem $(V, 0, 1, 1, c)$ is the perfect matching problem. For general degree bounds a and b, the problem $(V, 0, a, b, c)$ is known to be solvable in a polynomial time (see [14] for example). As an example of instances that have both degree bounds and edge-connectivity demand, the k-traveling salesperson problem is known (this problem is also called the vehicle routing problem). In the k-traveling salesperson problem, one special vertex $s \in V$ is specified as a depot. A solution of this problem consists of k cycles such that every cycle covers s and all vertices in $V - s$ are covered by one of those cycles (see Figure 1). The objective is to minimize the sum of the costs of all cycles. If $r = 2$, $a(s) = b(s) = 2k$ and $a(v) = b(v) = 2$ for $v \in V - s$, then $(V, 2, a, b, c)$ is equivalent to the k-traveling salesperson problem. Notice that the k-traveling salesperson problem contains the traveling salesperson problem as a special case. The traveling salesperson problem is inapproximable within any constant factor unless $P = NP$, and is shown to be approximable within a factor of 1.5 if c is metric (i.e., $c(uv) + c(vz) \ge c(uz)$ for every $u, v, z \in V$) by N. Christofides [1]. For the k-traveling salesperson problem, a primal-dual algorithm achieves 2-approximation [7]. In addition to these prior results, S. P. Fekete et al. [2] considered the problem $(V, 1, 0, b \ge 2, c)$, i.e., a problem to find a minimum cost spanning tree under the constraint that the degree of each vertex is bounded from the above. They proved that the problem $(V, 1, 0, b \ge 2, c)$ is approximable within a factor of $2 - \min_{v \in V, d(v;T) > 2}(b(v) - 2)/(d(v; T) - 2)$, where T is a minimum spanning tree. The Steiner tree problem with further constraint that all terminals must be leaves (i.e., $b(v) = 1$ for all $v \in \mathcal{T}$ and $b(v) = +\infty$ for all $v \in V - \mathcal{T}$) is called the terminal Steiner tree problem (or full Steiner tree problem). F. V. Martinez et al. [12] showed that this problem can be approximated within $2\rho - \rho/(3\rho - 2)$ if the Steiner tree problem can be approximated within ρ.

Although the problem (V, r, a, b, c) is a natural framework as a generalization of the above problems, a few results on this problem setting have been obtained so far. A. Frank [3] solved the problem of augmenting a given graph to an r-edge-connected graph by adding a smallest number of new edges under lower and upper bounds on degrees. This implies that the problem $(V, r, a, b, 1)$ admits a polynomial time algorithm. Moreover, an extended result by A. Frank [3] suggests that (V, r, a, b, c) is polynomially solvable in a special case where cost $c(e)$ for each edge $e = uv$ is given by $w(u) + w(v)$ for some vertex weight $w : V \to \mathbb{Q}_+$. L. C. Lau et al. [10] considered more restricted problem, in which solutions are restricted to subgraphs of a given graph. They presented an algorithm that computes a graph such that it cost is at most twice the optimal and $d(v)$ is at most $2b(v) + 3$ for all $v \in V$.

In this paper, we consider the problem (V, r, a, b, c) with a metric cost c, and show several conditions on functions r, a and b for which the problem admits an approximation algorithm. To design most algorithms proposed in this paper, we use splitting and detachment, which are graph transformations that split vertices into several copies of them while preserving edge-connectivity. Use of such operations to design a minimum cost graph is new. Only splitting has been used to solve the edge-connectivity augmentation problem. However, the way of using splitting in this paper is different from those methods in the edge-connectivity augmentation.

The paper is organized as follows. Section 2 derives new results on splitting and detachment, which will be the basis of our algorithms in this paper. It also considers the relationship between that result and parsimonious property of the Steiner network problem. Section 3 deals with the case of $b = +\infty$, i.e., each degree is bounded only from the below, and Section 4 considers the case of $a = 0$, i.e., each degree is bounded only from the above. Section 5 shows an approximation algorithm for the problem $(V, r \geq 2, a, \ell, c)$. Section 6 concludes this paper with some remarks.

2. Splitting and Detachment

Splitting is an operation that replaces two edges $e = us$ and $f = vs$ by a new edge uv. The resulting graph by splitting a pair $\{e, f\}$ of edges in a graph G is denoted by G^{ef}. Note that this operation decreases the degree of vertex s while keeping degrees of the other vertices. Also in a metric cost $c : \binom{V}{2} \to \mathbb{Q}_+$, it does not increase the cost of the graph since $c(us) + c(vs) \geq c(uv)$ holds by the triangle inequality. However, it may decrease the local-edge-connectivity between some pairs of vertices. A pair of edges is called *splittable* if splitting it preserves the local-edge-connectivity between every pair of vertices in $V - \{s\}$, i.e., $\lambda(x, y; G^{ef}) = \lambda(x, y; G)$ for each $x, y \in V - s$. W. Mader proved that, for a designated vertex $s \in V$, there always exists a splittable pair of edges incident to s except for the case in which $d(s) = 3$ or a cut-edge is incident to s.

Theorem 1 ([11]) *Let $G = (V, E)$ be a connected graph and $s \in V$ be a vertex with $d(s) \neq 3$. If no cut-edge is incident to s, then there is at least one splittable pair $\{e, f\}$ of edges incident to s.*

Furthermore, the following stronger result was proven by A. Frank.

Theorem 2 ([4]) *Let $G = (V, E)$ be a connected graph and s be a vertex in V. If no cut-edge is incident to s and $d(s)$ is even, then edges incident to s can be partitioned into $d(s)/2$ disjoint splittable pairs.*

Splitting has been used as a useful technique for solving many connectivity problems. In particular, it plays a key role to solve the edge-connectivity augmentation problem (see [3]). However, the above result on splitting has not been used to design a minimum cost graph even if a cost function is metric. Moreover, in Mader's theorem, the local-edge-connectivity between the designated vertex s and other vertices is not taken into account. Our algorithms discussed in the following sections first construct a graph that satisfies the connectivity demand and the lower degree constraint in a given instance, and then utilize splitting in order to make the graph satisfy the upper degree constraint. For this, we need to preserve the local-edge-connectivity between s and other vertices as well. In the following, we strengthen Theorem 1 so that the local-edge-connectivity between s and any other vertex is preserved (up to $d(s)-2$).

Let us define

$$r_G(x,y) = \begin{cases} \lambda(x,y;G) & \text{if } x,y \in V-s, \\ \min\{d(s)-2, \lambda(x,y;G)\} & \text{if } s \in \{x,y\}. \end{cases}$$

Notice that G is r_G-edge-connected. We call a pair of edges $e=us$ and $f=vs$ *strongly splittable* if G^{ef} is also r_G-edge-connected. The following lemma characterizes strongly splittable pairs.

Lemma 1 *A pair $\{e=us, f=vs\}$ of edges in G is strongly splittable if and only if there exists no set $X \subseteq V-s$ such that $u,v \in X$ and*

$$d(X;G) \le 1 + \max_{x\in X, y\in V-X} r_G(x,y).$$

Proof: First, suppose that there exists such X. Then $d(X;G^{ef}) = d(X;G)-2 < r_G(x,y)$ for some $x^* \in X$ and $y^* \in V-X$. This implies that G^{ef} is not r_G-edge-connected, i.e., $\{e,f\}$ is not strongly splittable. Hence the necessity is proven.

Conversely, suppose that $\{e,f\}$ is not strongly splittable. Then G^{ef} is not r_G-edge-connected, i.e., there exists a pair $\{x^*,y^*\}$ of vertices such that $\lambda(x^*,y^*;G^{ef}) < r_G(x^*,y^*)$. This implies the existence of $X \subset V$ such that $x^* \in X$, $y^* \in V-X$, $d(X;G^{ef}) < r_G(x^*,y^*)$. Let $s \in V-X$ without loss of generality. It holds $u,v \in X$ since otherwise $d(X;G) = d(X;G^{ef}) < r_G(x^*,y^*)$ contradicting to the fact that G is r_G-edge-connected. Hence $d(X;G) = d(X;G^{ef}) + 2 \le 1 + r_G(x^*,y^*) \le 1 + \max_{x\in X, y\in V-X} r_G(x,y)$, implying the sufficiency. □

The following theorem gives a condition for a graph to have a strongly splittable pair.

Theorem 3 *Let $G=(V,E)$ be a connected graph and $s \in V$ be a vertex with $d(s) \neq 3$. If no cut-edge is incident to s, then there is at least one strongly splittable pair $\{e=us, f=vs\}$ of edges, where e and f can be chosen so that $u \neq v$ unless s is adjacent to only one vertex. No new cut-edge will be created after splitting such $\{e,f\}$.*

Proof: In order to show the existence of a strongly splittable pair, we construct a new graph $G' = (V \cup s', E \cup E')$ from G by adding a new vertex s' together with a set E' of $d(s;G)-2$ new edges between s and s'. Then $\lambda(x,y;G') = r_G(x,y)$ for every $x,y \in V-s$, $\lambda(s',y;G') = r_G(s,y)$ for every $y \in V-s$, and $d(s;G') = 2d(s;G)-2$. Since $d(s;G')$ is even and no cut-edge is incident to s in G' by the assumption, we see by Theorem 2 that edges incident to s can be partitioned into $d(s;G)-1$ disjoint pairs such that splitting them preserves the local-edge-connectivity between any two vertices in $(V\cup s')-s$. Notice that at least one of those pairs consists only of edges in E. This is exactly a strongly splittable pair in G.

We show that a strongly splittable pair $\{us, vs\}$ in G can be chosen so that $u \neq v$ when s has more than one neighbor in G. From the above argument, G has a strongly splittable pair. If such a pair consists of two parallel edges $\{us, us\}$, then there is no set X such that $u \in X$, $s \notin X$, $d(X;G) - \max_{x\in X, y\in V-X} r(x,y) \leq 1$ by Lemma 1. This implies that any pair of us and another arbitrary edge vs $(v \neq u)$ is also strongly splittable, as required.

Finally we show that splitting a strongly splittable pair $\{e = us, f = vs\}$ will not generate a new cut-edge in G. Assume that G^{ef} contains a new cut-edge $e' = zw$. If e' is an existing edge in G, then $1 = \lambda(z,w;G^{ef}) = r_G(z,w)$, implying that $\lambda(z,w;G) = 1$ by the definition of $r_G(z,w)$, a contradiction to that e' was not a cut-edge in G. Next assume that e' is a new edge in G^{ef}, i.e., $zw = uv$. In this case, $G^{ef} - e'$ is not connected and has a component not containing s, implying that s and this component was joined by a cut-edge in G, a contradiction to the assumption. □

This theorem has a close relationship to the *parsimonious property* of the Steiner network problem, which tells that an LP relaxation of the Steiner network problem

$$\begin{array}{lll} \text{minimize} & \sum_{e\in\binom{V}{2}} c(e)x(e) & \\ \text{subject to} & \sum_{e\in\delta(X)} x(e) \geq \max_{u\in X, v\in V-X} r(u,v) & \text{for each } X \subset V,\ X \neq \emptyset, \\ & x(e) \in \mathbb{R}_+ & \text{for each } e \in \binom{V}{2}, \end{array}$$

has an optimal solution satisfying

$$\sum_{e\in\delta(v)} x(e) = \max_{u\in V-v} r(u,v) \quad \text{for each } v \in V,$$

when c is a metric cost. In fact, M. X. Goemans and D. J. Bertsimas [6] proved this property by showing that every Eulerian graph admits a strongly splittable pair of edges incident to a vertex, which is a weaker version of Theorem 3. However, it was not known whether the Steiner network problem has an integer programming version of the parsimonious property or not. Theorem 3 enables us to derive such an integer version of the parsimonious property.

Corollary 1 *The Steiner network problem* $(V, r \geq 2, 0, +\infty, c)$ *with a metric cost function* $c : \binom{V}{2} \to \mathbb{Q}_+$ *has an optimal solution* $x(e) \in \mathbb{Z}_+$, $e \in \binom{V}{2}$ *that satisfies*

$$\max_{u\in V-v} r(u,v) \leq \sum_{e\in\delta(v)} x(e) \leq 1 + \max_{u\in V-v} r(u,v) \quad \textit{for each } v \in V.$$

Proof: Let G be an optimal solution for $(V, r, 0, +\infty, c)$, and suppose that $d(v;G) > \max_{u\in V-v} r(u,v) + 1$ holds for some vertex $v \in V$. Since $r \geq 2$, no cut-edge is incident to v. Hence by Theorem 3, we can obtain another r-edge-connected graph G' with $d(v;G') = d(v;G) - 2 \geq \max_{u\in V-v} r(u,v)$ by splitting an appropriate pair of edges incident to v, and since c is metric, the cost of G' is at most that of G. Hence, by repeating this operation, we can obtain another optimal solution such that the degree of each vertex v is $\max_{u\in V-v} r(u,v)$ or $\max_{u\in V-v} r(u,v) + 1$. □

Now we introduce a *detachment*, an extension of splitting, in which each vertex $v \in V$ will be replaced with a set V_v of vertices with degree $\rho(v')$, $v' \in V_v$. A *degree specification* g on V consists of a family $\{V_v \mid v \in V\}$ of disjoint vertex sets, and a function $\rho : \bigcup_{v\in V} V_v \to \mathbb{Z}_+$ such that

$$\sum_{v'\in V_v} \rho(v') = d(v;G) \quad \text{for all } v \in V.$$

A *g-detachment* G' of G is a graph obtained from G by replacing each $v \in V$ with the vertices in V_v changing the end vertex of each edge $uv \in \delta(v)$ from v to a vertex $v' \in V_v$ so that $d(v';G') = \rho(v')$ holds for each $v' \in V_v$. Hence G is obtained from G' by contracting each V_v into a single vertex v. The following corollary can be derived from Theorem 3.

Corollary 2 *For a connected graph $G = (V, E)$, let g be a degree specification such that $\rho(v_1) = d(v; G) - 2(p_v - 1) \geq 2$ and $\rho(v_i) = 2$ $(i = 2, \ldots, p_v)$ hold for all $V_v = \{v_1, \ldots, v_{p_v}\}$, where $p_v = 1$ if $d(v; G) \leq 3$ or a cut-edge is incident to v. Let*

$$r_g(x, y) = \begin{cases} \min\{\lambda(u, v; G), \rho(x), \rho(y)\} & \text{if } x \in V_u,\ y \in V_v \text{ with } u \neq v, \\ 2 & \text{otherwise.} \end{cases}$$

Then there is an r_g-edge-connected g-detachment of G.

Furthermore, let $\ell \in \mathbb{Z}_+$ and $V' = \{v \in V \mid d(v; G) > \ell\}$. If $|V| \geq 3$ and $\ell - 1 \leq \rho(v_1) \leq \ell$ for all $v \in V'$, then we can choose an r_g-edge-connected g-detachment of G so that no two edges incident to $v_i \in V_v - \{v_1\}$ are parallel for any $v \in V'$.

Proof: Let s be a vertex such that $d(s; G) > 3$ holds and no cut-edge is incident to s (if such a vertex does not exist, the corollary is obvious). We show how to construct an r_g-edge-connected g-detachment of G without creating no new cut-edge when $p_v = 1$ (i.e., $V_v = \{v\}$) for every $v \in V - s$, $V_s = \{s_1, s_2\}$, $\rho(s_1) = d(s; G) - 2 \geq 2$ and $\rho(s_2) = 2$. With this construction, we can recursively prove the existence of an r_g-edge-connected g-detachment for an arbitrary g.

By Theorem 3, G has a strongly splittable pair $\{e = us, f = vs\}$ of edges incident to s. That is to say, $\lambda(x, y; G^{ef}) \geq \lambda(x, y; G)$ holds for all pairs $x, y \in V - s$, and $\lambda(s, y; G^{ef}) \geq \min\{d(s) - 2, \lambda(s, y; G)\}$ holds for all $y \in V - s$. Let G' be a graph obtained from G^{ef} by regarding s as s_1 and by replacing the edge uv with two edges us_2 and vs_2 introducing a new vertex s_2. Observe that $\lambda(x, y; G') = \lambda(x, y; G^{ef}) \geq r_g(x, y)$ for vertices x, y with $\{x, y\} \cap \{s_2\} = \emptyset$. We first show that $\lambda(s_1, s_2; G') \geq 2$. Assume $\lambda(s_1, s_2; G') \leq 1$; there is a minimal subset X with $s_2 \in X \subseteq (V - s) \cup \{s_2\}$ and $d(X; G') = \lambda(s_1, s_2; G') \leq 1$. Notice that $X \neq \{s_2\}$ since $d(s_2; G') = 2$. Hence there exists a vertex $w \in X - s_2$. By the minimality, X induces a connected component from G'. Suppose that h is an edge whose end vertices belong to X and $(V - X) \cup \{s_1\}$ respectively, i.e., $d(X; G') = 1$. Then removing h disconnects X from the other vertices in G' but does not do so in G. This implies h is a new cut-edge, a contradiction. If $d(X; G') = 0$, then $\lambda(s_1, w; G') = \lambda(s_1, w; G^{ef}) = 0$, which contradicts the strong splittability of $\{e, f\}$. Hence $\lambda(s_1, s_2; G') \geq 2$ holds. Finally we show that $\lambda(s_2, y; G') \geq r_g(s_2, y) = \min\{\lambda(s, y; G), \rho(s_2) = 2\}$ holds. Assume $\lambda(s_2, y; G') < \min\{\lambda(s, y; G), \rho(s_2) = 2\}$. Then $\lambda(s_2, y; G') \leq 1$. Then by $\lambda(s_1, s_2; G') \geq 2$, there is a subset $Y \in V - s$ with $y \in Y$ and $d(Y; G') = \lambda(s_2, y; G') \leq 1$. This also implies that $\lambda(s_1, y; G') \leq d(Y; G') = \lambda(s_2, y; G') < \min\{\lambda(s, y; G), \rho(s_2) = 2\}$, which is a contradiction to the strong splittability of $\{e, f\}$. Therefore G' is a desired g-detachment of G.

Next, we consider the case in which $|V| \geq 3$ and $\ell - 1 \leq \rho(v_1) \leq \ell$ for all $v \in V'$. Let $s \in V'$. If s is adjacent to only one vertex (say w), then w has another neighbor in $V - s$. Hence $d(w; G) > d(s; G) > \ell$. This implies that if $V' \neq \emptyset$, then V' contains a vertex which has more than two neighbors. Hence by repeatedly splitting such a vertex as above, we can obtain a desired g-detachment. □

3. Problem with Lower Degree Bound

In this section, we consider the problem $(V, r, a, +\infty, c)$. As stated in Section 1, this problem is equivalent to the Steiner network problem if $a = 0$, and the Steiner network problem is shown to be 2-approximable by K. Jain. Actually he proposed a 2-approximation algorithm for a generalization of the Steiner network problem; A set function $f : 2^V \to \mathbb{Q}_+$ with

$f(V) = f(\emptyset) = 0$ is called *weakly supermodular* if, for every $X, Y \subseteq V$,

$$f(X) + f(Y) \leq f(X \cap Y) + f(X \cup Y) \tag{1}$$

or

$$f(X) + f(Y) \leq f(X - Y) + f(Y - X) \tag{2}$$

holds. Given a set V of vertices, an edge cost $c : \binom{V}{2} \rightarrow \mathbb{Q}_+$ and a weakly supermodular function f, the *generalized Steiner network problem* asks to find a minimum cost multigraph G with no self-loop such that $d(X; G) \geq f(X)$ for every $X \subseteq V$. The Steiner network problem $(V, r, 0, +\infty, c)$ is a special case of the generalized Steiner network problem since set function f_r with $f_r(X) = \max_{u \in X, v \in V-X} r(u, v)$ is weakly supermodular.

Theorem 4 ([9]) *The generalized Steiner network problem is approximable within a factor of* 2.

The problem $(V, r, a, +\infty, c)$ is to find a minimum cost graph G such that $d(X; G) \geq f(X)$ for every $X \subseteq V$ by letting

$$f(X) = \begin{cases} 0 & \text{if } X = \emptyset \text{ or } V, \\ \max\{a(v), \max_{u \in V-v} r(u, v)\} & \text{if } X = \{v\} \text{ or } V - \{v\}, \\ \max_{u \in X, v \in V-X} r(u, v) & \text{otherwise.} \end{cases} \tag{3}$$

We show that $(V, r, a, +\infty, c)$ is 2-approximable by proving that the above f is weakly supermodular.

Theorem 5 *The problem $(V, r, a, +\infty, c)$ is approximable within a factor of* 2.

Proof: By Theorem 4 and the above-mentioned fact, it suffices to show that the set function f defined in (3) is weakly supermodular. Let $X \subseteq V$ and $Y \subseteq V$. It is easy to see that if $X \subseteq Y$ or $Y \subseteq X$, then (1) holds. Similarly (2) holds if $X \cap Y = \emptyset$. If $X \cup Y = V$, then (2) holds, since f is symmetric (i.e., $f(X) = f(V - X)$ for every $X \subseteq V$) and we have $f(X) + f(Y) = f(V - X) + f(V - Y) = f(Y - X) + f(X - Y)$. Then we only need to consider the case where each of $X - Y$, $Y - X$, $X \cap Y$ and $V - (X \cup Y)$ is nonempty. In this case, $|X| \notin \{1, |V| - 1\}$ and $|Y| \notin \{1, |V| - 1\}$ hold, and inequality (1) or (2) follows from the weakly supermodularity of $f_r(X) = \max_{u \in X, v \in V-X} r(u, v)$. □

4. Problem with Upper Degree Bound

In this section, we discuss the approximability of the problem $(V, r \geq 2, 0, b, c)$. Notice that the problem has no feasible solution if there is a vertex $v \in V$ with $b(v) < \max_{u \in V-v} r(u, v)$. Therefore we suppose that $b(v) \geq \max_{u \in V-v} r(u, v)$ for each $v \in V$. In addition, we can assume without loss of generality that $\sum_{v \in V} b(v)$ is even. In order to show this fact, let us assume that $\sum_{v \in V} b(v)$ is odd. For such b, any optimal solution $G = (V, E)$ to $(V, r, 0, b, c)$ has a vertex u^* with $d(u^*; G) < b(u^*)$ since $\sum_{v \in V} d(v; G)$ is even. Hence G is also optimal for $(V, r, 0, b', c)$, where $b'(u^*) = b(u^*) - 1$ and $b'(v) = b(v)$ for $v \in V - u^*$. Therefore, any approximation algorithm for instances with even $\sum_{v \in V} b(v)$ can be used to approximate those instances with odd $\sum_{v \in V} b(v)$; Apply the algorithm to at most $|V|$ instances each of which is obtained by decreasing $b(v)$ by 1 for a vertex $v \in V$, and then output the best of the obtained solutions.

Our algorithm for $(V, r \geq 2, 0, b, c)$ consists of the following two phases. The first phase finds a feasible solution G_r to $(V, r \geq 2, 0, +\infty, c)$, i.e., G_r is an r-edge-connected graph.

At this point, there may be some vertices v that violate the upper degree constraint (i.e., $d(v;G_r) > b(v)$). Moreover notice that G_r has no cut-edge since $r \geq 2$. For now, let us suppose that $b(v) - d(v;G_r)$ is even for each vertex v with $d(v;G_r) > b(v) = \max_{u \in V-v} r(u,v)$. The second phase reduces the degree of each vertex v with $d(v;G_r)$ to at most $b(v)$. This can be done by computing an r_g-edge-connected g-detachment of G_r for a degree specification g such that, for each $v \in V$, $V_v = \{v_1, \ldots, v_{p_v}\}$, $\rho(v_1) = d(v;G_r) - 2(p_v - 1)$, and $\rho(v_i) = 2$ $(v_i \in V_v - \{v_1\})$, where $p_v = 1 + \max\{0, \lceil (d(v;G_r) - b(v))/2 \rceil\}$. Observe that

$$\rho(v_1) = d(v;G_r) - 2\lceil (d(v;G_r) - b(v))/2 \rceil = \begin{cases} b(v) & \text{if } d(v;G_r) - b(v) \text{ is even,} \\ b(v) - 1 & \text{if } d(v;G_r) - b(v) \text{ is odd,} \end{cases}$$

holds for a vertex v with $d(v;G_r) > b(v)$. Since we are assuming that $b(v) - 1 \geq \max_{u \in V-v} r(u, v)$ if $d(v;G_r) - b(v)$ is odd, it holds that $\rho(v_1) \geq \max_{u \in V-v} r(u,v)$ for every $v \in V$. Let G' be the graph obtained from the g-detachment by neglecting all vertices $v_i \in V_v - \{v_1\}$ $(v \in V)$ (i.e., replacing the edges uv_i, v_iu' with uu'), and regard v_1 as v. Then $d(v;G') = \rho(v_1) \leq b(v)$ holds for all $v \in V$. Neglecting some vertices from the g-detachment may create self-loops, which can simply be eliminated. Although this may further reduce the degree of the vertex v, the resulting graph G' still satisfies the degree constraints since $a = 0$. Hence G' satisfies the degree bounds. On the other hand, we can let G' be r-edge-connected by Corollary 2 since $\lambda(u,v;G') \geq r_g(u_1,v_1) = \min\{\lambda(u,v;G), \rho(u_1), \rho(v_1)\} \geq r(u,v)$ for every $u, v \in V$. Therefore we can obtain a feasible solution G'.

If there exists a vertex v such that $b(v) - d(v;G_r)$ is odd and $d(v;G_r) > b(v) = \max_{u \in V-v} r(u,v)$, then $d(v;G') < \max_{u \in V-v} r(u,v)$, where G' is the graph constructed in the above. In this case, we use the detachment of another graph instead of G_r. Let $U = \{v \in V \mid |b(v) - d(v;G_r)| \text{ is odd}\}$, where $|U|$ is even since $\sum_{v \in V} b(v)$ and $\sum_{v \in V} d(v;G_r)$ are even. Furthermore, compute a minimum cost perfect matching M on U, (i.e., solves $(U, 0, 1, 1, c)$), and adds M to G_r to obtain a graph G'_r. Observe that $|b(v) - d(v;G'_r)|$ is even for all $v \in V$ here. Then the second phase transforms G'_r into a feasible solution G'.

Theorem 6 *Let us suppose that we can obtain an α-approximate solution G_r for $(V, r \geq 2, 0, +\infty, c)$. If $d(v;G_r) - b(v)$ is even for each vertex v with $d(v;G_r) > b(v) = \max_{u \in V-v} r(u, v)$, then the problem $(V, r \geq 2, 0, b, c)$ is approximable within α. Otherwise, the problem $(V, r \geq 2, 0, b, c)$ is approximable within $\alpha + 2/k$ for $k = \min_{u,v \in V} r(u,v)$.*

Proof: We have already seen that our algorithm outputs a feasible solution. The second phase does not increase the edge cost since c is metric. Hence it suffices to show that $c(G_r) \leq \alpha c(G^*)$ and $c(M) \leq 2/k \cdot c(G^*)$, where G^* is an optimal solution for $(V, r \geq 2, 0, b, c)$. An optimal solution for $(V, r \geq 2, 0, +\infty, c)$ has the cost at most $c(G^*)$. This implies that $c(G_r) \leq \alpha c(G^*)$. Hence in the following, we show that $c(M) \leq 2/k \cdot c(G^*)$.

Let $2G^*$ be the graph obtained by duplicating every edge in G^*. Since G^* is r-edge-connected and $r \geq k$, $2G^*$ is $2k$-edge-connected. It is known that any $2k$-edge-connected graph contains k edge-disjoint spanning trees $\{T_1, \ldots, T_k\}$ [8]. Let $j = \arg\min_{1 \leq i \leq k} c(T_i)$. Then $c(T_j) \leq c(2G^*)/k$. Observe that a spanning tree T_j has $|U|/2$ edge-disjoint paths whose end vertices are U. By shortcutting intermediate vertices in the paths, we can obtain a perfect matching on U whose cost is at most $c(T_j) \leq c(2G^*)/k = 2/k \cdot c(G^*)$, as required. □

By Theorem 4, we can let $\alpha = 2$ in Theorem 6.

5. Problem with Lower and Upper Degree Bounds

We now consider the problem $(V, r \geq 2, a, b, c)$ with lower and upper degree bounds. In general, self-loops may be created from a loopless graph during the second phase of our algorithm in the previous section. Removing those self-loops may violate the lower degree constraints for some vertices to which the self-loops are incident. Thus our algorithm cannot be applied to this general case. However in this section, we show that the problem $(V, r \geq 2, a, b, c)$ is approximable if an upper bound is uniform, i.e., $b(v) = \ell$, $v \in V$ for some $\ell \in \mathbb{Z}_+$. In what follows, we assume without loss of generality that $b(v) \geq a(v) \geq \max_{u \in V-v} r(u, v)$ for all $v \in V$ since otherwise no feasible solution exists.

Theorem 7 *Let us suppose that we can obtain an α-approximate solution $G_{r,a}$ for $(V, r \geq 2, a, +\infty, c)$ and $b(v) = \ell$, $v \in V$ for an $\ell \in \mathbb{Z}_+$. If $d(v; G_{r,a}) - b(v)$ is even for each vertex v with $d(v; G_{r,a}) > b(v) = a(v)$, then the problem $(V, r \geq 2, a, b, c)$ is approximable within α. Otherwise, the problem $(V, r \geq 2, a, b, c)$ is approximable within $\alpha + 2/k$ for $k = \min_{u,v \in V} r(u, v)$.*

Proof: Suppose $|V| \geq 3$ since otherwise the theorem is obvious. If $d(v; G_r) - b(v)$ is even for each vertex v with $d(v; G_{r,a}) > b(v) = a(v)$, we transform $G_{r,a}$ so that the degree upper constraints are satisfied as in the algorithm in Section 4. By Corollary 2, edges incident to $v_i \in V_v - v_1$ in the detachment are not parallel for any $v \in V$ and $2 \leq i \leq p_v$. This implies that neglecting v_i creates no self-loop. Hence we can obtain a feasible solution. Notice that the optimal cost for $(V, r, a, +\infty, c)$ is at most that of (V, r, a, b, c). Hence the solution obtained by this algorithm is an α-approximate solution for (V, r, a, b, c).

Let us consider the latter case in the following. Let $U = \{v \in V \mid a(v) = \ell$ and $|d(v; G_{r,a}) - a(v)|$ is odd$\}$. If $|U|$ is odd, force $|U|$ to be even by adding a vertex u with $a(u) < \ell$ to U. Such a vertex u exists by the following reason; Suppose $a(v) = b(v) = \ell$ for all $v \in V$. If ℓ is even, then $U = \{v \in V \mid d(v; G_{r,a})$ is odd$\}$, which leads to the contradiction that $\sum_{v \in V} d(v; G_{r,a})$ is odd. If ℓ is odd, $|V|$ must be even since otherwise the problem instance would be infeasible. Because $U = \{v \in V \mid d(v; G_{r,a})$ is even$\}$ and $|U|$ is odd, the size of $V - U = \{v \in V \mid d(v; G_{r,a})$ is odd$\}$ is also odd, which leads to the above contradiction again. Hence we can assume that $|U|$ is even.

Then, compute a minimum cost perfect matching M on U and let $G'_{r,a}$ be the union of $G_{r,a}$ and M. As in the former case, we can transform $G'_{r,a}$ into a feasible solution. Since $c(M)$ is at most $2/k$ times the optimal cost as stated in Theorem 6, the resulting feasible solution is an $(\alpha + 2/k)$-approximate solution. □

By Theorem 5, we can let $\alpha = 2$ in Theorem 7.

6. Concluding Remarks

In this paper, we formulated the problem (V, r, a, b, c) of finding a minimum cost undirected multigraph with a connectivity requirement under degree bounds. This framework contains a number of fundamental and important problems. To develop a unified treatment of the framework, we derived new results on splitting and detachment, and revealed several conditions on functions r, a, b and c for which the above problem admits a constant-factor approximation algorithm. If we maintain the number of selected edges between every two vertices by a variable, we can implement our algorithms so that they compute solutions in polynomial time of the input size.

We still have some open problems. One is to find an approximation algorithm for the problem (V, r, a, b, c) with general degree bounds a and b. For example, k-traveling salesperson problem with $k \geq 2$ cannot be approximated by the algorithms in this paper. For the general degree bounds, a naive application of splitting may generate a self-loop. Hence we may need to characterize a condition for a graph to have splitting that does not create any self-loops. In [5], we gave a $(3 + 2/k)$-approximation algorithm for the instances with $r(u, v) = k$ for all $u, v \in V$ and $a = b \geq 2$, which contains k-traveling salesperson problem.

In Section 4, we proved the constant-factor approximability of $(V, r, 0, b, c)$ under the assumption that $b \geq 2$. However, it is also still open to approximate the problem for the case where $b(v)$ may be 1 for some vertex $v \in V$.

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