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ENUMERATING SPANNING AND CONNECTED SUBSETS IN GRAPHS AND MATROIDS *†

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Abstract We show that enumerating all minimal spanning and connected subsets of a given matroid can be solved in incremental quasi-polynomial time. In the special case of graphical matroids, we improve this complexity bound by showing that all minimal 2-vertex connected subgraphs of a given graph can be generated in incremental polynomial time.

Keywords: Algorithm, matroid, enumeration, quasi-polynomial

1. Introduction

The level of connectivity in communications and computer networks is an important parameter influencing the reliability of the service such networks provide. The problem of computing network reliability, that is calculating the probability that the network is able to provide its services without interruption, assumes the enumeration of minimal subsets of links in the network which guarantee the required level of connectivity [6, 22].

In the simplest case the connectivity of an undirected graph is required. In this case minimal working subsets are the spanning trees, and reliability computations call for the enumeration of all spanning trees of the given graph. In case of a (directed) communication network, minimal working subnetworks are determined by subsets of the arcs which guarantee strong connectivity. Both minimal spanning trees and minimal strongly connected subgraphs can be efficiently enumerated [4, 18, 21].

Practical applications frequently demand higher levels of connectivity, resulting in higher reliability. Numerous research articles consider the problem of increasing efficiently the connectivity of a given (directed) graph, see e.g. [1, 11]. Determining the reliability of such highly connected networks requires the enumeration of all minimal edge (arc) sets $F \subseteq E$ of a given (directed) graph $G = (V, E)$ which guarantee the required level of connectivity of the subgraph (V, F).

In this paper we consider such enumeration problems, corresponding to the next levels of connectivity. An undirected graph $G = (V, E)$ is called 2-*vertex connected* if between any pair $u, v \in V$ of its vertices there are at least two paths connecting u and v and having no other vertex in common. Obviously, adding edges to a graph can only increase its connectivity. In other words, the property that for a subset $X \subseteq E$ the subgraph (V, X)

*Our friend and co-author, Leonid Khachiyan passed away with tragic suddenness, while we were working on this paper.

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is 2-vertex connected is a monotone property, i.e., if $X \subseteq X' \subseteq E$, and (V, X) is 2-vertex connected, then (V, X') must also be 2-vertex connected. Thus, determining the reliability of such a graph, that is the probability that it remains 2-vertex connected if its edges are deleted (fail) according to some probability distribution, requires the enumeration of all minimal 2-vertex connected subgraphs of G:

Minimal 2-Vertex-Connected Spanning Subgraphs: *Given a 2-vertex connected undirected graph $G = (V, E)$, enumerate all minimal edge sets $X \subseteq E$ such that $G' = (V, X)$ is still 2-vertex connected.*

Undirected graphs can be viewed as special cases of matroids (so called graphical or cycle matroids), and thus the above enumeration problems have a natural generalization for matroids. In our presentation we follow standard terminology of matroid theory (see e.g., [17] or [23]). Given a matroid M on ground set E, a subset $T \subseteq E$ is called *connected* if for every pair of distinct elements x, y of T there is a circuit C of M such that $T \supseteq C \supseteq \{x, y\}$. It is well-known that connectivity defines an equivalence relation on E, whose equivalence classes are called the *connected components* of M. Let us also recall that a subset $X \subseteq E$ is said to *span* the matroid M if $r(X) = r(E)$, where $r : E \to \mathbb{Z}_+$ denotes the *rank* function of M and we define $r(M) = r(E)$.

Note that in the cycle matroid of a graph $G = (V, E)$ spanning trees are the bases. Thus, the problem of enumerating all bases of a matroid includes as a special case the spanning tree enumeration problem. Note also that edge subsets $X \subseteq E$ for which (V, X) are 2-vertex connected are exactly the spanning and connected subsets in the cycle matroid of G (see e.g., [23, Theorem 3 on page 70]). Let us add that spanning and connected subsets in a matroid form a monotone system (see Lemma 2.3), while connected subsets of a matroid may not form a monotone system. The following enumeration problem generalizes naturally the problem of enumerating minimal 2-vertex-connected spanning subgraphs:

Minimal Spanning and Connected Subsets in Matroids: *Given a connected matroid M on ground set E, generate all minimal spanning and connected subsets of E.*

1.1. Main results

It is easy to see that in enumeration problems, such as the ones mentioned above, the size of the output may be exponential in terms of the input size. For such problems the efficiency of the enumeration method is measured both in the input and output sizes (see e.g., [15, 18, 22]). In particular, the enumeration procedure is said to run in *incremental (quasi-) polynomial*[‡] *time*, if generating k elements of the target (or generating all if it has less than k elements) can be done in (quasi-) polynomial time in the size of the input and k, for an arbitrary integer k.

Note that all of the above mentioned enumeration problems involve monotone systems. Among such monotone generation problems perhaps the most widely known is the so called *hypergraph transversal* problem (or equivalently, *monotone Boolean dualization*). For a hypergraph $\mathcal{H} \subseteq 2^V$ on finite set of vertices $V = \{1, 2, \ldots n\}$, a set $X \subseteq V$ is called *transversal* if $X \cap H \neq \emptyset$ for all $H \in \mathcal{H}$. The hypegraph transversal problem is to generate all (inclusion-wise) minimal transversals of a given hypergraph. This problem has numerous applications in several different areas (see e.g., [2, 7–9]).

Our first result shows that the problem of generating minimal spanning and connected subsets of a matroid generalizes the hypergraph transversal problem:

[‡]A function $f(x)$ is called *quasi-polynomial* if $f(x) = O(2^{polylog(x)})$.

Theorem 1.1 *The problem of enumerating all minimal spanning and connected subsets of a matroid includes, as a special case, the hypergraph transversal problem.*

This theorem implies that generating minimal spanning and connected subsets of a matroid is at least as hard as the hypergraph transversal problem, for which the most efficient currently known algorithm is incrementally quasi-polynomial [10]. Our next result shows that minimal spanning and connected subsets in a matroid can also be generated in incremental quasi-polynomial time.

Theorem 1.2 *All minimal spanning and connected subsets in a matroid can be generated in incremental quasi-polynomial time.*

As we noted above, the problem of generating minimal spanning and connected subsets in a graphical matroid coincides with the problem of generating minimal 2-vertex-connected subgraphs of the underlying undirected graph. Our third result shows that in this special case the problem can be solved more efficiently:

Theorem 1.3 *All minimal 2-vertex connected spanning subgraphs of a given graph can be generated in incremental polynomial time. The total running time of the algorithm is $O(m^2N^2)$, where $m = |E|$ and N is the total number of 2-vertex connected subgraphs.*

The remainder of the paper is organized as follows. We prove Theorems 1.1 and 1.2 in Section 2, and the proof of Theorem 1.3 is included in Section 3.

2. Minimal Spanning and Connected Subsets in Matroids

2.1. Proof of Theorem 1.1

Let $\mathcal{H}$ be a hypergraph on n vertices consisting of $m = |\mathcal{H}|$ hyperedges. We denote by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ the column vectors of the edge-vertex incidence matrix of $\mathcal{H}$.

We shall associate to $\mathcal{H}$ a binary matroid $M = M_{\mathcal{H}}$, defined by $n+2m+2$ binary vectors of dimension $2m+2$. For this, let us introduce $\mathbf{o} = (0, ..., 0)$ denoting the zero vector of dimension m, and let $\mathbf{e}_i$ denote the ith unit vector of dimension m, for $i = 1, ..., m$. We shall define the vectors of $M_{\mathcal{H}}$ by concatenations from the above vectors, as follows: Let $\mu(\mathbf{v}_j) = (\mathbf{v}_j, \mathbf{o}, 1, 0)$ for $j = 1, ..., n$, let $\mathbf{a}_i = (\mathbf{e}_i, \mathbf{e}_i, 0, 0)$ and $\mathbf{b}_i = (\mathbf{o}, \mathbf{e}_i, 0, 0)$ for $i = 1, ..., m$, and finally let $\mathbf{c}_1 = (\mathbf{o}, \mathbf{o}, 1, 1)$ and $\mathbf{c}_2 = (\mathbf{o}, \mathbf{o}, 0, 1)$.

We group the above defined vectors into four groups: $H = \{\mu(\mathbf{v}_j) \mid j = 1, ..., n\}$, $A = \{\mathbf{a}_i \mid i = 1, ..., m\}$, $B = \{\mathbf{b}_i \mid i = 1, ..., m\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$, and finally we consider the binary matroid $M = M_{\mathcal{H}}$ on the ground set $E = H \cup A \cup B \cup C$. For simplicity, we re-interpret $\mathcal{H}$ as a family of subsets of H.

Example 2.1 *Consider the hypergraph $\mathcal{H}$ defined by the incidence matrix*

$$(\mathbf{v}_1, \ldots, \mathbf{v}_5) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Then the binary matroid $M_{\mathcal{H}}$ on the ground set $H \cup A \cup B \cup C$ is represented by a matrix

$$\left(\begin{array}{ccccc|cccc|cccc|cc}
1&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&1&1&0&1&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0\\
\hline
1&1&1&1&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{array}\right)$$

The following two lemmas complete the proof of Theorem 1.1. Let us recall (e.g., from [23]) that in a matroid M on ground set E with rank function r a subset $X \subseteq E$ is spanning and connected if and only if for every nontrivial partition $X = Y \cup Z$ (i.e., for which $|Y| \geq 1$ and $|Z| \geq 1$) we have $r(Y) + r(Z) \geq r(M) + 1$.

Lemma 2.1 *Let X be a spanning and connected subset in M. Then $A \cup B \cup C \subseteq X$ and $X \cap H$ is a transversal of $\mathcal{H}$.*

Proof: First we show that for each $i = 1, \ldots, 2m + 2$ at least two vectors of X have their ith coordinates equal to 1. Indeed, since X is spanning and the matrix representing M has full row rank, there is at least one vector in X whose ith coordinate is 1. Suppose that there is exactly one such vector $\mathbf{x} \in X$. Then we consider the partition of X into $Y = \{\mathbf{x}\}$ and $Z = X \setminus \{\mathbf{x}\}$. For this partition we have $r(Y) = 1$, and $r(Z) < r(M)$ since all vectors of Z have their ith coordinates equal to 0. Consequently, $r(Y) + r(Z) \leq r(M)$, contradicting the assumption that X is spanning and connected.

The above implies that X must contain all vectors of A, B and C, in order to have two 1's in rows from $m + 1$ to $2m$ and in row $2m + 2$. In order to contain two 1's in the first m rows, $H \cap X$ must form a transversal of $\mathcal{H}$. □

Lemma 2.2 *If X is a transversal of $\mathcal{H}$, then $X \cup A \cup B \cup C$ is a connected and spanning subset in M.*

Proof: First note that $r(M) = 2m + 2$, and observe that $X \cup A \cup B \cup C$ is spanning, since $r(A \cup B \cup C) = 2m + 2$.

To prove the statement, we show that $r(Y) + r(Z) \geq r(M) + 1 = 2m + 3$ for every partition $Y \cup Z = X \cup A \cup B \cup C$ for which $|Y| \geq 1$ and $|Z| \geq 1$.

Suppose that all vectors of $A \cup B \cup C$ belong to Y. Then $r(Y) = 2m + 2$ and $r(Z) \geq 1$, since Z is nonempty. For all other nontrivial partitions of $X \cup A \cup B \cup C$ the vectors of $A \cup B \cup C$ must be split between Y and Z.

Without loss of generality assume that Y contains k vectors of $A \cup B \cup C$ including $\mathbf{c}_1$, where $1 \leq k \leq 2m+1$. Since $A \cup B \cup C$ are independent, $r(Y) \geq k$ and $r(Z) \geq 2m+2-k$. Let $I_i \subseteq \{1, \ldots, m\}$ denote the set of coordinates of $\mathbf{v}_i$ equal to 1. Observe that $\mu(\mathbf{v}_i) = \mathbf{c}_1 + \mathbf{c}_2 + \sum_{j \in I_i}(\mathbf{a}_j + \mathbf{b}_j)$ is the only combination of vectors in $A \cup B \cup C$ producing $\mu(\mathbf{v}_i)$. Depending on whether Z contains a vector $\mu(\mathbf{v}_i) \in X$, or not, we have two cases:

Case 1: Z contains at least one vector of X. Since vectors of X cannot be obtained as a linear combination of vectors from $A \cup B \cup C$, without $\mathbf{c}_1$, we must have $r(Z) \geq 2m + 3 - k$ implying thus $r(Y) + r(Z) \geq 2m + 3$.

Case 2: Y contains all vectors of X. Since X is a transversal of $\mathcal{H}$, we have $\bigcup_{\mu(\mathbf{v}_i)\in X} I_i = \{1,\ldots,m\}$. As Y does not contain all vectors of $A \cup B \cup C$, there is a vector in X which cannot be obtained as a combination of vectors in $Y \setminus X$. Thus $r(Y) \geq k+1$, which again implies $r(Y)+r(Z) \geq 2m+3$. □

The statement of the theorem follows from Lemmas 2.1 and 2.2.

2.2. Proof of Theorem 1.2

Let M be a matroid on ground set E with rank function $r : E \to \mathbb{Z}_+$. Since no $X \subseteq E$ is connected if X contains a loop, i.e., a singleton of rank 0, we assume that M contains no loop.

Let us show first that spanning and connected subsets of a matroid form a monotone family.

Lemma 2.3 *If $X \subseteq E$ is spanning and connected then for an arbitrary element $f \in E \setminus X$ the set $X \cup f$ is again spanning and connected.*

Proof: Let $X \subseteq E$ be a spanning and connected subset and let $f \in E \setminus X$. Clearly $X \cup f$ is spanning. According to [23], to see that $X \cup f$ is also connected it is enough to show that $r(Y)+r(Z) \geq r(X \cup f)+1$ holds for an arbitrary partition $Y \cup Z = X \cup f$ with $|Y|, |Z| \geq 1$. Note that, since X is spanning, $r(X)+1 = r(X \cup f)+1$. Without loss of generality assume that $f \in Z$. If $|Z| = 1$ we have $r(Y)+r(Z) = r(X)+r(f) = r(X)+1$, since we assume that all singletons of M have rank 1. In case $|Z| \geq 2$, we have $r(Y)+r(Z) \geq r(Y)+r(Z \setminus f) \geq r(X)+1$, since $r(Z) \geq r(Z \setminus f)$ and the sets Y and $Z \setminus f$ form a partition of X, with $|Y| \geq 1$, $|Z \setminus f| \geq 1$, completing the proof of our claim. □

To prove Theorem 1.2, we show that for every matroid M, the family $\mathcal{F}$ of all minimal spanning and connected subsets in M is exactly the set of minimal solutions of a polymatroid inequality $f(X) \geq t$ with polynomially bounded right hand side t. Here $f : 2^E \to \mathbb{Z}$ is called *polymatroid* if it is *nondecreasing* (i.e., $X \subseteq Y\ (\subseteq E)$ always implies $f(X) \leq f(Y)$), and *submodular* (i.e., $f(X)+f(Y) \geq f(X \cap Y)+f(X \cup Y)$ holds for any $X, Y \subseteq E$). In this paper, we call f polymatroid, even though $f(\emptyset) \neq 0$. For such inequalities, it is known that the generation of minimal feasible sets can be done in incremental quasi-polynomial time (see Theorem 3 in [3]).

Toward this end, let us define a set function $f(X)$ on subsets of E by:

$$f(X) = |E|\, r(X) - 1.$$

The *Dilworth truncation of* $f(X)$ is the set function $f_*(X)$ defined as follows:

$$f_*(\emptyset) = 0,$$

$$f_*(X) = \min\{f(X_1)+\ldots+f(X_k) \mid \{X_1,\ldots,X_k\} \text{ is a partition of } X\} \text{ for } X \neq \emptyset.$$

It is known [16] that if $f(X)$ is submodular then so is $f_*(X)$, and moreover $f_*(X)$ can be evaluated in polynomial time using *poly*$(|E|)$ calls to the membership oracle defining the matroid M. We next show that $f_*(X)$ is also nondecreasing, implying that $f_*(X)$ is polymatroid.

Lemma 2.4 *$f_*(X)$ is nondecreasing.*

Proof: We will show that $f_*(X \cup e) \geq f_*(X)$, where $X \subseteq E$ and $e \in E \setminus X$. Let $X_1, X_2, \ldots, X_k$ be an optimal partition for $X \cup e$, i.e., $f_*(X \cup e) = f(X_1) + f(X_2) + \ldots + f(X_k)$. Without loss of generality assume that $e \in X_1$. There are two cases:
Case 1: $X_1 \setminus e \neq \emptyset$. Then $\{X_1 \setminus e, X_2, \ldots, X_k\}$ is a partition of X. Hence

$$f_*(X) \leq f(X_1 \setminus e) + \sum_{i=2}^{k} f(X_i) \leq f(X_1) + \sum_{i=2}^{k} f(X_i) = f_*(X \cup e),$$

where the last inequality in the chain follows from $f(X_1 \setminus e) = |E| r(X_1 \setminus e) - 1 \leq |E| r(X_1) - 1 = f(X_1)$.
Case 2: $X_1 = \{e\}$. Consider the partition $\{X_2, \ldots, X_k\}$ of X, which again gives

$$f_*(X) \leq \sum_{i=2}^{k} f(X_i) \leq f(e) + \sum_{i=2}^{k} f(X_i) = f_*(X \cup e),$$

where the last inequality in the chain follows from the fact that $r(e) = 1$, thus $f(e) = |E| - 1 \geq 0$, for all $e \in X$. □

Consider now the polymatroid inequality

$$f_*(X) \geq |E|\, r(M) - 1.$$

Note that the right hand side of the above inequality is bounded by $|E|^2$. We prove that minimal connected spanning subsets are exactly minimal solutions to the above polymatroid inequality.
Lemma 2.5 *X is connected in M if and only if $f_*(X) \geq f(X)$.*
Proof: Let X be a connected subset in M. Consider a partition $\{X_1, \ldots, X_k\}$ of X into at least $k \geq 2$ sets. Since the rank function is submodular and by the definition of connectivity we have $r(A) + r(X \setminus A) > r(X)$ for every proper subset A of X, we obtain $r(X_1) + r(X_2) + \ldots + r(X_k) \geq r(X_1) + r(X_2 \cup \ldots \cup X_k) \geq r(X) + 1$. Hence

$$f(X_1) + f(X_2) + \ldots + f(X_k) \geq |E|\, r(X) + |E| - k > |E|\, r(X) - 1 = f(X).$$

Comparing that with the trivial partition $X = X_1$ for $k = 1$, we conclude that $f_*(X) = f(X)$.

On the other hand, if X is not connected, then we can decompose X into two disjoint sets X_1 and X_2 such that $r(X_1) + r(X_2) = r(X)$. Hence $f(X_1) + f(X_2) = |E|\, r(X) - 2$, and consequently, $f_*(X) < |E| r(X) - 1 = f(X)$. □

Lemma 2.6 *X is connected and spanning subset in M if and only if $f_*(X) \geq |E|\, r(M) - 1$.*

Proof: If X is connected and spanning, the claim follows from Lemma 2.5 and the fact that $r(X) = r(M)$.

Conversely, suppose X satisfies $f_*(X) \geq |E|\, r(M) - 1$ and also suppose that X is not spanning. Then since $r(X) < r(M)$ for the trivial partition $X = X_1$, we obtain $f(X_1) = |E| r(X_1) - 1 < |E| r(M) - 1$, which implies $f_*(X) < |E| r(M) - 1$, a contradiction. Thus X must be spanning and by Lemma 2.5 X must also be connected. □

This completes the proof of Theorem 1.2.

2.3. The $X - e + Y$ method

In the next section we present another proof of Theorem 1.2, which is somewhat more direct, based on a general approach, called the $X - e + Y$ method. This approach will be particularly useful in the special case of graphical matroids, i.e., for the proof of Theorem 1.3.

Let us first recall the $X - e + Y$ method from [12, 13], which is a variant of the so called *supergraph approach* introduced in [19]. Let E be a finite set, and $\pi : 2^E \to \{0, 1\}$ be a monotone Boolean function, i.e., one for which $X \subseteq Y$ implies $\pi(X) \le \pi(Y)$. We assume that $\pi(\emptyset) = 0$ and $\pi(E) = 1$. We also assume that an efficient algorithm for evaluating $\pi(X)$ in polynomial time in the size of E is available for every $X \subseteq E$. Let

$$\mathcal{F} = \{X \mid X \subseteq E \text{ is a minimal set satisfying } \pi(X) = 1\}.$$

Our goal is to generate all sets belonging to $\mathcal{F}$.

Let us remark first that for every $X \subseteq E$ for which $\pi(X) = 1$ we can derive a subset $Y \subseteq X$ such that $Y \in \mathcal{F}$, by evaluating π exactly $|X|$ times. This can be accomplished (typically in many different ways) by deleting one-by-one elements of X the removal of which does not change the value of π. To formalize this, let us denote by $\mu(X) = Y$ such a minimal subset of X, i.e., μ is a mapping from $\{X \subseteq E \mid \pi(X) = 1\}$ to $\mathcal{F}$ such that $\mu(X) \subseteq X$.

We next introduce a directed graph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ on vertex set $\mathcal{F}$, where the edge set $\mathcal{E}$ is defined by specifying the neighborhood $N(X)$ for every $X \in \mathcal{F}$ as

$$N(X) = \{\mu((X \setminus e) \cup Y) \mid e \in X, Y \in \mathcal{Y}_{X,e}\},$$

where

$$\mathcal{Y}_{X,e} = \{Y \mid Y \subseteq E \setminus X \text{ is a minimal set satisfying } \pi((X \setminus e) \cup Y) = 1\}.$$

In other words, for every set $X \in \mathcal{F}$ and for every element $e \in X$ (since $X \in \mathcal{F}$, we have $\pi(X \setminus e) = 0$) we extend $X \setminus e$ in all possible ways to a set $X' = (X \setminus e) \cup Y$ for which $\pi(X') = 1$, and introduce each time a directed arc from X to $\mu(X')$. We call the obtained directed graph $\mathcal{G}$ a *supergraph* of our generation problem.

Proposition 2.1 ([12, 13]) *The supergraph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ is strongly connected.* □

Since $\mathcal{G}$ is strongly connected by performing a breadth-first search in $\mathcal{G}$ we can generate all elements of $\mathcal{F}$ as follows:

Traversal($\mathcal{G}$)

Find an initial vertex $X^0 \leftarrow \mu(E)$, initialize a queue $\mathcal{Q} = \emptyset$ and a dictionary of output vertices $\mathcal{D} = \emptyset$.

Perform a breadth-first search of $\mathcal{G}$ starting from X^o:

```
output X^0 and insert it to Q and to D
while Q ≠ ∅ do
   take the first vertex X out of the queue Q
   for every e ∈ X do
      for every Y ∈ Y_{X,e} do
         compute the neighbor X' ← μ((X ∖ e) ∪ Y)
         if X' ∉ D then
              output X' and insert it to Q and to D
```

Proposition 2.2 ([12, 13]) *If the sets of $\mathcal{Y}_{X,e}$ can be generated in incremental polynomial time for every $X \in \mathcal{F}$ and $e \in X$, then Traversal($\mathcal{G}$) generates all elements of $\mathcal{F}$ in incremental polynomial time.* □

2.4. Second proof of Theorem 1.2

In this section, we present another proof of Theorem 1.2 by applying the $X - e + Y$ method. For the problem of generating all minimal connected spanning subsets of a matroid M, let us define $\pi(X) = 1$ if X is connected and spanning of M, and 0 otherwise. By Lemma 2.3, π is monotone, and the corresponding family $\mathcal{F}$ of all minimal subsets X for which $\pi(X) = 1$ is exactly the family of minimal spanning and connected subsets of M.

We apply the $X - e + Y$ method for the generation of $\mathcal{F}$, as described in the previous subsection. By Proposition 2.2, it is sufficient to prove the following statement.

Proposition 2.3 *All elements of $\mathcal{Y}_{X,e}$ can be generated in incremental quasi-polynomial time.*

Proof: First we show that $X \setminus e$ is spanning. Since X is connected there is a circuit C in X containing e. Thus the independent set $C \setminus e$ must be contained in a maximal independent set $B \subseteq X$ such that $e \notin B$. Since X is spanning and $B \subseteq X$, B is a basis. Thus $X \setminus e$ is spanning.

Since X is a minimal connected spanning subset of E and $X \setminus e$ is spanning, $X \setminus e$ must be partitioned into connected components $K_1, \ldots, K_r$, $r \geq 2$. For a cycle C we denote by $I(C)$ the set of indices of components having some elements belonging to C:

$$I(C) = \{i \in \{1, ..., r\} \mid K_i \cap C \neq \emptyset\}.$$

For $y \in E \setminus X$ let

$$\mathcal{C}_y = \{C \mid C \text{ is a cycle, } y \in C \text{ and } C \subseteq (X \setminus e) \cup y\}$$

be a family of all cycles containing y and some elements of $X \setminus e$. Note that C_y is nonempty for every $y \in E \setminus X$, since $X \setminus e$ is spanning.

We show that for every cycle $C \in \mathcal{C}_y$ the set $I(C)$ contains the same indices.

Lemma 2.7 *There is $I_y \subseteq \{1, \ldots, r\}$ such that $I(C) = I_y$ for every $C \in \mathcal{C}_y$.*

Proof: Let $I_y = I(C)$, where $C \in \mathcal{C}_y$ is an arbitrary cycle. Suppose that there is a cycle $C' \in \mathcal{C}_y$ such that $I(C') \setminus I_y \neq \emptyset$. Let $i \in I(C') \setminus I_y$. Then there is an element $z \in K_i$ belonging to C' but no element of K_i belongs to C. By the strong circuit axiom there is a cycle $C'' \subseteq (C \cup C') \setminus y$ such that $z \in C''$. Since $y \notin C''$, all elements of C'' belong to $K_1, \ldots, K_r$. Observe that C'' must contain at least one element of C, since $C' \setminus y$ is independent. Thus C'' contains elements of two different connected components, a contradiction. □

We construct a hypergraph $\mathcal{H}$ on vertex set $\{1, \ldots, r\}$ whose edges are the sets I_y for $y \in E \setminus X$. A *connected set cover* Y of $\mathcal{H}$ is a subset of hyperedges satisfying:

(i) $\bigcup_{y \in Y} I_y = \{1, \ldots, r\}$ and

(ii) there is no partition of Y into Y_1 and Y_2 such that $\bigcup_{y \in Y_1} I_y$ and $\bigcup_{y \in Y_2} I_y$ are disjoint subsets of vertices.

We now prove that generating all minimal connected set covers of $\mathcal{H}$ is equivalent to the generation of all elements of $\mathcal{Y}_{X,e}$.

Lemma 2.8 *Let $Y \subseteq E \setminus X$. Then $\{I_y \mid y \in Y\}$ is a connected set cover of $\mathcal{H}$ if and only if $(X \setminus e) \cup Y$ is connected.*

Proof: If $\{I_y \mid y \in Y\}$ is a connected set cover of $\mathcal{H}$ then by Lemma 2.7 and by the definition of a connected set cover it reconnects all components $K_1, \dots, K_r$.

Conversely suppose that $(X \setminus e) \cup Y$ is connected. Let $C \subseteq (X \setminus e) \cup Y$ be a cycle containing at least one element of Y. We prove that $I(C) \subseteq \bigcup_{y \in C \cap Y} I_y$ by induction on the size of $C \cap Y$. If $C \cap Y = \{z\}$, then by Lemma 2.7 $I(C) = I_z$. Suppose that there is $j \in I(C) \setminus \bigcup_{y \in C \cap Y} I_y$. Then there is an element $z \in K_j$ belonging to C but no element of K_j belongs to cycles in $\mathcal{C}_y$, for each $y \in C \cap Y$. Pick an arbitrary $w \in C \cap Y$. Let $C' \in \mathcal{C}_w$. By the strong circuit axiom, there is a cycle $C'' \subseteq (C \cup C') \setminus w$ such that $z \in C''$. If C'' does not contain any elements of Y, then C'' connects K_j to some other component, a contradiction. Thus $C'' \cap Y$ is nonempty. Since $|C'' \cap Y| < |C \cap Y|$, we have $I(C'') \subseteq \bigcup_{y \in C'' \cap Y} I_y$. As $z \in C''$, we obtain $j \in \bigcup_{y \in C'' \cap Y} I_y$, a contradiction. Hence we have $I(C) \subseteq \bigcup_{y \in C \cap Y} I_y$.

By the above and since all components are connected by the cycles contained in $(X \setminus e) \cup Y$, we have $\bigcup_{y \in Y} I_y = \{1, \dots, r\}$. Hence Y satisfies (i).

Suppose that Y does not satisfy (ii), thus there is a partition of Y into Y_1 and Y_2 such that $\bigcup_{y \in Y_1} I_y = R_1$ and $\bigcup_{y \in Y_2} I_y = R_2$, where R_1 and R_2 are disjoint subsets of vertices. Since $X \setminus e \cup Y$ is connected, there is a cycle C containing elements of components in R_1 and R_2 with $|C \cap Y|$ minimal. Observe that C must contain at least two elements of Y. Pick $y \in C \cap Y$ and $C_y \in \mathcal{C}_y$. Assume $y \in Y_1$. Let $w \in C$ be an element of a component in R_2. By the strong circuit axiom, there is a cycle $C'' \subseteq (C \cup C_y) \setminus y$ such that $w \in C''$. C'' must contain an element of C_y, belonging to a component in R_1, since $C \setminus y$ is independent. Thus C'' is the cycle containing elements of components in R_1 and R_2 and $|C'' \cap Y| < |C \cap Y|$, because $y \notin C''$, a contradiction to the selection of C. □

To prove that all minimal connected set covers of $\mathcal{H}$ can be generated in incremental quasi-polynomial time, we show that this problem reduces to generating all minimal spanning collections of a graph, which can be done in incremental quasi-polynomial time (see Section 2.6 in [3]). Let $R = \{1, \dots, r\}$ be a vertex set. For every $y \in E \setminus X$, let $E_y = \{(i,j) \mid i, j \in I_y, i \neq j\}$ be a clique on vertices of I_y. Recall that $f(Y) = r - k(Y) \geq t$ is a polymatroid inequality, where $k(Y)$ be the number of connected components in the graph $(R, \bigcup_{y \in Y} E_y)$. Thus we can generate elements of the family $\mathcal{B}_t$ of all minimal subsets $Y \subseteq E \setminus X$ satisfying the inequality $f(Y) \geq t$ in incremental quasi-polynomial time for every $t \in \{1, \dots, r\}$. Now observe that elements of $\mathcal{B}_{r-1}$ are minimal subsets of $E \setminus X$ such that the graph $(R, \bigcup_{y \in Y} E_y)$ is connected. Hence they are minimal connected set covers of $\mathcal{H}$.

This completes the proof of Theorem 1.2. □

Let us finally remark that our first proof of Theorem 1.2, which shows that the family $\mathcal{F}$ of minimal spanning and connected subsets of a matroid is exactly the family of all minimal feasible solutions of a corresponding polymatroid inequality, implies (by [3]) quasi-polynomially *dual-boundedness*, that is, that the inequality $|\mathcal{F}^*| \leq q(|\mathcal{F}|)$ holds for a quasi-polynomial function $q()$, where $\mathcal{F}^*$ denotes the family of all maximal non-connected spanning subsets of M. Thus the so-called *joint generation* approach would also provide us with a quasi-polynomial generation algorithm (see [5] for details and precise definitions). However, it is still open whether the problem is polynomially dual-bounded.

3. Minimal 2-Connected Spanning Subgraphs

3.1. Proof of Theorem 1.3

We apply the $X-e+Y$ method to the generation of all minimal 2-vertex connected spanning subgraphs of a 2-vertex connected graph $G=(V,E)$.

For $X \subseteq E$, we define a Boolean function π as follows:

$$\pi(X) = \begin{cases} 1, & \text{if (V,X) is 2-vertex connected;} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly π is monotone and it can be evaluated in $O(|V|+|E|)$ time [20]. Then $\mathcal{F} = \{X \mid X \subseteq E$ is a minimal set satisfying $\pi(X)=1\}$ is a family of edge sets of all minimal 2-vertex connected spanning subgraphs of (V,E). It is not difficult to see that we can compute from G a minimal 2-vertex connected spanning subgraph X in $O(|V|+|E|)$ time. This implies that we have a mapping $\mu : \{X \subseteq E \mid \pi(X)=1\} \to \mathcal{F}$ such that $\mu(X) \subseteq X$, in which $\mu(X)$ is linearly computable from X. For $X \in \mathcal{F}$ and $e \in X$ we define

$$\mathcal{Y}_{X,e} = \{Y \mid Y \subseteq E \setminus X \text{ is a minimal set satisfying } \pi((X \setminus e) \cup Y) = 1\}.$$

Therefore by Proposition 2.2 we only need to prove that we can generate all elements of $\mathcal{Y}_{X,e}$ in incremental polynomial time. In fact, we show that we can do it, more efficiently, with *polynomial delay*, i.e., in which the generation of the first k elements can be accomplished in time polynomial in the input size and linear in k.

Recall that a maximal connected subgraph without a cutvertex is called a *block*. Thus, every block of a connected graph H is either a maximal 2-vertex connected subgraph, or a bridge (with its ends). Different blocks overlap in at most one vertex, which is a cutvertex of H. Hence, every edge of the graph lies in a unique block.

Let A denote the set of cutvertices of H and let $\mathcal{B}$ denote the set of its blocks. We then have a natural bipartite graph on vertex set $A \cup \mathcal{B}$ in which two vertices $B \in \mathcal{B}$, $a \in A$ are connected if a is a cutvertex of H belonging to B. We call such graph a *block graph* of H. Observe that the block graph of a connected graph is a tree.

Proposition 3.1 *All elements of $\mathcal{Y}_{X,e}$ can be generated with polynomial delay.*

Proof: Let (V,X) be a minimal 2-vertex connected spanning subgraph of (V,E) (see Figure 1).

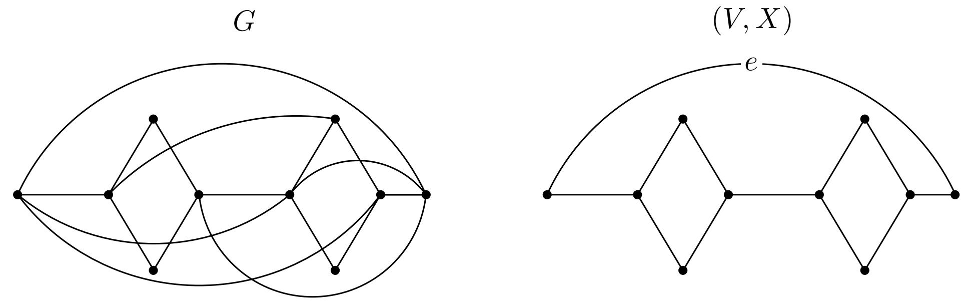

Figure 1: 2-vertex connected graph $G=(V,E)$ and a minimal 2-connected spanning subgraph (V,X) of G.

First we show that the block graph of $(V, X \setminus e)$ is a path such that endpoints of e belong to its ends. As we observed above the block graph of $(V, X \setminus e)$ is a tree. Suppose it has a leaf B that does not contain an endpoint of e. Let a be a cutvertex of $(V, X \setminus e)$ adjacent to B in the block graph. But removing the vertex a from the 2-vertex connected graph (V, X) disconnects vertices of B from other vertices, a contradiction. Thus the block graph of $(V, X \setminus e)$ has only two leaves, each containing one endpoint of e.

We denote by $B_1, \ldots, B_r$ the blocks of $(V, X \setminus e)$ and by $a_1, \ldots, a_{r-1}$ its cutvertices. Without loss of generality we assume that the block graph of $(V, X \setminus e)$ is a path $B_1 a_1 B_2 \ldots a_{r-1} B_r$ (see Figure 2).

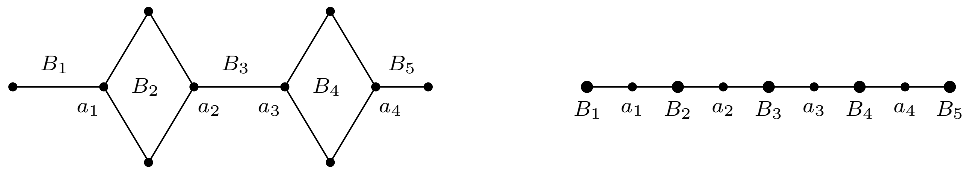

Figure 2: $(V, X \setminus e)$ and its block graph

Let $f = uv$ be an edge of $E \setminus X$, such that u belongs to the block B_i and v belongs to B_j, where $i < j$. We define

$$\alpha(f) = \begin{cases} i, & \text{if } v \in B_i \setminus a_i \ ; \\ i+1, & \text{if } v = a_i, \end{cases} \quad \text{and} \quad \beta(f) = \begin{cases} j, & \text{if } v \in B_j \setminus a_{j-1} \ ; \\ j-1, & \text{if } v = a_{j-1}. \end{cases}$$

Then we construct a directed multigraph D on vertex set $B_1, \ldots, B_r$ whose arc set is defined as follows:

- for each $i = 1, \ldots, r-1$, we add an arc $B_{i+1}B_i$,
- for each edge $f \in E \setminus X$, such that $\alpha(f) < \beta(f)$, we add an arc $B_{\alpha(f)}B_{\beta(f)}$ (see Figure 3).

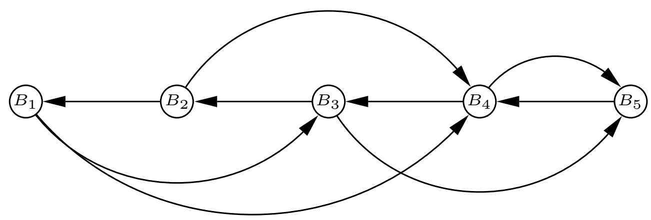

Figure 3: Directed multigraph D.

Claim 3.1 *The directed multigraph D has at most $|V|$ vertices and $|V| + |E|$ arcs and it can be constructed in $O(|V| + |E|)$.*

Proof: The number r of vertices of D is equal to the number of blocks of $(V, X \setminus e)$, which is at most $|V|$. D has exactly $r - 1$ arcs between subsequent vertices and at most $|E|$ arcs corresponding to edges of $E \setminus X$. Thus D has at most $|V| + |E|$ arcs.

We can construct D as follows:

- find all blocks of $(V, X \setminus e)$ in $O(|V| + |E|)$ time [20],
- create r vertices and add $r - 1$ arcs between subsequent vertices and at most $|E|$ arcs corresponding to edges of $E \setminus X$; this step also takes $O(|V| + |E|)$ time.

□

Now we show that the generation of elements $Y \in \mathcal{Y}_{X,e}$ is equivalent to the generation of minimal directed B_1-B_r paths in D.

For every cutvertex a_k there is an edge $f \in Y$ such that $\alpha(f) \leq k < \beta(f)$. By minimality of Y, edges of $E \setminus X$ whose both endpoints belong to the same block cannot be in Y. We conclude that $Y = \{f_1, \ldots, f_s\}$ such that

$$1 = \alpha(f_1) < \alpha(f_2) \leq \beta(f_1) < \alpha(f_3) \leq \ldots < \alpha(f_s) \leq \beta(f_{s-1}) < \beta(f_s) = r.$$

Thus Y corresponds to a directed path

$$B_{\alpha(f_1)}B_{\beta(f_1)}B_{\beta(f_1)-1}\ldots B_{\alpha(f_2)+1}B_{\alpha(f_2)}B_{\beta(f_2)}B_{\beta(f_2)-1}\ldots B_{\alpha(f_3)+1}B_{\alpha(f_3)}B_{\beta(f_3)}\ldots B_{\beta(f_s)}$$

(see Figure 4).

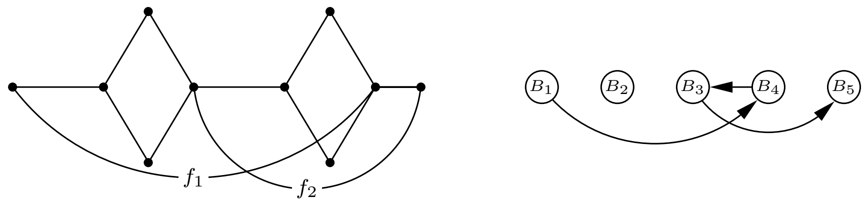

Figure 4: $Y = \{f_1, f_2\}$ and corresponding directed path $B_1B_4B_3B_5$.

Since all minimal directed paths between two vertices can be generated via backtracking with polynomial delay [18], Proposition 3.1 follows. $\square$

This completes the proof of Theorem 1.3.

3.2. Complexity

In this section we analyze the total running time of the procedure *Traversal*($\mathcal{G}$). Let $n = |V|$, $m = |E|$ and $N = |\mathcal{F}|$. We observe $m \geq n$, since G is 2-vertex connected.

As noticed in the previous section, the initial vertex X^o of the supergraph can be computed in $O(m)$ time. Outputting a vertex of the supergraph takes $O(m)$ time and we output each vertex only once. Thus the total time of outputting vertices is $O(mN)$.

Each vertex of the supergraph is inserted to the queue $\mathcal{Q}$ and removed from $\mathcal{Q}$ only once. The operations of enqueuing and dequeuing take $O(1)$ time, so the total time devoted to queue operations is $O(N)$. We implement the dictionary $\mathcal{D}$ for $\mathcal{F}$ as a binary tree with depth m. Here leaves in the tree store elements $X \in \mathcal{F}$, and the components of X correspond to the path from the root to the leaf. Since the insert and find operations in the tree take $O(m)$ time, the total time devoted to insert operations is $O(mN)$.

Since a vertex is removed from $\mathcal{Q}$ every time we execute the while loop (lines **2-8**) and it will never be reinserted to $\mathcal{Q}$, the while loop is executed at most N times. As a vertex of the supergraph is a subset of edges of the input graph, the for loop (lines **4-8**) is executed at most m times. We now analyze the time *Traversal*($\mathcal{G}$) spends performing lines of the main loop.

line 5: As we noted above we generate at most mN families $\mathcal{Y}_{X,e}$. Also note that we always have $|\mathcal{Y}_{X,e}| \leq N$. By the claim, the construction of the directed multigraph D takes $O(m)$ time and this graph has at most n vertices and $n + m$ arcs. Since the generation of elements of $\mathcal{Y}_{X,e}$ is equivalent to the generation of directed paths in D we obtain that for a given X and e we can compute $\mathcal{Y}_{X,e}$ in $O(mN)$ time [18]. Thus the total time spent generating families $\mathcal{Y}_{X,e}$ is $O(m^2N^2)$.

line 6: We compute μ at most mN^2 times. Thus the total time of computing μ is $O(m^2N^2)$.

line 7: We test if $\mathcal{D}$ contains X' at most mN^2 times. Thus the total running time of the test is $O(m^2N^2)$.

Therefore, in total *Traversal*($\mathcal{G}$) runs in $O(m^2N^2)$ time.

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