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LINKING SYSTEMS AND MATROID PENCILS

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Abstract A matroid pencil is a pair of linking systems having the same ground sets in common. It provides a combinatorial abstraction of matrix pencils. This paper investigates the properties of matroid pencils analogous to the theory of Kronecker canonical form. As an application, we give a simple alternative proof for a theorem of Murota on power products of linking systems.

Keywords: Combinatorial optimization, matroid theory

1. Introduction

Linking systems (or bimatroids) were introduced by Kung [3] and Schrijver [7] as a combinatorial abstraction of matrices. They naturally provide combinatorial counterparts of linear algebraic notions such as multiplications of matrices. In this paper, we introduce matroid pencils as a combinatorial abstraction of matrix pencils.

A matrix pencil is a pair of matrices of the same size. It is often treated as a polynomial matrix whose nonzero entries are of degree at most one. Based on the theory of elementary divisors, Weierstrass established a criterion for strict equivalence, as well as a canonical form, of regular matrix pencils. Somewhat later, Kronecker investigated singular pencils to obtain a canonical form for matrix pencils in general under strict equivalence transformations, which is now called the Kronecker canonical form (or the Kronecker-Weierstrass normal form). The Kronecker canonical form of matrix pencils plays fundamental roles in application areas such as differential algebraic equations and control theory.

The Kronecker canonical form is characterized by the structural indices determined by the ranks of expanded matrices (Theorems 2.1 and 2.2). For matroid pencils, we define associated linking systems corresponding to the expanded matrices. Then we show that the ranks of these linking systems have the same properties as the expanded matrices (Lemmas 4.1–4.11), which enables us to define "structural indices" of matroid pencils. In particular, we will reveal that the ranks of a certain type of the associated linking systems are determined by some periodic structure (Theorem 5.1). This result in turn brings about an alternative proof of a theorem of Murota [4] on power products of linking systems.

The outline of this paper is as follows. Section 2 is devoted to a brief description of the Kronecker canonical form of matrix pencils. Section 3 provides a preliminary on linking systems. In Section 4, we introduce matroid pencils and describe their properties. Section 5 investigates the periodic structure. Finally, in Section 6, we present an alternative proof for the theorem on power products of linking systems.

2. The Kronecker Canonical Form of Matrix Pencils

Let $D(s) = sA + B$ be an $m \times n$ matrix pencil of rank r. A matrix pencil $\bar{D}(s)$ is said to be strictly equivalent to $D(s)$ if there exists a pair of nonsingular constant matrices U and V such that $\bar{D}(s) = UD(s)V$. A matrix pencil $D(s) = sA + B$ is said to be regular if $\det D(s) \neq 0$ as a polynomial in s. It is strictly regular if both A and B are nonsingular matrices.

For a positive integer μ, we consider $\mu \times \mu$ matrix pencils N_μ and K_μ defined by

$$N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & s \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \qquad K_\mu = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & s & 1 \\ 0 & \cdots & \cdots & 0 & s \end{pmatrix}.$$

For a positive integer ε, we further denote by L_ε an $\varepsilon \times (\varepsilon + 1)$ matrix pencil

$$L_\varepsilon = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s & 1 \end{pmatrix}.$$

We also denote by $L_\eta^\top$ the transpose matrix of L_η.

The following theorem establishes the Kronecker canonical form of matrix pencils under strict equivalence transformations. See [1, §XII.4] and [5, §5.1.3] for its proofs.

Theorem 2.1 (Kronecker, Weierstrass) *For any matrix pencil $D(s)$, there exists a pair of nonsingular constant matrices U and V such that $\bar{D}(s) = UD(s)V$ is in a block-diagonal form*

$$\bar{D}(s) = \text{block-diag}(H_\nu, K_{\rho_1}, \cdots, K_{\rho_c}, N_{\mu_1}, \cdots, N_{\mu_d}, L_{\varepsilon_1}, \cdots, L_{\varepsilon_p}, L_{\eta_1}^\top, \cdots, L_{\eta_q}^\top, O),$$

where $\rho_1 \geq \cdots \geq \rho_c > 0$, $\mu_1 \geq \cdots \geq \mu_d > 0$, $\varepsilon_1 \geq \cdots \geq \varepsilon_p > 0$, $\eta_1 \geq \cdots \geq \eta_q > 0$, and H_ν is a strictly regular matrix pencil of size ν. The numbers c, d, p, q, ν, $\rho_1, \cdots, \rho_c$, $\mu_1, \ldots, \mu_d$, $\varepsilon_1, \cdots, \varepsilon_p$, $\eta_1, \cdots, \eta_q$ are uniquely determined.

The block-diagonal matrix pencil $\bar{D}(s)$ in Theorem 2.1 is referred to as the Kronecker canonical form of $D(s)$. The numbers $\mu_1, \ldots, \mu_d$ are called the indices of nilpotency. The numbers $\varepsilon_1, \cdots, \varepsilon_p$ and $\eta_1, \cdots, \eta_q$ are the minimal column and row indices, respectively. These numbers together with $\nu, \rho_1, \cdots, \rho_c$ are collectively called the structural indices of $D(s)$.

For an $m \times n$ matrix pencil $D(s) = sA + B$, we construct a $(k+1)m \times kn$ matrix $\Psi_k(D)$ and a $km \times (k+1)n$ matrix $\Phi_k(D)$ defined by

$$\Psi_k(D) = \begin{pmatrix} A & O & \cdots & O \\ B & A & \ddots & \vdots \\ O & B & \ddots & O \\ \vdots & \ddots & \ddots & A \\ O & \cdots & O & B \end{pmatrix}, \qquad \Phi_k(D) = \begin{pmatrix} B & A & O & \cdots & O \\ O & B & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O \\ O & \cdots & O & B & A \end{pmatrix}.$$

We denote $\psi_k(D) = \operatorname{rank}\Psi_k(D)$ and $\varphi_k(D) = \operatorname{rank}\Phi_k(D)$. We also construct a pair of $km \times kn$ matrices $\Theta_k(D)$ and $\Omega_k(D)$ defined by

$$\Theta_k(D) = \begin{pmatrix} A & O & \cdots & \cdots & O \\ B & A & \ddots & & \vdots \\ O & B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A & O \\ O & \cdots & O & B & A \end{pmatrix}, \qquad \Omega_k(D) = \begin{pmatrix} B & A & O & \cdots & O \\ O & B & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O \\ \vdots & & \ddots & B & A \\ O & \cdots & \cdots & O & B \end{pmatrix}.$$

We denote $\theta_k(D) = \operatorname{rank}\Theta_k(D)$ and $\omega_k(D) = \operatorname{rank}\Omega_k(D)$. Then it is easy to see that the ranks of these expanded matrices are expressed by the structural indices as follows.

Theorem 2.2 *Let $(\nu, \rho_1, \cdots, \rho_c, \mu_1, \ldots, \mu_d, \varepsilon_1, \ldots, \varepsilon_p, \eta_1, \ldots, \eta_q)$ be structural indices of a matrix pencil $D(s)$. Then we have*

$$\psi_k(D) = rk + \sum_{i=1}^{p} \min\{k, \varepsilon_i\}, \qquad \varphi_k(D) = rk + \sum_{i=1}^{q} \min\{k, \eta_i\},$$
$$\theta_k(D) = rk - \sum_{i=1}^{d} \min\{k, \mu_i\}, \qquad \omega_k(D) = rk - \sum_{i=1}^{c} \min\{k, \rho_i\},$$

where r is the rank of $D(s)$.

This theorem enables us to determine the structural indices from the sequences ψ_k, φ_k, θ_k, and ω_k for $k = 1, \ldots, r$. In particular, we have the following corollary.

Corollary 2.1 *The size ν of the strictly regular block in the Kronecker canonical form of $D(s)$ is given by*

$$\nu = r - \varphi_r - \varphi_r + \theta_r + \omega_r,$$

where r is the rank of $D(s)$.

3. Linking Systems

Let S and T be a pair of finite sets. Let Λ be a nonempty collection of pairs of subsets of S and T. Then the triple (S, T, Λ) is a linking system if it satisfies the following axioms.

(L1) If $(X, Y) \in \Lambda$, then $|X| = |Y|$.
(L2) If $(X, Y) \in \Lambda$ and $x \in X$, then there exists $y \in Y$ such that $(X \setminus \{x\}, Y \setminus \{y\}) \in \Lambda$.
(L3) If $(X, Y) \in \Lambda$ and $y \in Y$, then there exists $x \in X$ such that $(X \setminus \{x\}, Y \setminus \{y\}) \in \Lambda$.
(L4) If $(X, Y) \in \Lambda$ and $(X', Y') \in \Lambda$, then there exists $(X^\circ, Y^\circ) \in \Lambda$ such that $X \subseteq X^\circ \subseteq X \cup X'$ and $Y' \subseteq Y^\circ \subseteq Y \cup Y'$.

A member of Λ is called a linked pair. The sets S and T are respectively called the row set and the column set of Λ.

The rank function $\lambda : 2^S \times 2^T \to \mathbf{Z}$ of $\mathbf{L} = (S, T, \Lambda)$ defined by

$$\lambda(X, Y) = \max\{|W| \mid (W, Z) \in \Lambda, W \subseteq X, Z \subseteq Y\} \qquad (X \subseteq S, Y \subseteq T)$$

satisfies the following properties.

(R1) $0 \leq \lambda(X, Y) \leq \min\{|X|, |Y|\}$ for any $X \subseteq S$ and $Y \subseteq T$.
(R2) $\lambda(X, Y) \leq \lambda(X', Y')$ for any $X \subseteq X' \subseteq S$ and $Y \subseteq Y' \subseteq T$.
(R3) $\lambda(X, Y) + \lambda(X', Y') \geq \lambda(X \cup X', Y \cap Y') + \lambda(X \cap X', Y \cup Y')$ for any $X, X' \subseteq S$ and $Y, Y' \subseteq T$.

In particular, (R3) is referred to as linking bisubmodularity. The rank of $\mathbf{L}$, denoted by $r(\mathbf{L})$, is the maximum size $|X|$ of a linked pair $(X, Y) \in \Lambda$, i.e., $r(\mathbf{L}) = \lambda(S, T)$.

Alternatively, we may define linking systems in terms of rank functions satisfying the above (R1)–(R3). Then the family Λ of linked pairs is determined by

$$\Lambda = \{(X, Y) \mid \lambda(X, Y) = |X| = |Y|, X \subseteq S, Y \subseteq T\}.$$

A principal example of linking systems comes from matrices. Let A be a matrix with row set S and column set T. For a pair of $X \subseteq S$ and $Y \subseteq T$, we denote by $A[X, Y]$ the submatrix of A indexed by X and Y. Then $\mathbf{L}(A) = (S, T, \Lambda(A))$ is a linking system, where

$$\Lambda(A) = \{(X, Y) \mid \operatorname{rank} A[X, Y] = |X| = |Y|, X \subseteq S, Y \subseteq T\}.$$

The rank function λ of $\mathbf{L}(A)$ is given by

$$\lambda(X, Y) = \operatorname{rank} A[X, Y],$$

which satisfies(R1)–(R3).

Another interesting example of linking systems comes from graphs. Consider a directed graph G with a pair of disjoint vertex subsets S and T. Let $\Lambda(G)$ be the family of pairs (X, Y) of $X \subseteq S$ and $Y \subseteq T$ with $|X| = |Y|$ such that there exist $|X|$ pairwise disjoint directed paths from X to Y. Then $(S, T, \Lambda(G))$ forms a linking system.

For a pair of linking systems $\mathbf{L} = (S, T, \Lambda)$ and $\mathbf{L}' = (S', T', \Lambda')$, the union $\mathbf{L} \vee \mathbf{L}' = (S \cup S', T \cup T', \Lambda \vee \Lambda')$ defined by

$$\Lambda \vee \Lambda' = \{(X \cup X', Y \cup Y') \mid X \cap X' = \emptyset, Y \cap Y' = \emptyset, (X, Y) \in \Lambda, (X', Y') \in \Lambda'\}$$

is a linking system. Note that $S \cap S'$ and $T \cap T'$ can be nonempty.

Lemma 3.1 *Let λ and λ' be the rank functions of $\mathbf{L} = (S, T, \Lambda)$ and $\mathbf{L}' = (S', T', \Lambda')$. Then the rank function $\lambda \vee \lambda'$ of $\mathbf{L} \vee \mathbf{L}'$ is given by*

$$(\lambda \vee \lambda')(X, Y) = \min_{W \subseteq X, Z \subseteq Y} \{\lambda(W \cap S, Z \cap T) + \lambda'(W \cap S', Z \cap T') + |X \setminus W| + |Y \setminus Z|\}.$$

The union of linking systems is analogous to the addition of matrices. Similarly, multiplication of linking systems is defined as follows. For a pair of linking systems $\mathbf{A} = (R, S, \Lambda)$ and $\mathbf{B} = (S, T, \Xi)$, the multiplication is defined by $\mathbf{A} * \mathbf{B} = (R, T, \Lambda * \Xi)$ with

$$\Lambda * \Xi = \{(W, Y) \mid \exists X \subseteq S, (W, X) \in \Lambda, (X, Y) \in \Xi\}.$$

Let $\mathbf{I} = (S, S, \Delta)$ denote the diagonal linking system with $\Delta = \{(X, X) \mid X \subseteq S\}$. Then we have the following lemma.

Lemma 3.2 *The rank of $\mathbf{A} * \mathbf{B}$ satisfies*

$$r(\mathbf{A} * \mathbf{B}) = r(\mathbf{A} \vee \mathbf{I} \vee \mathbf{B}) - |S|.$$

4. Matroid Pencils

A matroid pencil is a pair of linking systems having the row/column sets in common. Consider a matroid pencil $(\mathbf{A}, \mathbf{B})$ with $\mathbf{A} = (S, T, \Lambda)$ and $\mathbf{B} = (S, T, \Xi)$. The rank of $(\mathbf{A}, \mathbf{B})$ is defined by the rank of $\mathbf{A} \vee \mathbf{B}$, which we denote by r throughout this section.

We now introduce combinatorial counterparts of expanded matrices. For a positive integer j, let S_j and T_j be distinct copies of S and T, respectively. Furthermore, let $\mathbf{A}_j = (S_j, T_j, \Lambda_j)$ and $\mathbf{B}_j = (S_{j+1}, T_j, \Xi_j)$ be the copies of $\mathbf{A}$ and $\mathbf{B}$, respectively.

For each positive integer k, consider the unions:

$$\begin{aligned}
\Psi_k(\mathbf{A},\mathbf{B}) &= \mathbf{A}_1 \vee \mathbf{B}_1 \vee \mathbf{A}_2 \vee \cdots \vee \mathbf{A}_k \vee \mathbf{B}_k, \\
\Phi_k(\mathbf{A},\mathbf{B}) &= \mathbf{B}_1 \vee \mathbf{A}_2 \vee \mathbf{B}_2 \vee \cdots \vee \mathbf{B}_k \vee \mathbf{A}_{k+1}, \\
\Theta_k(\mathbf{A},\mathbf{B}) &= \mathbf{A}_1 \vee \mathbf{B}_1 \vee \mathbf{A}_2 \vee \cdots \vee \mathbf{B}_{k-1} \vee \mathbf{A}_k, \\
\Omega_k(\mathbf{A},\mathbf{B}) &= \mathbf{B}_1 \vee \mathbf{A}_2 \vee \mathbf{B}_2 \vee \cdots \vee \mathbf{A}_k \vee \mathbf{B}_k.
\end{aligned}$$

We denote the ranks of $\Psi_k(\mathbf{A},\mathbf{B})$, $\Phi_k(\mathbf{A},\mathbf{B})$, $\Theta_k(\mathbf{A},\mathbf{B})$, and $\Omega_k(\mathbf{A},\mathbf{B})$ by ψ_k, φ_k, θ_k, and ω_k, respectively. Note that φ_k is equal to the rank of $\Psi_k(\mathbf{B},\mathbf{A})$ and ω_k is the rank of $\Theta_k(\mathbf{B},\mathbf{A})$. For $k = 0$, we set $\psi_0 = \varphi_0 = \theta_0 = \omega_0 = 0$. Obviously, these four sequences are monotone nondecreasing in k. The following lemmas show that ψ_k and φ_k are concave in k while θ_k and ω_k are convex in k.

Lemma 4.1 *For any $k > 0$, we have $2\psi_k \geq \psi_{k-1} + \psi_{k+1}$ and $2\varphi_k \geq \varphi_{k-1} + \varphi_{k+1}$.*

Proof. Let σ be the rank function of $\Psi_{k+1}(\mathbf{A},\mathbf{B})$. Let S^* and T^* denote the row and column sets of $\Psi_{k+1}(\mathbf{A},\mathbf{B})$, respectively. For $Z = T_2 \cup \cdots \cup T_k$, we have

$$\sigma(S^*, T_1 \cup Z) + \sigma(S^*, Z \cup T_{k+1}) \geq \sigma(S^*, T^*) + \sigma(S^*, Z)$$

by the linking bisubmodularity of σ. Note that $\psi_k = \sigma(S^*, T_1 \cup Z) = \sigma(S^*, Z \cup T_{k+1})$, $\psi_{k-1} = \sigma(S^*, Z)$ and $\psi_{k+1} = \sigma(S^*, T^*)$. Thus we obtain $2\psi_k \geq \psi_{k-1} + \psi_{k+1}$. By interchanging the roles of $\mathbf{A}$ and $\mathbf{B}$, we also obtain $2\varphi_k \geq \varphi_{k-1} + \varphi_{k+1}$. ■

Lemma 4.2 *For any $k > 0$, we have $2\theta_k \leq \theta_{k-1} + \theta_{k+1}$ and $2\omega_k \leq \omega_{k-1} + \omega_{k+1}$.*

Proof. Let σ denote the rank function of $\Theta_{k+1}(\mathbf{A},\mathbf{B})$. Let S^* and T^* denote the row and column sets of Θ_{k+1}. For $X = S_1 \cup \cdots \cup S_k$ and $Y = T_2 \cup \cdots \cup T_{k+1}$, we have

$$\sigma(X, Y) + \sigma(S^*, T^*) \geq \sigma(X, T^*) + \sigma(S^*, Y)$$

by the linking bisubmodularity of σ. Note that $\theta_{k-1} = \sigma(X, Y)$, $\theta_{k+1} = \sigma(S^*, T^*)$ and $\theta_k = \sigma(X, T^*) = \sigma(S^*, Y)$. Thus we obtain $2\theta_k \leq \theta_{k-1} + \theta_{k+1}$. By interchanging the roles of $\mathbf{A}$ and $\mathbf{B}$, we obtain $2\omega_k \leq \omega_{k-1} + \omega_{k+1}$. ■

Let λ and ξ be the rank functions of $\mathbf{A} = (S, T, \Lambda)$ and $\mathbf{B} = (S, T, \Xi)$, respectively. Then the rank r of $(\mathbf{A},\mathbf{B})$ is given by

$$r = \min_{W \subseteq S, Z \subseteq T} \{\lambda(W, Z) + \xi(W, Z) + |S \setminus W| + |T \setminus Z|\}.$$

A pair (W, Z) that attains the minimum in the right hand side is called a minimum cover of $\mathbf{A} \vee \mathbf{B}$. A pair of $(X, Y) \in \Lambda$ and $(X', Y') \in \Xi$ is called a maximum linking if it satisfies $X \cap X' = \emptyset$, $Y \cap Y' = \emptyset$ and $|X| + |X'| = r$.

Lemma 4.3 *Let (W, Z) be a minimum cover of $\mathbf{A} \vee \mathbf{B}$. Then we have $\psi_k \leq rk + |S \setminus W|$ and $\varphi_k \leq rk + |T \setminus Z|$.*

Proof. Let S^* and T^* denote the row and column sets of $\Psi_k(\mathbf{A},\mathbf{B})$. That is, $S^* = S_1 \cup \cdots \cup S_{k+1}$ and $T^* = T_1 \cup \cdots \cup T_k$. Let $W_j \subseteq S_j$ be the copies of W for $j = 1, \ldots, k+1$ and

$Z_j \subseteq T_j$ the copies of Z for $j = 1, \ldots, k$. Put $W^* = W_1 \cup \cdots \cup W_{k+1}$ and $Z^* = Z_1 \cup \cdots \cup Z_k$. Then we have

$$\begin{aligned}\psi_k &\leq (\lambda_1 \vee \cdots \vee \lambda_k)(W^*, Z^*) + (\xi_1 \vee \cdots \vee \xi_k)(W^*, Z^*) + |S^* \setminus W^*| + |T^* \setminus Z^*| \\ &= k\,\lambda(W, Z) + k\,\xi(W, Z) + (k+1)\,|S \setminus W| + k\,|T \setminus Z| = rk + |S \setminus W|.\end{aligned}$$

By interchanging the roles of $\mathbf{A}$ and $\mathbf{B}$, we obtain $\varphi_k \leq rk + |T \setminus Z|$. ∎

Lemma 4.4 *Let (W, Z) be a minimum cover of $\mathbf{A} \vee \mathbf{B}$. Then we have $\theta_k \leq rk - \xi(W, Z)$ and $\omega_k \leq rk - \lambda(W, Z)$ for any k.*

Proof. Let S^* and T^* denote the row and column sets of $\Theta_k(\mathbf{A}, \mathbf{B})$. That is, $S^* = S_1 \cup \cdots \cup S_k$ and $T^* = T_1 \cup \cdots \cup T_k$. Let $W_j \subseteq S_j$ and $Z_j \subseteq T_j$ be the copies of W and Z, respectively. Then $W^* = W_1 \cup \cdots \cup W_k$ and $Z^* = Z_1 \cup \cdots \cup Z_k$ satisfy

$$\begin{aligned}\theta_k &\leq (\lambda_1 \vee \cdots \vee \lambda_k)(W^*, Z^*) + (\xi_1 \vee \cdots \vee \xi_{k-1})(W^*, Z^*) + |S^* \setminus W^*| + |T^* \setminus Z^*| \\ &= k\,\lambda(W, Z) + (k-1)\,\xi(W, Z) + k\,|S \setminus W| + k\,|T \setminus Z| = rk - \xi(W, Z).\end{aligned}$$

By interchanging the roles of $\mathbf{A}$ and $\mathbf{B}$, we obtain $\omega_k \leq rk - \lambda(W, Z)$. ∎

Lemma 4.5 *Let $(X, Y) \in \Lambda$ and $(X', Y') \in \Xi$ be a maximum linking. Then we have $\psi_k \geq rk$ and $\varphi_k \geq rk$ for any k.*

Proof. Let $X_j, X'_j \subseteq S_j$ be the copies of $X, X' \subseteq S$ for $j = 1, \ldots, k+1$ and $Y_j, Y'_j \subseteq T_j$ the copies of $Y, Y' \subseteq T$ for $j = 1, \ldots, k$. Put $X^* = X_1 \cup \cdots \cup X_k \cup X'_2 \cup \cdots \cup X'_{k+1}$ and $Y^* = Y_1 \cup \cdots \cup Y_k \cup Y'_1 \cup \cdots \cup Y'_k$. Then (X^*, Y^*) is a linked pair in $\Psi_k(\mathbf{A}, \mathbf{B})$. Hence we have $\psi_k \geq k\,|X| + k\,|X'| = rk$. By interchanging the roles of $\mathbf{A}$ and $\mathbf{B}$, we obtain $\varphi_k \geq rk$. ∎

Lemma 4.6 *Let $(X, Y) \in \Lambda$ and $(X', Y') \in \Xi$ be a maximum linking. Then we have $\theta_k \geq rk - |X'|$ and $\omega_k \geq rk - |X|$.*

Proof. Let $X_j, X'_j \subseteq S_j$ be the copies of $X, X' \subseteq S$ and $Y_j, Y'_j \subseteq T_j$ the copies of $Y, Y' \subseteq T$ for $j = 1, \ldots, k$. Put $X^* = X_1 \cup \cdots \cup X_k \cup X'_2 \cup \cdots \cup X'_k$ and $Y^* = Y_1 \cup \cdots \cup Y_k \cup Y'_1 \cup \cdots \cup Y'_{k-1}$. Then (X^*, Y^*) is a linked pair in $\Theta_k(\mathbf{A}, \mathbf{B})$. Hence we have $\theta_k \geq k\,|X| + (k-1)\,|X'| = rk - |X'|$. By interchanging the roles of $\mathbf{A}$ and $\mathbf{B}$, we obtain $\omega_k \geq rk - |X|$. ∎

Lemma 4.7 *For any k, we have $\theta_{k+1} - \theta_k \leq r$ and $\omega_{k+1} - \omega_k \leq r$.*

Proof. This is immediate from Lemmas 4.2 and 4.4. ∎

Lemma 4.8 *For any k, we have $\psi_{k+1} - \psi_k \geq r$ and $\varphi_{k+1} - \varphi_k \geq r$.*

Proof. This is immediate from Lemmas 4.1 and 4.5. ∎

Lemma 4.9 *If $k \geq r$, we have $\psi_{k+1} - \psi_k = \varphi_{k+1} - \varphi_k = r$.*

Proof. Since ψ_k is concave in k by Lemma 4.1, it follows from Lemmas 4.3 and 4.8 that there exists an integer h such that $\psi_{k+1} - \psi_k = r$ holds for any $k \geq h$. Let ℓ be the smallest such h. Then by Lemma 4.1, we have $\psi_\ell \geq (r+1)\ell$. On the other hand, a minimum cover (W, Z) of $\mathbf{A} \vee \mathbf{B}$ satisfies $\psi_\ell \leq r\ell + |S \setminus W|$ by Lemma 4.3. Therefore, we have $\ell \leq |S \setminus W| \leq r$. Thus we obtain $\psi_{k+1} - \psi_k = r$ for $k \geq r$. Similarly, we also obtain $\varphi_{k+1} - \varphi_k = r$ for $k \geq r$. ∎

Lemma 4.10 *If $k \geq r$, we have $\theta_{k+1} - \theta_k = \omega_{k+1} - \omega_k = r$.*

Proof. Since θ_k is convex in k by Lemma 4.2, it follows from Lemmas 4.6 and 4.7 that there exists an integer h such that $\theta_{k+1} - \theta_k = r$ holds for any $k \geq h$. Let ℓ be the smallest such h. Then by Lemma 4.2, we have $\theta_\ell \leq (r-1)\ell$. On the other hand, for a maximum linking $(X, Y) \in \Lambda$ and $(X', Y') \in \Xi$, we have $\theta_\ell \geq r\ell - |X'|$ by Lemma 4.6. Therefore, we have $\ell \leq |X| \leq r$. Thus we obtain $\theta_{k+1} - \theta_k = r$ for $k \geq r$. Similarly, we also obtain $\omega_{k+1} - \omega_k = r$ for $k \geq r$. ■

Lemma 4.11 *For any k, we have $\psi_k + \varphi_k - \theta_k - \omega_k \leq r$.*

Proof. Let S^* and T^* denote the row and column sets of $\Theta_{k+1}(\mathbf{A}, \mathbf{B})$. That is, $S^* = S_1 \cup \cdots \cup S_{k+1}$ and $T^* = T_1 \cup \cdots \cup T_{k+1}$. We also denote $S^\circ = S_2 \cup \cdots \cup S_{k+1}$ and $T^\circ = T_1 \cup \cdots \cup T_k$. By the linking bisubmodularity of the rank function σ of $\Theta_{k+1}(\mathbf{A}, \mathbf{B})$, we have

$$\sigma(S^*, T^*) + \sigma(S^\circ, T^\circ) \geq \sigma(S^*, T^\circ) + \sigma(S^\circ, T^*).$$

Since $\theta_{k+1} = \sigma(S^*, T^*)$, $\omega_k = \sigma(S^\circ, T^\circ)$, $\psi_k = \sigma(S^*, T^\circ)$ and $\varphi_k = \sigma(S^\circ, T^*)$, this can be rewritten as $\theta_{k+1} + \omega_k \geq \psi_k + \varphi_k$. Therefore, we have $\psi_k + \varphi_k - \theta_k - \omega_k \leq \theta_{k+1} - \theta_k \leq r$ by Lemma 4.7. ■

Lemma 4.11 leads us to the definition of $\nu(\mathbf{A}, \mathbf{B}) = r - \psi_r - \varphi_r + \theta_r + \omega_r \geq 0$, which is analogous to the size of the strictly regular block in the Kronecker canonical form shown in Corollary 2.1. For a matrix pencil $D(s) = sA + B$, consider a matroid pencil $(\mathbf{L}(A), \mathbf{L}(B))$. It is not always true that $\nu(\mathbf{A}, \mathbf{B})$ is equal to the size of the strictly regular block in the Kronecker canonical form $\bar{D}(s)$ of $D(s)$. A recent result in [2] implies that the equality holds if $D(s)$ is a generic matrix pencil, i.e., if the nonzero entries in A and B are independent parameters.

5. Periodic Linking

In this section, we investigate a periodic structure of $\Theta_k(\mathbf{A}, \mathbf{B})$. Recall that a linked pair (X^*, Y^*) in $\Theta_k(\mathbf{A}, \mathbf{B})$ consists of disjoint sums $X^* = X_1 \cup \cdots \cup X_k \cup X'_2 \cup \cdots \cup X'_k$ and $Y^* = Y_1 \cup \cdots \cup Y_k \cup Y'_1 \cup \cdots \cup Y'_{k-1}$ such that $(X_j, Y_j) \in \Lambda_j$ for $j = 1, \ldots, k$ and $(X'_{j+1}, Y'_j) \in \Xi_j$ for $j = 1, \ldots, k-1$. Then a linked pair (X^*, Y^*) with such a decomposition is said to be a periodic linking if (X_j, Y_j) are the copies of the same $(X, Y) \in \Lambda$ for $j = 1, \ldots, k$ and (X'_{j+1}, Y'_j) are the copies of the same $(X', Y') \in \Xi$ for $j = 1, \ldots, k-1$. This section is to show that a maximum size $|X^*| = |Y^*|$ of a periodic linking (X^*, Y^*) in Θ_k is equal to the rank θ_k.

Let $(X \cup X', Y \cup Y')$ be a linked pair of $\mathbf{A} \vee \mathbf{B}$ such that $(X, Y) \in \Lambda$ and $(X', Y') \in \Xi$. Then the periodic linking (X^*, Y^*) determined by (X, Y) and (X', Y') is of size $k\,|X| + (k-1)\,|X'|$.

Given a linked pair $(X, Y) \in \Lambda$, we construct an auxiliary directed graph $G_{\mathbf{A}}(X, Y) = (S \cup T, E)$ with vertex set $S \cup T$ and arc set $E = E_S \cup E_T \cup E_+ \cup E_-$ defined by

$$\begin{aligned}
E_S &= \{(u, v) \mid u \in S \setminus X, v \in X, (X \cup \{u\} \setminus \{v\}, Y) \in \Lambda\}, \\
E_T &= \{(u, v) \mid u \in Y, v \in T \setminus Y, (X, Y \cup \{v\} \setminus \{u\}) \in \Lambda\}, \\
E_+ &= \{(u, v) \mid u \in S \setminus X, v \in T \setminus Y, (X \cup \{u\}, Y \cup \{v\}) \in \Lambda\}, \\
E_- &= \{(u, v) \mid u \in Y, v \in X, (X \setminus \{v\}, Y \setminus \{u\}) \in \Lambda\}.
\end{aligned}$$

Similarly, for a linked pair $(X', Y') \in \Xi$, we also construct an auxiliary directed graph $G_{\mathbf{B}}(X', Y') = (S \cup T, F)$. Furthermore, the auxiliary directed graph for a linked pair $(X \cup X', Y \cup Y')$ in $\mathbf{A} \vee \mathbf{B}$ is the superposition of $G_{\mathbf{A}}(X, Y)$ and $G_{\mathbf{B}}(X', Y')$. For simplicity we denote this graph by $G_{\mathbf{A} \vee \mathbf{B}} = (S \cup T, E \cup F)$.

The linked pair $(X \cup X', Y \cup Y')$ in $\mathbf{A} \vee \mathbf{B}$ determines a periodic linking (X^*, Y^*). Let S^* and T^* denote the row and column sets of $\Theta_k(\mathbf{A}, \mathbf{B})$. That is, $S^* = S_1 \cup \cdots \cup S_k$ and $T^* = T_1 \cup \cdots \cup T_k$. For $j = 1, \ldots, k$, let E_j denote the edge set of $G_{\mathbf{A}_j}(X_j, Y_j)$. For $j = 1, \ldots, k-1$, let F_j denote the edge set of $G_{\mathbf{B}_j}(X_{j+1}, Y_j)$. The auxiliary directed graph $G_{\Theta_k(\mathbf{A},\mathbf{B})} = (S^* \cup T^*, E^* \cup F^*)$ for (X^*, Y^*) is given by $E^* = E_1 \cup \cdots \cup E_k$ and $F^* = F_1 \cup \cdots \cup F_{k-1}$.

Lemma 5.1 *Suppose* $(X, Y) \in \Lambda$ *and* $(X', Y') \in \Xi$ *form a linking in* $\mathbf{A} \vee \mathbf{B}$ *that maximizes* $k\,|X| + (k-1)\,|X'|$. *Then the periodic linking* (X^*, Y^*) *that consists of the copies of* (X, Y) *and* (X', Y') *is a maximum linking in* $\Theta_k(\mathbf{A}, \mathbf{B})$.

Proof. If (X^*, Y^*) is not a maximum linking, there exists a directed path from $S^* \setminus X^*$ to $T^* \setminus Y^*$ in $G_{\Theta_k(\mathbf{A},\mathbf{B})}$. Let P^* be such a path with minimum number of arcs. The corresponding set of arcs in $G_{\mathbf{A}\vee\mathbf{B}}$ forms a directed path from $S \setminus X$ to $T \setminus Y$. Note that P does not include a cycle. Let $S(P)$ and $T(P)$ denote the sets of vertices in S and T, respectively, along P. Then the symmetric differences $X_P = X \triangle S(P)$, $Y_P = Y \triangle T(P)$, $X'_P = X' \triangle S(P)$ and $Y'_P = Y' \triangle T(P)$ form new linked pairs $(X_P, Y_P) \in \Lambda$ and $(X'_P, Y'_P) \in \Xi$.

Let s and t be the initial and terminal vertices of P. Suppose that P^* starts from S_h and terminates in T_ℓ. Then we have $k\,|X_P| + (k-1)\,|X'_P| = k\,|X| + (k-1)\,|X'| + k + \ell - h$. If $s \notin X'$ and $t \notin Y'$, then $X_P \cap X'_P = \emptyset$ and $Y_P \cap Y'_P = \emptyset$, which implies that (X_P, Y_P) and (X'_P, Y'_P) form a linking in $\mathbf{A} \vee \mathbf{B}$. Since $k + \ell - h > 0$, this contradicts the choice of (X, Y) and (X', Y'). If $s \in X'$ and $t \notin Y'$, we have $X_P \cap X'_P = \{s\}$, $Y_P \cap Y'_P = \emptyset$, and $h = 1$. By (L2), there exists $v \in Y'_P$ such that $(X'_P \setminus \{s\}, Y'_P \setminus \{v\}) \in \Xi$. Thus (X_P, Y_P) and $(X'_P \setminus \{s\}, Y'_P \setminus \{v\})$ form a linking in $\mathbf{A} \vee \mathbf{B}$ with $k\,|X_P| + (k-1)\,|X'_P \setminus \{s\}| = k\,|X| + (k-1)\,|X'| + \ell$, which contradicts the choice of (X, Y) and (X', Y'). Similarly, if $s \notin X'$ and $t \in Y'$, we have $X_P \cap X'_P = \emptyset$, $Y_P \cap Y'_P = \{t\}$, and $\ell = k$. By (L3), there exists $u \in X'$ such that $(X'_P \setminus \{u\}, Y'_P \setminus \{t\}) \in \Xi$. Thus (X_P, Y_P) and $(X'_P \setminus \{u\}, Y'_P \setminus \{t\})$ form a linking in $\mathbf{A} \vee \mathbf{B}$ with $k\,|X_P| + (k-1)\,|X'_P \setminus \{u\}| = k\,|X| + (k-1)\,|X'| + k + 1 - h$, which contradicts the choice of (X, Y) and (X', Y'). Finally, if $s \notin X'$ and $t \in Y'$, we have $X_P \cap X'_P = \emptyset$, $Y_P \cap Y'_P = \{t\}$, $h = 1$ and $\ell = k$. It follows from (L2) and (L3) that $(X'_P \setminus \{s\}, Y'_P \setminus \{t\}) \in \Xi$ or there exist $u \in X'$ and $v \in Y'$ such that $(X'_P \setminus \{s, u\}, Y'_P \setminus \{t, v\}) \in \Xi$. In the former case, (X_P, Y_P) and $(X'_P \setminus \{s\}, Y'_P \setminus \{t\})$ form a linking in $\mathbf{A} \vee \mathbf{B}$ with $k\,|X_P| + (k-1)\,|X'_P \setminus \{s\}| = k\,|X| + (k-1)\,|X'| + k$. In the latter case, (X_P, Y_P) and $(X'_P \setminus \{s, u\}, Y'_P \setminus \{t, v\})$ form a linking in $\mathbf{A} \vee \mathbf{B}$ with $k\,|X_P| + (k-1)\,|X'_P \setminus \{s, u\}| = k\,|X| + (k-1)\,|X'| + 1$. In either case, we have contradiction to the choice of (X, Y) and (X', Y'). Thus we may conclude that (X^*, Y^*) is a maximum size linking in $\Theta_k(\mathbf{A}, \mathbf{B})$. ∎

Theorem 5.1 *For a matroid pencil* $(\mathbf{A}, \mathbf{B})$, *we have*

$$\theta_k(\mathbf{A}, \mathbf{B}) = \max\{k\,|X| + (k-1)\,|X'| \mid (X, Y) \in \Lambda, (X', Y') \in \Xi, X \cap X' = \emptyset, Y \cap Y' = \emptyset\}.$$

6. Eigensets and Power Products

In this section, we give an alternative proof to a theorem of Murota [4] on maximum eigensets and the ranks of power products of linking systems.

Let $\mathbf{A} = (S, S, \Lambda)$ be a linking system whose row set and column set are identical. Murota [4] introduced the concept of eigenset of such a linking system and investigated its connection to power products. A subset $X \subseteq S$ is called an eigenset if $(X, X) \in \Lambda$. Let $\mathbf{A}^k$ denote the k-th power product $\mathbf{A} * \cdots * \mathbf{A}$ of $\mathbf{A}$. Then $r(\mathbf{A}^k)$ is monotone nonincreasing and convex in k. Hence there exists $\ell \leq |S|$ such that $r(\mathbf{A}^k) = r(\mathbf{A}^{k+1})$ holds for $k \geq \ell$.

We denote this rank by $r(\mathbf{A}^\infty)$. The following theorem characterizes $r(\mathbf{A}^\infty)$ in terms of eigensets.

Theorem 6.1 (Murota [4]) *For a linking system* $\mathbf{A} = (S, S, \Lambda)$*, we have*

$$r(\mathbf{A}^\infty) = \max\{|X| \mid (X, X) \in \Lambda\}.$$

We now present an alternative proof of this result using Theorem 5.1. Consider a matroid pencil $(\mathbf{A}, \mathbf{I})$ with diagonal linking system $\mathbf{I} = (S, S, \Delta)$ and denote the rank of $\Theta_k(\mathbf{A}, \mathbf{I})$ by $\theta_k(\mathbf{A}, \mathbf{I})$. Then it follows from Lemma 3.2 that

$$r(\mathbf{A}^k) = \theta_k(\mathbf{A}, \mathbf{I}) - (k-1)\,|S|.$$

Therefore, the following lemma completes the proof of Theorem 6.1.

Lemma 6.1 *For* $k \geq |S|$*, we have*

$$\theta_k(\mathbf{A}, \mathbf{I}) = (k-1)\,|S| + \max\{|X| \mid (X, X) \in \Lambda\}.$$

Proof. Applying Theorem 5.1 to $(\mathbf{A}, \mathbf{I})$, we obtain

$$\theta_k(\mathbf{A}, \mathbf{I}) = \max\{k\,|X| + (k-1)\,|Z| \mid (X, Y) \in \Lambda, X \cap Z = \emptyset, Y \cap Z = \emptyset\}.$$

Taking $(X, Y) = (\emptyset, \emptyset)$ and $Z = S$ in the right hand side, we observe $\theta_k \geq (k-1)\,|S|$. If $|X| + |Z| < |S|$, we have $k\,|X| + (k-1)\,|Z| \leq (k-1)\,|S| - (k-1) + |X| \leq (k-1)\,|S|$, where the last inequality follows from $|X| < |S| \leq k$. This implies that the maximum of the right hand side must be attained by some $X \subseteq S$ and $Z = S \setminus X$. Thus we obtain

$$\theta_k(\mathbf{A}, \mathbf{I}) = \max\{k\,|X| + (k-1)\,|S \setminus X| \mid (X, X) \in \Lambda\},$$

which is obviously equivalent to the desired formula. ■

For a square matrix A, consider a linking system $\mathbf{A} = \mathbf{L}(A)$. It should be noted that $\mathbf{A}^k$ can be different from $\mathbf{L}(A^k)$. A theorem of Poljak [6], however, shows that rank $A^k = r(\mathbf{A}^k)$ holds if A is a generic matrix, i.e., if the nonzero entries of A are independent parameters. An alternative proof for this theorem is also described in [2].

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