Journal of the Operations Research
Society of Japan
2007, Vol. 50, No. 3, 178-200

A MULTI-STAGE SEARCH ALLOCATION GAME WITH THE PAYOFF OF DETECTION PROBABILITY

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(Received March 8, 2005; Revised April 9, 2007)

Abstract This paper deals with a multi-stage two-person zero-sum game called the *multi-stage search allocation game* (MSSAG), in which a searcher and an evader participate. The searcher distributes his searching resources in a discrete search space to detect the evader, while the evader moves under an energy constraint to evade the searcher. At each stage of the search, the searcher is informed of the evader's position and his moving energy, and the evader knows the rest of the searcher's budget, by which the searcher allocates searching resources. A payoff of the game is the probability of detecting the evader during the search. There have been few search games that have dealt with the MSSAG. We formulate the problem as a dynamic programming problem. Then, we solve the game to obtain a closed form of equilibrium point, and to investigate the properties of the solution theoretically and numerically.

Keywords: Search, game theory, nonlinear programming, dynamic programming

1. Introduction

Search theory originates from an analysis of military operation, in which two hostile sides participate with totally opposite rewards. This is why most researchers discuss the search game as a two-person zero-sum game. In the early history of search theory, what researchers were interested in were the one-sided search problems of optimizing the searcher's plan. Koopman [14] was also interested in this problem and his work put together much military research about anti-submarine warfare (ASW) in World War II. Stone [21] made a big contribution to the generalization of the one-sided search problem. After that, the search problem was extended to search games, in which not only the searcher's strategies but also the target's or evader's strategies are optimized.

The most important information the searcher wants to know about the target is its position, which is referred to as datum. Thus, the search game that starts with the exposure of the target position is called a datum search game. Meinardi [15] discussed the game with payoff being the probability of detecting the target on a line. At the beginning of the search, a searcher knows only the datum of a target and successively chooses points to search, after a time lag. The target starts from the datum point and selects a point to hide among neighborhood points near his existing place, knowing which positions have already been searched. This problem is formulated as a multi-stage game (MSG) because the history of the searched positions is available to the target in the process of the game. It is difficult to apply Meinardi's approach to other search games because he aimed to find a way to make the target distribution as uniform as possible on the line.

From the view of modeling the search game, a simpler problem is a single-stage game (SSG) with a stationary target. Danskin [4] investigated an ASW game, where submarine's strategy was to choose a fixed course and speed for diffusively moving from a datum point, and the strategy of the ASW helicopter was to choose a sequence of points where he would dip his sonar. He took advantage of a specific space called a speed circle so that the target course and speed were represented by a point in the space, and then the problem was formulated as a SSG for the stationary target. Gal [6] published a book in which the main theme was the SSG for stationary targets and moving targets on geometric spaces, such as lines or disks. He also dealt with the moving strategy on the searcher's side, in contrast with the early works of one-sided problems in which the distribution plan of the searching effort was the popular optimized strategy. Besides these studies, many papers have dealt with the moving strategy of the searcher because that strategy seems to be natural for modeling realistic search operations. Baston and Bostock [1] studied a SSG in which a helicopter dropped bombs to destroy a submarine. Eagle and Washburn [5] also discussed a SSG that optimized the cumulative reward determined by the consecutive positions of a target and a searcher. The so-called Ruckle problem discussed in Ruckle [19] or Garnaev [7] is also a SSG in a special situation, in which a bird tries to cross a square field in safety and a hunter sets up ambushes consisting of several lines of nets to capture the bird. There are several multi-stage search games, in which the searcher chooses positions for moving or trapping. This is similar to Meinardi's work mentioned above. Washburn [22] set up traveling time as the payoff of a MSG, which continued until a target and a searcher chose the same position. Nakai [16] considered the effect of a safety zone in a MSG similar to Meinardi's modeling.

Concerning the strategy of distributing searching resources, most of the related papers deal with single-stage games. Nakai [17] made a contribution toward the SSG of stationary targets. Iida et al. [13] and Hohzaki and Iida [10] expanded the SSG to the moving target game. Hohzaki and Iida [11] then proposed a method to solve a more generalized game. Washburn and Hohzaki [23], Hohzaki et al. [12] and Hohzaki [9] were interested in the SSG, in which the target has constraints on his moving energy. The problem presented by Baston and Garnaev [2] is also a single-stage two-person zero-sum game, where both of the two players distribute their resources, and it can be regarded as a convex game.

As seen in this survey of search game research, few studies are related to a multi-stage game, in which the searcher's strategy is to distribute his searching resources over search space every stage of the game. We refer to the evasion-search game played by the searcher distributing searching resources and the moving evader as a search allocation game (SAG), as Garnaev [8] calls it. In this paper, we consider the multi-stage search allocation game (MSSAG). The first one of our motives is to model the MSSAG, formulate it and propose a methodology for solution, for the first time. In this sense, this paper is methodology-oriented or theory-oriented. At the same time, the problem is located in the extension of the former researches of the single-stage search allocation game [9, 12, 23], where we made a single-stage datum search game more realistic by introducing practical constraints such as energy constraints on the target motion.

One of the direct applications of the MSSAG model is a sequence of the datum-search games. We want to analyze a multi-stage game, where a single-stage datum search game restarts with new datum information as long as the target is not detected.

When the game has a large number of stages, we are interested in whether the game converges to any stationary state and what it is if exists. The following example is a metaphor that the MSSAG could be a game with an infinite number of stages. When the software doesn't give the desired output, or its execution terminates abnormally, the programmer begins to search for the bug, probably starting in the neighborhood of the superficially defective program statements judging from the point of view of the software semantics. A software bug affects some chains of program components and makes them look malfunctioning and defective superficially. The programmer has to chase the superficial errors to reach real bugs. He would throw several types of input data set for debugging, which cause superficial errors probabilistically. The debugging load usually must be measured in terms of manpower. The programmer must make a decision about how much manpower he should use up and where he puts that manpower to check possible types of bugs. We could estimate a worst-case of the workload by assuming that the bug puts the trace of his influence in the program as a malicious decision maker. Until the programmer can specify the real bugs, he must repeat the bugging game, which may look endless.

Another application is the inspection process in a factory. When a product consisting of many components fails to pass inspection, an inspection planner must organize a schedule of checking some components, including how many minutes he spends to check each component, and which components he should examine. If he cannot find the defects for a long time, he feels as if they are playing the role of hostile persons. The MSSAG would be a good model for the above problems.

In the next section, we describe some assumptions of the MSSAG and formulate the game as a dynamic programming problem. First, we find an equilibrium solution for a single-stage game as a preliminary study in Section 3.1. Using the result, we derive a closed form of the value of the MSSAG in Section 3.2. We also discuss some properties of the solution and a stationary solution at infinite stages in Section 3.3. In Section 4, we investigate some properties of the game by some numerical examples.

2. Description of Assumptions and Formulation

Here we consider the following problem of a multi-stage stochastic search game:

A1. A search space is a discrete cell space $\boldsymbol{K} = \{1, \cdots, K\}$. Time space is also assumed to be discrete. Time point n indicates the residual time until the stopping time of the search so that $n = 0$ is the stopping time point.

A2. Two players, a searcher and an evader, join the game. At the initial time, the searcher has his total budget Φ. Using the budget, he distributes searching resources in the search space to detect the evader. The evader possesses initial energy e_0. He moves in the search space under some constraints on energy and other factors. The strategies and information sets of the players, a payoff function and the process of the game are as follows:

  (1) At the beginning of time point n, the searcher obtains the information about the evader position, say cell k, and his residual energy. At the same time, the evader is informed of the searcher's residual budget.

  (2) Then the evader makes the decision to move from the cell k, probabilistically. But his movement is constrained as follows: From cell k, he can move only to cells $N(k) \subseteq \boldsymbol{K}$, which we refer to as the neighborhood cells of k. He spends energy $\mu(i, j)$ by moving from cell i to j, where $\mu(i, j)$ is positive for $i \neq j$. That is why the cells he can move to depend on his residual energy in addition to the neighborhood-cell constraints. It is assumed that $\mu(k, k) = 0$ and $k \in N(k)$. The evader is forced to stay at his current cell after his energy is exhausted.

  (3) The searcher distributes his searching resources based on his guess as to the cell the evader moves to. However, this distribution must be done by taking his residual budget into account. Cost $c_i > 0$ is necessary to allocate unit resource into cell i.

(4) Provided that the evader is in cell i, and x searching resources are distributed there, the searcher detects the evader with probability $1 - q_i(x)$. Namely, $q_i(x)$ indicates the non-detection probability of the evader, and it is assumed to be given by

$$q_i(x) = \exp(-\alpha_i \, x) \; . \tag{1}$$

In the case of no resource, the searcher cannot detect the evader at all. As the amount of allocated resources increases, the non-detection probability decreases in a convex form, and it reaches zero only for infinite resources. Parameter $\alpha_i > 0$ is an indicator of how effective unit resource in cell i is for the detection of the evader. When he detects the evader, the searcher receives payoff 1, and the evader loses the same amount. At that moment, the game is terminated.

(5) Unless the detection occurs at the time point n, the game proceeds toward the next stage of time point, $n - 1$.

A3. The game ends when the evader is detected or the time point reaches $n = 0$. In the game, the searcher acts as a maximizer and the evader as a minimizer.

The problem is a multi-stage stochastic game with the payoff of the detection probability of the evader. Now we extend the neighborhood-cell concept so that it includes the energy constraint. The neighborhood cells to which the evader with energy e is able to move from cell k are given by the following $N(k, e)$:

$$N(k,e) \equiv \{i \in N(k) \mid \mu(k,i) \le e\} \; . \tag{2}$$

Suppose that the game starts from a state (k, e, Φ), which indicate that the evader is in cell k with residual energy e and the searcher has budget Φ left, at the beginning of Stage n. Two players know all values n, k, e and Φ, as assumed in A2(1). In this situation, we represent an evader strategy by variables $\{p(k, i; n, e, \Phi), i \in \boldsymbol{K}\}$, $\{p(k, i), i \in \boldsymbol{K}\}$ for short, where $p(k, i) \ge 0$ indicates the probability that he moves to cell i from cell k. It holds that $p(k, i) = 0$ for $i \in \boldsymbol{K} - N(k, e)$ and $\sum_{i \in N(k,e)} p(k, i) = 1$, of course. Now the feasible region for moving strategies of the evader with energy e, who is currently in cell k, is given by

$$\begin{aligned} P_k(e) \;=\; & \Bigg\{ \{p(k,i), i \in \boldsymbol{K}\} \,|p(k,i) \ge 0, i \in \boldsymbol{K}, p(k,i) = 0, i \in \boldsymbol{K} - N(k,e), \\ & \sum_{i \in N(k,e)} p(k,i) = 1 \Bigg\} . \end{aligned} \tag{3}$$

On the other hand, we represent the searcher's strategy at time point n by $\{\varphi(i; n, k, e, \Phi), i \in \boldsymbol{K}\}$, $\{\varphi(i), i \in \boldsymbol{K}\}$ for short, where $\varphi(i) \ge 0$ is searching resources to be distributed in cell i. The feasibility condition of the strategy is $\sum_i c_i \varphi(i) \le \Phi$ if the searcher's residual budget is Φ. It is evidently useless to distribute searching resources in cells other than $N(k, e)$ and a feasible region of the searcher's strategy $\Psi(\Phi; n, k, e)$ is given by

$$\begin{aligned} \Psi(\Phi; n, k, e) \;=\; & \Bigg\{ \{\varphi(i), i \in \boldsymbol{K}\} \;|\; \varphi(i) \ge 0, i \in \boldsymbol{K}, \; \varphi(i) = 0, i \in \boldsymbol{K} - N(k,e), \\ & \sum_{i \in N(k,e)} c_i \varphi(i) \le \Phi \Bigg\} . \end{aligned} \tag{4}$$

We will use notation $\Psi(\Phi)$, which stands for $\Psi(\Phi; n, k, e)$.

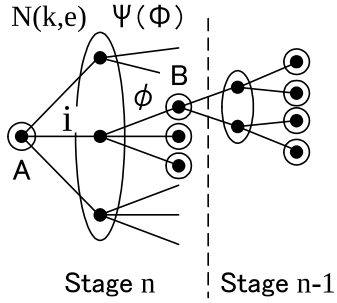

Figure 1: Game tree

Here we give an extensive form of the game to illustrate the strategies and information sets of the players, by Figure 1.
A root (Point A) represents the current state (n, k, e, Φ). The evader chooses a cell among $N(k, e)$, say Cell i. The searcher does not know the evader's choice, of course. Therefore a set of evader strategies $N(k, e)$ is enclosed by a circle of information set. The next turn is for the searcher to select a feasible distribution of searching resources among $\Psi(\Phi)$, say φ. Each node of $N(k, e)$ has the same branches of $\Psi(\Phi)$ but please note that an infinite number of branches are there for the searcher. At the end of Stage n, the game terminates if the evader is detected. Otherwise, after the choices of i and φ are revealed to both players, the game starts again from a point B at Stage $n-1$, as it branches from Point A at first.

Here we assume that there exists a value of the game. For the evader with energy e in cell k, and the searcher with his budget Φ at the beginning of time point n, we denote the value of the game by $v(n, k, e, \Phi)$. If the evader moves to cell i with probability $p(k, i)$, searching resource $\varphi(i)$ brings detection probability $1-q_i(\varphi(i))$. Unless the detection occurs, the game moves forward to the next stage of time point $n-1$, and then the evader enters a state $(i, e-\mu(k, i))$ of his cell and energy. The searcher's budget decreases to $\Phi' = \Phi - \sum_{i \in N(k,e)} c_i \varphi(i)$. The game is a stochastic game, where state (n, k, e, Φ) transfers to state $(n-1, i, e-\mu(k, i), \Phi')$, $i \in N(k, e)$ with probability $p(k, i) q_i(\varphi(i)) \geq 0$, and it terminates with probability $\sum_{i \in N(k,e)} p(k, i)(1-q_i(\varphi(i))) < 1$. Only the termination brings the searcher unit reward.

A general stochastic game may be played forever, but it terminates with certainty under the assumption that it has positive probability of termination at any stage and the value of the game is uniquely determined, as Shapley [20] and Owen [18] showed. However, our game is also a multi-stage game, where it certainly terminates at a final stage $n = 0$ unless it ends by the occurrence of the detection. For such a multi-stage stochastic game, the value is recursively determined by replacing the game of a state with the value of the game according to the transition law of the game, which is reviewed above. Considering the transition from state (n, k, e, Φ) and the termination of the game, as discussed above, the value of the game $v(n, k, e, \Phi)$ satisfies the following recursive equation, although its existence is assumed:

$$\begin{aligned} v(n, k, e, \Phi) &= \max_{\varphi \in \Psi(\Phi)} \min_{p \in P_k(e)} \sum_{i \in N(k,e)} p(k, i) \\ & \quad \times \left\{ 1 - q_i(\varphi(i)) + q_i(\varphi(i)) v(n-1, i, e-\mu(k, i), \Phi - \sum_{i \in N(k,e)} c_i \varphi(i)) \right\} \\ &= 1 - \min_{\varphi \in \Psi(\Phi)} \max_{p \in P_k(e)} \sum_{i \in N(k,e)} p(k, i) \end{aligned}$$

$$\times \left\{1 - v(n-1, i, e-\mu(k,i), \Phi - \textstyle\sum_j c_j\varphi(j))\right\} \exp(-\alpha_i\varphi(i)). \tag{5}$$

From properties of the problem, its initial conditions and boundary conditions are as follows:

$$v(0,k,e,\Phi) = 0, \quad v(n,k,e,0) = 0 \ . \tag{6}$$

If $e = 0$, which is a case that the evader inevitably stays in cell k forever, the searcher is expected to distribute his resources there forever. His available resources of Φ/c_k bear the resultant detection probability

$$v(n,k,0,\Phi) = 1 - \exp(-\alpha_k\Phi/c_k) \tag{7}$$

all through time points $\{n, n-1, \cdots, 1\}$, even if the resource is divided in any way during the time periods. After we transform Equation (5) by introducing a new value $h(n,k,e,\Phi) \equiv 1 - v(n,k,e,\Phi)$ to make it easy to handle, we have

$$h(n,k,e,\Phi) =$$
$$\min_{\varphi\in\Psi(\Phi)} \ \max_{p\in P_k(e)} \sum_{i\in N(k,e)} p(k,i) h(n-1, i, e-\mu(k,i), \Phi - \sum_{j\in N(k,e)} c_j\varphi(j)) \exp(-\alpha_i\varphi(i)). \tag{8}$$

In this case, the initial conditions and boundary conditions (6) and (7) are exchanged for

$$h(0,k,e,\Phi) = 1, \quad h(n,k,e,0) = 1, \ \ h(n,k,0,\Phi) = \exp(-\alpha_k\Phi/c_k) \ . \tag{9}$$

The proof of an existence theorem of the value of the game remains as our job, which will be completed later on. At present, we are going to deal with a min-max optimization problem (8). First, introducing Φ_n as the searching budget used at time point n, $\Phi_n = \sum_{i\in N(k,e)} c_i\varphi(i)$, let us write the problem in the nested optimization structure as follows:

$$h(n,k,e,\Phi) =$$
$$\min_{0\le\Phi_n\le\Phi} \ \min_{\varphi\in\Psi_n(\Phi_n)} \ \max_{p\in P_k(e)} \sum_{i\in N(k,e)} p(k,i) h(n-1, i, e-\mu(k,i), \Phi - \Phi_n) \exp(-\alpha_i\varphi(i)), \tag{10}$$

where $\Psi_n(\Phi_n; k, e)$, $\Psi_n(\Phi_n)$ for short, is defined by

$$\Psi_n(\Phi_n;k,e) \ = \ \Bigg\{ \{\varphi(i), i\in \boldsymbol{K}\} \ | \ \varphi(i)\ge 0, i\in\boldsymbol{K}, \varphi(i) = 0, i \in \boldsymbol{K} - N(k,e), \ \Phi_n = \sum_{i\in N(k,e)} c_i\varphi(i) \Bigg\}.$$

Before we proceed to derive an equilibrium point or optimal strategies of players for the multi-stage game, we itemize notation for later reference.

Notation

$\boldsymbol{K}$: search cell space, $= \{1, 2, \cdots, K\}$
n: index for time point
e_0: initial energy of the evader
e: index for evader's energy
Φ: index for searcher's budget for search
$N(k)$: cells that the evader can move to from cell k

$N(k,e)$: cells that the evader with energy e can move to from cell k
$\mu(i,j)$: energy that it takes to move from cell i to j
c_i: budget that it costs to distribute unit searching resources in cell i
α_i: effectivity that unit searching resource distributed in cell i has on the detection of the evader
c_i/α_i: called "cost-of-detection coefficient" (COD coefficient) of Cell i
$q_i(x) := \exp(-\alpha_i x)$: non-detection probability of the evader by x searching resources in cell i given that the evader is there
(n,k,e) or (n,k,e,Φ): triplet or quadruplet indicating a state of stage n, evader's current cell k, evader's energy e and searcher's budget Φ
$p(k,i;n,e,\Phi)$ or $p(k,i)$: probability that the evader selects cell i in state (n,k,e,Φ)
$P_k(e)$: feasible region for evader's strategy $p(k,i)$ in state (n,k,e,Φ)
$\varphi(i)$ or $\varphi(i;n,k,e,\Phi)$: searching resources to be distributed in cell i in state (n,k,e,Φ)
$\Psi(\Phi;n,k,e)$ or $\Psi(\Phi)$: feasible region for searcher's strategy $\varphi(i)$ in state (n,k,e,Φ)
$E(\boldsymbol{p},\varphi)$: expected payoff for an evader's strategy $\boldsymbol{p}$ and a searcher's strategy φ
$v(n,k,e,\Phi)$: value of the game in state (n,k,e,Φ) on the criterion of detection probability
$h(n,k,e,\Phi)$: value of the game in state (n,k,e,Φ) on the criterion of non-detection probability

3. Equilibrium Solution for Game

3.1. Solution for single-stage game

Here we focus on the effective distribution of the total amount Φ_n of budget at time n in Problem (10). For simplicity, we substitute β_i for $h(n-1,i,e-\mu(k,i),\Phi-\Phi_n)$. Symbols p_i, φ_i and A are substitutes for $p(k,i)$, $\varphi(i)$ and $N(k,e)$, respectively. Now we consider the following problem:

$$\min_{\{\varphi_i\}} \max_{\{p_i\}} \sum_{i\in A} p_i\beta_i \exp(-\alpha_i\varphi_i) \ . \tag{11}$$

From our previous work of Hohzaki and Iida [11], we already know that an above min-max value equals a max-min value, and then it gives the value of the game. This single-stage game can be described as a hide-and-search game for a stationary evader as follows: At the beginning of the search, a stationary evader chooses a cell i to hide with probability p_i. A searcher decides on a distribution plan of searching resources, which totals to Φ_n in terms of cost, trying to detect the evader. There have been already several related studies of the single-stage-game, such as Danskin [3] and Garnaev [8]. Now let us find an equilibrium point for the single-stage game.

The feasible conditions of $\varphi=\{\varphi_i\}$ and $\boldsymbol{p}=\{p_i\}$ are given by $\varphi_i\geq 0,\ i\in A,\ \sum_{i\in A} c_i\varphi_i = \Phi_n$ for φ_i and $p_i\geq 0,\ i\in A,\ \sum_{i\in A} p_i = 1$ for p_i, respectively. The objective function of Problem (11) is $E(\boldsymbol{p},\varphi)\equiv\sum_{i\in A} p_i\beta_i\exp(-\alpha_i\varphi_i)$. The optimality of φ^* and $\boldsymbol{p}^*$ is given by the fact that inequality $E(\boldsymbol{p},\varphi^*)\leq E(\boldsymbol{p}^*,\varphi^*)\leq E(\boldsymbol{p}^*,\varphi)$ holds for any feasible solution φ and $\boldsymbol{p}$, that is,

$$E(\boldsymbol{p}^*,\varphi^*) \ = \ \max_{\boldsymbol{p}} E(\boldsymbol{p},\varphi^*) \quad s.t. \ \ p_i\geq 0,\ i\in A,\ \sum_{i\in A} p_i = 1 \tag{12}$$

$$E(\boldsymbol{p}^*,\varphi^*) \ = \ \min_{\varphi} E(\boldsymbol{p}^*,\varphi) \quad s.t. \ \ \varphi_i\geq 0,\ i\in A,\ \sum_{i\in A} c_i\varphi_i = \Phi_n \ . \tag{13}$$

The first problem (12) is easy to solve by noting the following transformation:

$$\max_{\boldsymbol{p}} \sum_{i\in A} p_i\beta_i\exp(-\alpha_i\varphi_i^*) = \max_{i\in A}\beta_i\exp(-\alpha_i\varphi_i^*). \tag{14}$$

If we denote the last optimal value by ρ, optimal solution is given by

$$p_i^* = 0 \quad for\ i \in \{i \mid \beta_i \exp(-\alpha_i \varphi_i^*) < \rho\}\ ,\quad p_i^* \geq 0,\ i \in A,\ \sum_{i \in A} p_i^* = 1\ ,$$

which are equivalent to the following conditions:

$$\beta_i \exp(-\alpha_i \varphi_i^*) \leq \rho,\ i \in A, \qquad p_i^*(\rho - \beta_i \exp(-\alpha_i \varphi_i^*)) = 0,\ i \in A, \tag{15}$$

$$p_i^* \geq 0,\ i \in A,\ \sum_{i \in A} p_i^* = 1\ . \tag{16}$$

The second problem (13) is a convex minimization problem. Therefore, we can easily derive the necessary and sufficient conditions for optimality from Karush-Kuhn-Tucker conditions.

$$\alpha_i p_i^* \beta_i \exp(-\alpha_i \varphi_i^*) = c_i \lambda \ \ if\ \varphi_i^* > 0\ , \qquad \alpha_i p_i^* \beta_i \exp(-\alpha_i \varphi_i^*) \leq c_i \lambda \ \ if\ \varphi_i^* = 0, \tag{17}$$

$$\varphi_i^* \geq 0,\ i \in A,\ \sum_{i \in A} c_i \varphi_i^* = \Phi_n\ , \tag{18}$$

where λ is a Lagrangean multiplier corresponding to the second equality in the constraints of Problem (13). We can easily confirm that the following solution satisfies conditions (15)-(18), which is therefore optimal.

$$\varphi_i = \frac{1}{\alpha_i}\left[\log \frac{\beta_i}{\rho}\right]^+, \tag{19}$$

$$p_i = \begin{cases} c_i/\alpha_i \Big/ \sum_{j,\rho \leq \beta_j} c_j/\alpha_j\ , & \rho \leq \beta_i \\ 0, & \rho > \beta_i, \end{cases} \tag{20}$$

$$\lambda = \frac{\rho}{\sum_{j,\rho \leq \beta_j} c_j/\alpha_j}, \tag{21}$$

where symbol $[\ \ ]^+$ means $[x]^+ = \max\{x, 0\}$ and ρ is uniquely determined by equation

$$\sum_{i \in A} \frac{c_i}{\alpha_i}\left[\log \frac{\beta_i}{\rho}\right]^+ = \Phi_n. \tag{22}$$

We can easily apply the above results of the single-stage game to the optimization problem beginning with $\min_{\varphi \in \Psi_n(\Phi_n)} \max_{p \in P_k(e)}$ in Problem (10), and we have the following recursive formula:

$$h(n, k, e, \Phi) = \min_{\Phi_n, 0 \leq \Phi_n \leq \Phi} \rho(n, k, e, \Phi_n)\ , \tag{23}$$

where $\rho(n, k, e, \Phi_n)$ is a solution ρ of the following equation, using $A_n(k, e, \Phi_n, \Phi) \equiv \{i \in N(k, e) \mid \rho \leq h(n-1, i, e - \mu(k, i), \Phi - \Phi_n)\}$.

$$\sum_{i \in N(k,e)} \frac{c_i}{\alpha_i}\left[\log \frac{h(n-1, i, e - \mu(k, i), \Phi - \Phi_n)}{\rho}\right]^+ = \Phi_n \tag{24}$$

$$or \quad \sum_{i \in A_n(k,e,\Phi_n,\Phi)} \frac{c_i}{\alpha_i} \log \frac{h(n-1, i, e - \mu(k, i), \Phi - \Phi_n)}{\rho} = \Phi_n\ . \tag{25}$$

At the n-th stage, optimal solutions of φ^* and p^* are given by the following:

$$\varphi^*(i) = \begin{cases} 1/\alpha_i \cdot \log\left(h(n-1, i, e - \mu(k, i), \Phi - \Phi_n)/\rho\right), & i \in A_n(k, e, \Phi_n, \Phi) \\ 0, & otherwise\ , \end{cases} \tag{26}$$

$$p^*(k,i)=\begin{cases} c_i/\alpha_i \Big/ \sum_{j\in A_n(k,e,\Phi_n,\Phi)} c_j/\alpha_j\ , & i\in A_n(k,e,\Phi_n,\Phi) \\ 0, & otherwise\ . \end{cases} \tag{27}$$

Before closing this section, let us obtain the value of the game $h(1,k,e,\Phi)$ at the first stage $n=1$. Noting that $h(0,k,e,\Phi)=1$, we can solve Problem (8) with $n=1$ by applying $\beta_i=1$ to the single-stage game (11).

$$Value\ of\ the\ game\ :\quad h(1,k,e,\Phi)=\exp(-\Phi \Big/ \sum_{j\in N(k,e)} c_j/\alpha_j) \tag{28}$$

$$Optimal\ strategy\ of\ searcher\ :\quad \varphi(i)=\Phi/\alpha_i \Big/ \sum_{j\in N(k,e)} c_j/\alpha_j\ ,\quad i\in N(k,e) \tag{29}$$

$$Optimal\ strategy\ of\ evader\ :\quad p(k,i)=c_i/\alpha_i \Big/ \sum_{j\in N(k,e)} c_j/\alpha_j\ ,\quad i\in N(k,e). \tag{30}$$

3.2. Solution for multi-stage game

Now we are ready to discuss our multi-stage game. We could easily make a guess as to some properties of the value of the game $h(n,k,e,\Phi)$. The value will become smaller as the searcher's budget Φ increases because the searcher has more available searching resources on hand. On the other hand, more energy e gives the evader more mobility to expand his reachable area and flatten his probability over the larger area. The expansion and the flatness force the searcher to scatter his searching resources widely, which is disadvantageous to him, and then increase $h(n,k,e,\Phi)$. We have another plausible explanation as to why the value of the game is nondecreasing for e. If the evader has more energy than e, he has the option of not using the extra energy and can play as if he had only e, so surely he cannot do worse by having more than e. The following theorem states such properties of the game.

Theorem 1 *(i)* $\log h(n,k,e,\Phi)$ *is a monotone nonincreasing convex function for budget* Φ. *(ii)* $h(n,k,e,\Phi)$ *is monotone nondecreasing for energy* e.

Proof: (i) The nonincreasingness is evident. Concerning the convexity, we can verify that it holds in the case of $n=1$ from Equation (28). Now let us assume that it also holds for $h(n-1,k,e,\Phi)$. From problem (8), the value $h(n,k,e,\Phi)$ is defined by the following formulation, where we tentatively abbreviate $p(k,i)$ to $p(i)$ and $h(n-1,i,e-\mu(k,i),\Phi)$ to $h(n-1,i,\Phi)$ because the transformations appeared below are always valid regardless of items concerning evader's energy.

$$h(n,k,e,\Phi)=\min_{\varphi\in\Psi(\Phi)}\ \max_{p\in P_k(e)} \sum_{i\in N(k,e)} p(i)h(n-1,i,\Phi-\sum_{j\in N(k,e)} c_j\varphi(j))\exp(-\alpha_i\varphi(i)).$$

Considering the maximization of $\max_{p\in P_k(e)}$ and the transformation (14), we can see that $h(n,k,e,\Phi)$ coincides with $h(n-1,i,\Phi-\sum_j c_j\varphi(j))\exp(-\alpha_i\varphi(i))$ for some cell i and then $\log h(n,k,e,\Phi)$ is given by the following optimization problem:

$$\log h(n,k,e,\Phi)=\min_{\varphi\in\Psi(\Phi)}\ \max_{p\in P_k(e)} \sum_{i\in N(k,e)} p(i)\log\left(h(n-1,i,\Phi-\sum_j c_j\varphi(j))\exp(-\alpha_i\varphi(i))\right).$$

Therefore, for $0\le\beta\le1$ and $0\le\Phi_1,\ \Phi_2$, we have the following transformation:

$$\beta\log h(n,k,e,\Phi_1)+(1-\beta)\log h(n,k,e,\Phi_2)$$

$$
\begin{aligned}
&= \beta \min_{\varphi_1 \in \Psi(\Phi_1)} \max_{p_1 \in P_k(e)} \sum_{i \in N(k,e)} p_1(i) \log\left(h(n-1, i, \Phi_1 - \sum_j c_j \varphi_1(j)) \exp(-\alpha_i \varphi_1(i))\right) \\
&+ (1-\beta) \min_{\varphi_2 \in \Psi(\Phi_2)} \max_{p_2 \in P_k(e)} \sum_{i \in N(k,e)} p_2(i) \log\left(h(n-1, i, \Phi_2 - \sum_j c_j \varphi_2(j)) \exp(-\alpha_i \varphi_2(i))\right) \\
&\geq \min_{\varphi_1 \in \Psi(\Phi_1), \varphi_2 \in \Psi(\Phi_2)} \max_{p \in P_k(e)} \\
&\quad \sum_{i \in N(k,e)} p(i) \left\{ \beta \log\left(h(n-1, i, \Phi_1 - \sum_j c_j \varphi_1(j)) \exp(-\alpha_i \varphi_1(i))\right) \right. \\
&\quad \left. +(1-\beta) \log\left(h(n-1, i, \Phi_2 - \sum_j c_j \varphi_2(j)) \exp(-\alpha_i \varphi_2(i))\right) \right\}.
\end{aligned}
$$

In the last transformation, we take a common variable p for p_1 and p_2. Because $\log h(n-1, i, \Phi - \sum_j c_j \varphi(j))$ is convex for $\Phi - \sum_j c_j \varphi(j)$, we can proceed further.

$$
\begin{aligned}
&\geq \min_{\varphi_1 \in \Psi(\Phi_1), \varphi_2 \in \Psi(\Phi_2)} \max_{p \in P_k(e)} \\
&\quad \sum_{i \in N(k,e)} p(i) \left\{ \log h(n-1, i, \beta(\Phi_1 - \sum_j c_j \varphi_1(j)) + (1-\beta)(\Phi_2 - \sum_j c_j \varphi_2(j))) \right. \\
&\quad \left. -\alpha_i(\beta \varphi_1(i) + (1-\beta)\varphi_2(i)) \right\} \\
&= \min_{\varphi_1 \in \Psi(\Phi_1), \varphi_2 \in \Psi(\Phi_2)} \max_{p \in P_k(e)} \\
&\quad \sum_{i \in N(k,e)} p(i) \left\{ \log h(n-1, i, \beta\Phi_1 + (1-\beta)\Phi_2 - \sum_j c_j(\beta \varphi_1(j) + (1-\beta)\varphi_2(j))) \right. \\
&\quad \left. -\alpha_i(\beta \varphi_1(i) + (1-\beta)\varphi_2(i)) \right\}.
\end{aligned}
$$

Noting $\beta\varphi_1 + (1-\beta)\varphi_2 \in \Psi(\beta\Phi_1 + (1-\beta)\Phi_2)$, we set $\varphi \equiv \beta\varphi_1 + (1-\beta)\varphi_2$ to obtain the following transformation:

$$
\begin{aligned}
&\geq \min_{\varphi \in \Psi(\beta\Phi_1 + (1-\beta)\Phi_2)} \max_{p \in P_k(e)} \\
&\quad \sum_{i \in N(k,e)} p(i) \log\left(h(n-1, i, \beta\Phi_1 + (1-\beta)\Phi_2 - \sum_j c_j \varphi(j)) \exp(-\alpha_i \varphi(i))\right) \\
&= \log h(n, k, e, \beta\Phi_1 + (1-\beta)\Phi_2).
\end{aligned}
$$

Now the convexity of $\log h(n, k, e, \Phi)$ has been proved.

(ii) We can prove the nondecreasingness of $h(\cdot)$ for energy e by mathematical induction. If $e < e'$, it holds that $N(k, e) \subseteq N(k, e')$. It helps us verify the nondecreasingness from Equation (28) in the case of $n = 1$. Noting that $P_k(e) \subseteq P_k(e')$ from Equation (3), the assumption of $h(n-1, k, e - \mu(k, i), \Phi - \Phi_n) \leq h(n-1, k, e' - \mu(k, i), \Phi - \Phi_n)$ leads us to the following inequality:

$$
\begin{aligned}
&h(n, k, e, \Phi) \\
&\leq \min_{\Phi_n, 0 \leq \Phi_n \leq \Phi} \min_{\varphi \in \Psi_n(\Phi_n)} \max_{p \in P_k(e')} \sum_{i \in N(k,e')} p(k, i) h(n-1, i, e' - \mu(k, i), \Phi - \Phi_n) \exp(-\alpha_i \varphi(i)) \\
&= h(n, k, e', \Phi),
\end{aligned}
$$

which can be derived from the recursive formula (10). Now we have completed the proof. □

$\log h(n,k,e,\Phi)$ is convex for Φ, as stated in Theorem 1, and then function $h(n,k,e,\Phi)$ is convex too. The following theorem states a concrete form of the function.

Theorem 2 *For time point $n \geq 2$, current cell k, residual energy e of the evader and residual budget Φ of the searcher, function $h(n,k,e,\Phi)$ has the following expression:*

$$h(n,k,e,\Phi) = \exp(-\Phi/\gamma_n(k,e)) \ . \tag{31}$$

Coefficient $\gamma_n(k,e)$ and optimal budget Φ_n^* to be used at time point n are determined in the following manner: First, sort coefficients $\{\gamma_{n-1}(j,e-\mu(k,j)),\ j\in N(k,e)\}$ of stage $n-1$ in descending order and number cells of $N(k,e)$ like $k_1,k_2,\cdots,k_m$ such that $\gamma_{n-1}(k_1,e-\mu(k,k_1)) \geq \gamma_{n-1}(k_2,e-\mu(k,k_2)) \geq \cdots \geq \gamma_{n-1}(k_m,e-\mu(k,k_m))$, where m is the number of cells belonging to $N(k,e)$.

(i) If $1 > \sum_{i\in N(k,e)} c_i/\alpha_i/\gamma_{n-1}(i,e-\mu(k,i))$, the coefficient is calculated by

$$\gamma_n(k,e) = \sum_{i\in N(k,e)} \frac{c_i}{\alpha_i} \ . \tag{32}$$

The optimal strategies of the searcher and the evader are determined for current stage n, as follows:

$$\Phi_n^* = \Phi \ , \tag{33}$$

$$\varphi^*(i) = \frac{\Phi/\alpha_i}{\sum_{j\in N(k,e)} c_j/\alpha_j},\ i\in N(k,e) \ , \tag{34}$$

$$p^*(k,i) = \frac{c_i/\alpha_i}{\sum_{j\in N(k,e)} c_j/\alpha_j},\ i\in N(k,e) \ . \tag{35}$$

(ii) Otherwise, using $s_n^*\in\{1,\cdots,m\}$ of

$$s_n^* = \min\left\{ s \ \middle|\ 1 \leq \sum_{\tau=1}^{s} \frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau, e-\mu(k,k_\tau))} \right\} \ , \tag{36}$$

the coefficient is calculated by

$$\gamma_n(k,e) = \gamma_{n-1}(k_{s_n^*}, e-\mu(k,k_{s_n^*}))\left(1 - \sum_{\tau=1}^{s_n^*-1} \frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau,e-\mu(k,k_\tau))}\right) + \sum_{\tau=1}^{s_n^*-1}\frac{c_{k_\tau}}{\alpha_{k_\tau}} \ , \tag{37}$$

and the optimal strategies at stage n are

$$\Phi_n^* = \frac{\eta_{n-1}(k,s_n^*,e)}{1+\eta_{n-1}(k,s_n^*,e)}\Phi, \tag{38}$$

$$\begin{aligned}\varphi^*(i) &= \frac{\Phi/\alpha_i}{1+\eta_{n-1}(k,s_n^*,e)}\left(\frac{1}{\gamma_{n-1}(k_{s_n^*},e-\mu(k,k_{s_n^*}))} - \frac{1}{\gamma_{n-1}(i,e-\mu(k,i))}\right), \\ &\qquad\qquad i\in\{k_1,\cdots,k_{s_n^*}\} \\ &= 0,\ \textit{otherwise},\end{aligned} \tag{39}$$

$$\begin{aligned}p^*(k,i) &= c_i/\alpha_i \Bigg/ \sum_{\tau=1}^{s_n^*} c_{k_\tau}/\alpha_{k_\tau} \ ,\ i\in\{k_1,\cdots,k_{s_n^*}\} \\ &= 0,\ \textit{otherwise},\end{aligned} \tag{40}$$

where $\eta_{n-1}(k,s,e)$ *is given by*

$$\eta_{n-1}(k,s,e) \equiv \frac{\sum_{\tau=1}^{s-1} c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_s, e-\mu(k,k_s))} - \sum_{\tau=1}^{s-1} \frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau, e-\mu(k,k_\tau))} \ .$$

For $n=1$, $\gamma(\cdot)$ *is initialized by*

$$\gamma_1(k,e) = \sum_{j \in N(k,e)} c_j/\alpha_j, \tag{41}$$

and an optimal solution is given by Equation (28)–(30).

Proof: From the value of the game $h(1,k,e,\Phi)$ of Equation (28), we know that formulae (31) and (41) are valid for $n=1$. Let us verify that the theorem holds for $n \geq 2$ by mathematical induction. Now we assume $\log h(n-1,i,e,\Phi) = -\Phi/\gamma_{n-1}(i,e)$. Exchange ρ in Equation (24) for $y \equiv -\log\rho \geq 0$, and then we obtain

$$\sum_{i \in N(k,e)} \frac{c_i}{\alpha_i} \left[y - \frac{\Phi - \Phi_n}{\gamma_{n-1}(i, e-\mu(k,i))} \right]^+ = \Phi_n \ . \tag{42}$$

The minimization problem (23) with respect to ρ can be replaced with a maximization problem of y involved in the above equation. From this point, we are going to deal with the maximization problem.

Let us sort cells in $N(k,e)$ in the descending order of values $\gamma_{n-1}(i,e-\mu(k,i))$ to obtain $\gamma_{n-1}(k_1, e-\mu(k,k_1)) \geq \cdots \geq \gamma_{n-1}(k_m, e-\mu(k,k_m))$. For $s \in \{1,2,\cdots,m-1\}$, if y lies between two values like

$$\frac{\Phi-\Phi_n}{\gamma_{n-1}(k_s, e-\mu(k,k_s))} \leq y \leq \frac{\Phi-\Phi_n}{\gamma_{n-1}(k_{s+1}, e-\mu(k,k_{s+1}))} \ , \tag{43}$$

equation (42) can be simplified as follows:

$$\sum_{\tau=1}^{s} \frac{c_{k_\tau}}{\alpha_{k_\tau}} y = \sum_{\tau=1}^{s} \frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau, e-\mu(k,k_\tau))} \Phi + \left(1 - \sum_{\tau=1}^{s} \frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau, e-\mu(k,k_\tau))} \right) \Phi_n \ . \tag{44}$$

Delete y by replacing the above expression for y in inequality (43) and find a domain for Φ_n to satisfy the inequality (43), and then the result is

$$\frac{\eta_{n-1}(k,s,e)}{1+\eta_{n-1}(k,s,e)} \Phi \leq \Phi_n \leq \frac{\eta_{n-1}(k,s+1,e)}{1+\eta_{n-1}(k,s+1,e)} \Phi \ . \tag{45}$$

The domains (45) of Φ_n touch to each other at their extreme points in the order of $s = 1,\cdots,m-1$. The function y of (44) is linear for Φ_n, which means that a maximum of y is taken at a left extreme point if its gradient is negative and is taken at a right extreme point otherwise. Denoting the maxima by y^L_{max} and y^R_{max} in these respective cases, the maxima are given by

$$\begin{aligned} y^L_{max} &= \Phi \Bigg/ \left\{ \gamma_{n-1}(k_s, e-\mu(k,k_s)) \left(1 - \sum_{\tau=1}^{s-1} \frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau, e-\mu(k,k_\tau))} \right) + \sum_{\tau=1}^{s-1} \frac{c_{k_\tau}}{\alpha_{k_\tau}} \right\} \\ y^R_{max} &= \Phi \Bigg/ \left\{ \gamma_{n-1}(k_{s+1}, e-\mu(k,k_{s+1})) \left(1 - \sum_{\tau=1}^{s} \frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau, e-\mu(k,k_\tau))} \right) + \sum_{\tau=1}^{s} \frac{c_{k_\tau}}{\alpha_{k_\tau}} \right\} . \end{aligned} \tag{46}$$

Besides the intervals of (43), we have to take into account the following intervals:

(1) In the case of $0 \le y \le (\Phi - \Phi_n)/\gamma_{n-1}(k_1, e - \mu(k, k_1))$, we see that only a point $\Phi_n = 0$ is feasible from Equation (42). This implies that the interval (45) with $s = 1$ can be substituted for this case because $\eta_{n-1}(k, 1, e) = 0$.

(2) In the case of $(\Phi - \Phi_n)/\gamma_{n-1}(k_m, e - \mu(k, k_m)) \le y$, y is expressed by a line (44) with $s = m$ within its feasible domain of $\Phi\eta_{n-1}(k, m, e)/(1 + \eta_{n-1}(k, m, e)) \le \Phi_n \le \Phi$. Therefore, at $\Phi_n = \Phi$, y possibly takes its maximal value given by

$$\Phi / \sum_{i \in N(k,e)} c_i/\alpha_i. \tag{47}$$

Because the gradient of the line y is decreasing for $s = 1, 2, \cdots$, the function y is a piecewise linear concave function for $0 \le \Phi_n \le \Phi$. Therefore, the function reaches the maximum at an extreme point just before the gradient becomes negative. Now, from Equation (46) and (47), we can say that $h(n, k, e, \Phi)$ is expressed by the form of (31) and its coefficient $\gamma_n(k, e)$ is evaluated by recursive formula (32) or (37). Optimal budget Φ_n^* to be expended at Stage n is given by Equation (33) and (38). Using these equations and substituting $\beta_i = h(n-1, i, e - \mu(k, i), \Phi - \Phi_n^*)$ and $\rho = h(n, k, e, \Phi)$ in expressions (19) and (20), we have equations of (34) and (35) in Case (i), or (39) and (40) in Case (ii) with respect to optimal strategies $\varphi^*(i)$ and $p^*(k, i)$. $\square$

We give the initial value of $\gamma_n(k, e)$ by Equation (41). Assuming that $h(0, k, e, \Phi) = 1$ also has formula (31) with $n = 0$ leads us to $\gamma_0(k, e) = \infty$ for any (k, e). If so, Case (i) of Theorem 2 is always valid for $n = 1$. Furthermore, the resultant value $\gamma_1(k, e)$ of Equation (32) coincides with Equation (41). This means that the initial value

$$\gamma_0(k, e) = \infty \tag{48}$$

is valid for $n = 0$.

To make Theorem 2 understandable, we are going to demonstrate optimal strategies in a simple case. Assume that the searcher has initial budget Φ_0, and the evader has infinite energy $e = \infty$ and no neighborhood-cell constraint, i.e. $N(k) = \boldsymbol{K}$. Cell i has cost c_i and detectability α_i, and then COD coefficient $d_i \equiv c_i/\alpha_i$. In this case, $\gamma_1(k, e)$ is the same for all cell k at the last stage $n = 1$ because $\gamma_1(k, e) = \sum_{j \in \boldsymbol{K}} d_j$ from Equation (41). At stage $n = 2$, s_n^* given by Equation (36) is K and then the evader selects every cell $i \in \boldsymbol{K}$ as his next cell with positive probability proportional to d_i, which is seen by Equation (40). Optimal evader strategy becomes the same even if the evader is in any cell k at the beginning of the stage. We see that $\gamma_2(k, e)$ given by Equation (37) equals to $\gamma_1(k, e)$ and optimal evader strategy remains unchanged every stage. The evader seems to be stationary as if he hid himself in cell i with probability $d_i / \sum_{j \in \boldsymbol{K}} d_j$ at the beginning although he actually moves every stage. In this case, the searcher does not distribute any resource all stages but the last one $n = 1$, which is known by Equation (39). At the last stage, he expends distribution cost $c_i\varphi(i) = \Phi_0 d_i / \sum_{j \in \boldsymbol{K}} d_j$ for searching in cell i, proportional to its COD coefficient, as known by Equation (34). However more consideration gives us another optimal plan, where the searcher picks up a certain stage and executes the above optimal strategy then instead of doing it at the last stage. The stationary-like movement of the evader allows this kind of distribution plans to be optimal.

3.3. Properties of the solution

From the discussion in Theorem 2, we can say that $\gamma_n(k,e)$ is given as a minimum of denominator of expressions (46) or (47). Hence, by a definition of

$$G_n(k,e,s) \equiv \gamma_{n-1}(k_s, e-\mu(k,k_s))\left(1-\sum_{\tau=1}^{s-1}\frac{c_{k_\tau}/\alpha_{k_\tau}}{\gamma_{n-1}(k_\tau, e-\mu(k,k_\tau))}\right)+\sum_{\tau=1}^{s-1}\frac{c_{k_\tau}}{\alpha_{k_\tau}}\ , \tag{49}$$

we have

$$\gamma_n(k,e) = \min\left[\min_{s\in\{1,\cdots,m\}} G_n(k,e,s),\ \sum_{i\in N(k,e)}\frac{c_i}{\alpha_i}\right]. \tag{50}$$

In Theorem 2, cells of $N(k,e)$ are sorted, say $\{k_1,k_2,\cdots,k_m\}$, in the descending order of values $\{\gamma_{n-1}(i,e-\mu(k,i)), i\in N(k,e)\}$. Because $k\in N(k,e)$ for any energy e, we have an index $\tilde{s}_n$, such as $k=k_{\tilde{s}_n}$. From now on, we always assign cell k a smallest index number $\tilde{s}_n$ among cells with the same value of $\gamma_{n-1}(\cdot)$. Namely, $\gamma_{n-1}(k_{\tilde{s}_n-1}, e-\mu(k,k_{\tilde{s}_n-1})) > \gamma_{n-1}(k_{\tilde{s}_n}, e-\mu(k,k_{\tilde{s}_n}))$ for $\tilde{s}_n>1$.

As seen from expression (31), we can regard coefficient $\gamma_n(k,e)$ as a comprehensive effectiveness of unit budget on the non-detection probability of the evader, taking account of the number of residual stages n, the evader's existing cell k and his energy e. The effectiveness is calculated cumulatively based on value c_i/α_i of each cell i. Considering that c_i is cost for unit searching resource and α_i is effectiveness of unit searching resource upon the detection probability, we may call rate c_i/α_i "cost-of-detection coefficient", or COD coefficient for short. The evader is more likely to choose cells with higher COD coefficient, as seen in Equation (27). It is interesting that the region of $\{k_1,\cdots,k_{s_n^*}\}$, where the searching effort must be distributed at stage n, is determined based not on how much budget the searcher has but on coefficients $\gamma_{n-1}(\cdot)$ of the next stage $n-1$. The coefficients decide even the ratio of the optimal expense Φ_n^* to the total budget Φ, as seen from Equation (38).

Now we move forward to elucidate the properties of coefficient $\gamma(\cdot)$. For example, let us ask ourselves what the coefficient becomes when the number of stages n is larger. The increment of the number gives more chances for the searcher to attain the effective division of the total budget Φ. At the same time, it gives the evader more chances of expanding his possible area and reaching cells with high COD coefficients. Which effect is larger than the other? The following corollary answers this question.

Corollary 1 *(i) For any k and e, the value of the game $h(n,k,e,\Phi)$ and coefficient $\gamma_n(k,e)$ are monotone nonincreasing for the number of stage n, that is,*

$$h(n,k,e,\Phi) \geq h(n+1,k,e,\Phi) \tag{51}$$
$$\gamma_n(k,e) \geq \gamma_{n+1}(k,e)\ . \tag{52}$$

(ii) Values $\gamma_n(k_s, e-\mu(k,k_s)),\ s=1,\cdots,m$, sorted in descending order, have the monotone nonincreasingness for n. Namely,

$$\gamma_n(k_s,e-\mu(k,k_s)) \geq \gamma_{n+1}(k'_s, e-\mu(k,k'_s)),\ s=1,\cdots,m\ .$$

(iii) If optimal strategies are determined by Case (ii) of Theorem 2, one of the following conditions is valid:

$$\tilde{s}_n = s_n^* = 1\ ,\ \textit{or}\ \ \tilde{s}_n < s_n^*\ . \tag{53}$$

Proof: (i) Because inequality (51) is equivalent to (52), it is enough to prove the former. The inequality is satisfied for $n = 0$, as seen in Equation (9) and (28). Assuming $h(n-1, i, e, \Phi) \geq h(n, i, e, \Phi)$ for any i, e and Φ, we can verify inequality

$$h(n-1, i, e-\mu(k,i), \Phi - \sum_j c_j\varphi_j)\exp(-\alpha_i\varphi_i) \geq h(n, i, e-\mu(k,i), \Phi - \sum_j c_j\varphi_j)\exp(-\alpha_i\varphi_i)$$

for any $\{\varphi_j, j \in \boldsymbol{K}\} \in \Psi(\Phi)$. Considering optimization problem (8), we conclude that $h(n, k, e, \Phi) \geq h(n+1, k, e, \Phi)$.
(ii) Assume that there is some s of $\gamma_n(k_s, e-\mu(k, k_s)) < \gamma_{n+1}(k'_s, e-\mu(k, k'_s))$. Then each of cells $\{k_{s+1}, \cdots, k_m\}$ does not belong to another cell set $\{k'_1, \cdots, k'_s\}$ from inequality (52). Namely, we can say that $\{k_{s+1}, \cdots, k_m\}$ coincides with $\{k'_{s+1}, \cdots, k'_m\}$, and then cell k_s is in $\{k'_1, \cdots, k'_s\}$. It follows that $\gamma_n(k_s, e-\mu(k, k_s)) < \gamma_{n+1}(k'_s, e-\mu(k, k'_s)) \leq \gamma_{n+1}(k'_{s-1}, e-\mu(k, k'_{s-1})) \leq \cdots \leq \gamma_{n+1}(k'_1, e-\mu(k, k'_1))$, which is contradictory to inequality (52). Therefore, $\gamma_n(k_s, e-\mu(k, k_s)) \geq \gamma_{n+1}(k'_s, e-\mu(k, k'_s))$, $s = 1, \cdots, m$, is verified.
(iii) If $\widetilde{s}_n > s^*_n$, from Equation (37),

$$\begin{aligned} \gamma_n(k, e) &= \gamma_{n-1}(k_{s^*_n}, e-\mu(k, k_{s^*_n})) + \gamma_{n-1}(k_{s^*_n}, e-\mu(k, k_{s^*_n}))\eta_{n-1}(k, s^*_n, e) \\ &\geq \gamma_{n-1}(k_{s^*_n}, e-\mu(k, k_{s^*_n})) > \gamma_{n-1}(k_{\widetilde{s}_n}, e-\mu(k, k_{\widetilde{s}_n})) = \gamma_{n-1}(k, e)\ , \end{aligned}$$

which contradicts property (i). Now we have $\widetilde{s}_n \leq s^*_n$. If $\widetilde{s}_n = s^*_n$, an additional assumption of $1 < \widetilde{s}_n$ leads us $\gamma_n(k, e) > \gamma_{n-1}(k_{\widetilde{s}_n}, e-\mu(k, k_{\widetilde{s}_n})) = \gamma_{n-1}(k, e)$ from Equation (37), and the contradiction happens. Then $\widetilde{s}_n = s^*_n$ implies $\widetilde{s}_n = s^*_n = 1$. $\square$

Property (i) of Corollary 1 says that $h(n, k, e, \Phi)$ is nonincreasing for the number of stages of the game. The increment of the number gives some advantage for both of the searcher and the evader, as we mentioned before. The corollary elucidates that the advantage for the searcher overcomes that for the evader. This advantage may well come from the characteristic of the game in which the evader's existing cell is exposed to the searcher at each stage. Because $\sum_{i \in N(k,e)} c_i/\alpha_i \geq \gamma_n(k, e) \geq \gamma_{n+1}(k, e)$ from property (i) and Equation (50), we can see that once the optimal strategies are given by Case (i) of Theorem 2 at a stage, Case (i) must have borne the optimality all through the smaller stages for a given pair of cell k and energy e. As the number of stages increases, the optimality possibly transfers from Case (i) to (ii). But once the transfer happens, it does not return to Case (i) any more at larger number of stages. Case (ii) could be explained as the case that only a part of cells $N(k, e)$ is given positive transfer probability of the evader and positive searching resources of the searcher. At the last stage of $n = 1$, only Case (i) can happen, that is, the remaining searching budget must be exhausted all over the cells to which the evader can move, as seen from Equation (29) and (30). Property (iii) of Corollary 1 indicates that the evader's existing cell k has to be searched at the current stage.

An extremely important question still remains, which relates to the existence of an equilibrium point. From Theorem 2, the recursive formula (8) is written in the form of

$$h(n, k, e, \Phi) = \min_{\varphi \in \Psi(\Phi)} \max_{p \in P_k(e)} \sum_{i \in N(k,e)} p(k, i) \exp\left(-\frac{\Phi - \sum_j c_j \varphi(j)}{\gamma_{n-1}(i, e-\mu(k, i))} - \alpha_i \varphi(i)\right)\ .$$

This is just the game, where its payoff function is defined by $\exp\{-(\Phi - \sum_j c_j\varphi(j))/\gamma_{n-1}(i, e-\mu(k, i)) - \alpha_i\varphi(i)\}$ when the evader selects his next cell i with probability $p(k, i)$ as his mixed strategy and the searcher distributes his searching resources $\varphi(i)$ in cell i as his pure strategy. The payoff is convex for the continuous strategy φ. For a game in which a maximizer has a finite number of discrete strategies, a minimizer has an infinite continuous strategy and the payoff is convex for the continuous strategy, an equilibrium point exists within the region of the maximizer's mixed strategy and the minimizer's pure strategy, as seen in our previous work of Hohzaki and Iida [11]. Therefore, the min-max expected payoff $h(n, k, e, \Phi)$ coincides with the max-min value, which is just the value of the game. Now we can state the following theorem:

Theorem 3 *The MSSAG has an equilibrium point. Value $h(n, k, e, \Phi)$ given by (8) and (9) is the value of the game with non-detection probability as payoff.*

In Corollary 1(i), we prove the monotonicity of values $h(\cdot)$ and $\gamma(\cdot)$, which indicates that these values approach some convergence points as $n \to \infty$. Formula (32) or (37) in Theorem 2 is so simple that we can easily calculate $\gamma_n(k, e)$ to obtain its convergence point, even though the sizes of the whole cell space $\boldsymbol{K}$ or energy e are large. However, we discuss the range of the value $\gamma_n(k, e)$ here.

Corollary 2 *For any n, k and e, a lower bound of $\gamma_n(k, e)$ is given by*

$$\gamma_n(k, e) \geq \frac{c_k}{\alpha_k}\ . \tag{54}$$

Proof: At initial stage $n = 1$, inequality (54) holds because

$$\gamma_n(k, e) = \sum_{i \in N(k,e)} \frac{c_i}{\alpha_i} \geq \frac{c_k}{\alpha_k}\ .$$

Similarly, we verify the inequality in the case that $\gamma_n(k, e)$ is given by Case (i) of Theorem 2. Now we suppose that for $\gamma_{n-1}(k, e)$, inequality (54) is satisfied. For $\gamma_n(k, e)$ determined by Case (ii) of Theorem 2, we can see that $\gamma_n(k, e) = \gamma_{n-1}(k, e) \geq c_k/\alpha_k$ if $s_n^* = \widetilde{s}_n = 1$, from the property (iii) of Corollary 1 and Equation (37). If $s_n^* > 1$,

$$\gamma_n(k, e) \geq \sum_{\tau=1}^{s_n^*-1} \frac{c_{k_\tau}}{\alpha_{k_\tau}} \geq \frac{c_{k_{\widetilde{s}_n}}}{\alpha_{k_{\widetilde{s}_n}}} = \frac{c_k}{\alpha_k}. \qquad \square$$

From now on, we simplify the discussion about the range of $\gamma_n(k, e)$ by setting up $e_0 = \infty$. The simplification means that there is no longer any constraint on the evader's energy. All the theoretical results of Theorem 1∼3 and Corollary 1, 2 remain intact, even though we delete the element of energy state e from the used notation, $N(k, e)$, $P_k(e)$, $v(n, k, e, \Phi)$, $h(n, k, e, \Phi)$, $\gamma_n(k, e)$, $\eta_n(k, s, e)$, $G_n(k, e, s)$, where $N(k, e)$ becomes the original neighborhood cells of cell k. Now we look on each cell as a node in a graph and draw a directed arc from cell k to j if $j \in N(k)$. When there are a finite natural number r and a sequence of cells $l_1, l_2, \cdots, l_r, j$ to satisfy $l_1 \in N(k)$, $l_2 \in N(l_1), \cdots$, $j \in N(l_r)$, we say that cell j is reachable from cell k and denote the relation by $k \mapsto j$. We also define reachable cells of cell k by $R(k) \equiv \{j \in \boldsymbol{K} | k \mapsto j\}$. Noting that a relation $k \sim j \equiv (k \mapsto j) \cap (j \mapsto k)$ is an equivalence relation, we can classify the whole cell $\boldsymbol{K}$ into some equivalence classes $L_1,\ L_2,\ \cdots,\ L_u$, where $\boldsymbol{K} = L_1 \cup L_2 \cup \cdots \cup L_u$, $L_i \cap L_j = \emptyset\ (i \neq j)$. Furthermore we can easily extend the reachability of cell $\mapsto$ to the equivalence class and execute the so-called topological sort on the whole class by the reachability. As a result, let there be equivalence classes $L_{u_1},\ \cdots,\ L_{u_w}$, from which there is no other reachable class. Now we have a lemma about the range of $\gamma_n(k)$.

Lemma 1 *(i) There is a lower bound for* $\gamma_n(k)$.

$$\gamma_n(k) \geq \min_{j \in R(k)} \gamma_1(j) \ . \tag{55}$$

(ii) Letting $\gamma_1(i^*_{u_k}) = \min_{j \in L_{u_k}} \gamma_1(j)$ *for each of* $k = 1, \cdots, w$*, the value of* $\gamma_n(i^*_{u_k})$ *remains unchanged for any* n*, that is,*

$$\gamma_n(i^*_{u_k}) = \gamma_1(i^*_{u_k}) = \sum_{j \in N(i^*_{u_k})} \frac{c_j}{\alpha_j} \ , \ k = 1, \cdots, w \ .$$

(iii) In the special case that the neighborhood cell of each cell consists of itself, namely $N(k) = \{k\}$*, it holds* $\gamma_n(k) = c_k/\alpha_k$ *for any* n*. In the other special case that each cell has the whole cell space as its neighborhood cells, namely* $N(k) = \boldsymbol{K}$*,* $\gamma_n(k)$ *becomes constant of being* $\sum_{j \in \boldsymbol{K}} c_j/\alpha_j$ *for any* n *and cell* k*.*

Proof: (i) In the case of $n = 1$ and the case that $\gamma_n(k)$ is given by (i) of Theorem 2, because $k \in N(k) \subseteq R(k)$, it follows that

$$\gamma_n(k) = \sum_{j \in N(k)} \frac{c_j}{\alpha_j} = \gamma_1(k) \geq \min_{j \in R(k)} \gamma_1(j) \ .$$

Now suppose that inequality (55) is satisfied for $\gamma_{n-1}(k)$. For $\gamma_n(k)$ given by Case (ii) of Theorem 2, we can also prove the validity of inequality (55) by

$$G_n(k,s) \geq \gamma_{n-1}(k_s) \geq \min_{j \in N(k)} \gamma_{n-1}(j) \geq \min_{j \in N(k)} \min_{l \in R(j)} \gamma_1(l) = \min_{l \in R(k)} \gamma_1(l) \ ,$$

using definition (49).
(ii) Noting that $L_{u_k} = R(i^*_{u_k})$, it follows that $\gamma_n(i^*_{u_k}) \geq \min_{j \in L_{u_k}} \gamma_1(j) = \gamma_1(i^*_{u_k})$ from property (i) of this lemma. At the same time, $\gamma_1(i^*_{u_k}) \geq \gamma_n(i^*_{u_k})$ from Corollary 1(i).
(iii) It is self-evident from Theorem 2. □

From the above discussion, we can say that $\gamma_n(k)$ varies in a monotonic manner and converges to a value lying between $\sum_{j \in N(k)} c_j/\alpha_j$ and $\max\{c_k/\alpha_k, \ \min_{j \in R(k)} \gamma_1(j)\}$ as $n \to \infty$. In some special cases, such as (ii) and (iii) of Lemma 1, we can easily anticipate the convergence point. Applying the value $\lim_{n \to \infty} \gamma_n(k)$ to Theorem 2, we can find a stationary solution to the game.

4. Numerical Examples

Here we take several examples to investigate the characteristics of the optimal solution of the game, a part of which has been clarified theoretically so far. Let us set up a search space of $\boldsymbol{K} = \{1, 2, \cdots, 10\}$ and parameters $c_i, \ \alpha_i, \ i \in \boldsymbol{K}$ as follows:

Table 1: Cell space and parameter setting

Cell #	1	2	3	4	5	6	7	8	9	10
c_i	1	0.5	2	0.4	3	2.5	1.5	3.5	4	6
α_i	0.5	0.3	0.1	0.05	0.2	0.7	0.45	0.4	0.2	0.9
c_i/α_i	2	1.67	20	8	15	3.57	3.33	8.75	20	6.67

Initial value of $\gamma_1(\cdot)$ is calculated based on c_i/α_i, which is listed in the last row of the table. The evader tends to move to cells with higher COD coefficient. Cells $1, 2, \cdots, 10$ are located in a one-dimensional line in this order. The neighborhood-cell constraint is set up to be 2-neighbored cells, $N(k) = \{k-2, k-1, k, k+1, k+2\} \cap \boldsymbol{K}$. The evader consumes his energy of the square of his moving distance, that is, $\mu(i,j) = (i-j)^2$.

(1) Optimal strategies stemming from state $(n,k,e) = (3,1,5)$

The first example is a simulation of a multi-stage datum search game beginning with a state $(n,k,e) = (3,1,5)$. That is, at the beginning of Stage 3 the evader with his initial energy 5 is in Cell 1. The energy gives him the mobility up to 3-neighbored cells at the farthest until the end of the last stage $n = 1$. There are too many scenarios in terms of the evader movement that we cannot exhaust them. Let us check some of them illustrated in Figure 2. A black dot represents a state (n,k,e). An arrow branching from it indicates a destination cell i, an optimal evader strategy $p(k,i)$ and a ratio of optimal searching resources to the current available budget, $\varphi(i)/\Phi$, which are calculated from Equation (35), (34) or (40), (39). An arrow drawn from state (n,k,e) to cell i generates a new state $(n-1, i, e-\mu(k,i)$. For example, state $(2,3,1)$ is a resultant state of moving from $(3,1,5)$ to cell $i = 3$. The top table is for state $(3,1,5)$. We attach symbols 'i', '$p(k,i)$', '$\varphi(i)/\Phi$' to the table for explanation but omit them for other tables. We calculate what percentage of available budget is used at the current stage, $\sum_i c_i\varphi(i)/\Phi$, and add it in the row of the searcher's strategy.

From $(n,k,e) = (3,1,5)$, the evader transfers to three cells $N(k,e) = \{1,2,3\}$ with probabilities proportional to COD coefficients, as shown by Equation (40). At stage $n = 3$, the searcher scarcely uses his budget: only 0.78% of available budget in total and zero resource in Cell 3, even though the evader movement focuses on Cell 3. This may come from small $\gamma_2(3,1)$. The searcher can estimate that the evader arriving at Cell 3 would not have energy enough to be so active and the searcher can do an effective search for the evader after the next stage. Actually the evader would have energy $e = 1$ left at the next stage.

A evader movement to Cell 2 generates state $(n,k,e) = (2,2,4)$, which has transferable cells $N(k,e) = \{1,2,3,4\}$. However an optimal strategy tells us that only three cells $\{2,3,4\}$ can be options as the next evader cell, which are cells $\{k_1, \cdots, k_{s_n^*}\}$ in Equation (40). The probabilities of selecting the three cells are still proportional to COD coefficients of respective cells. In this state $(2,2,4)$, the searcher distributes comparatively large resources in Cell 3, which is exchanged for large cost $0.306 * 2$, taking account of a small number of residual stages and high transfer probability of the evader. 66.5% of the residual budget are used here. A evader movement to Cell 3 from $(3,1,5)$ generates state $(2,3,1)$. The evader must select one out of cells $\{3,4\}$ although he can move to cells $N(k,e) = \{2,3,4\}$. For the state, we can describe some interpretation about optimal strategies of players similar to the state $(2,2,4)$.

At the final stage $n = 1$, we are going to discuss four states $(1,3,3)$, $(1,4,0)$ and $(1,3,1)$, $(1,4,0)$. Optimal solutions are given by Equation (33)–(35). The searcher consumes all residual budget and the evader selects all of cells $N(k,e)$ with positive probabilities. Because of $N(3,3) = N(3,1) = \{2,3,4\}$, optimal strategies given by Equation (34) and (35) are the same for two states $(1,3,3)$ and $(1,3,1)$. But please note that when the evader reaches these two states, the residual budget of the searcher are different. Let us assume that the searcher has budget Q at the first stage $n = 3$. The searcher might have budget $(1-0.6646)(1-0.0078)Q = 0.333Q$ left in state $(1,3,3)$ but $(1-0.6461)(1-0.0078)Q = 0.351Q$ in state $(1,3,1)$. Similarly, two states of $(1,4,0)$ on the right and the left hand are the same situation for the evader but their searcher's residual budgets are different.

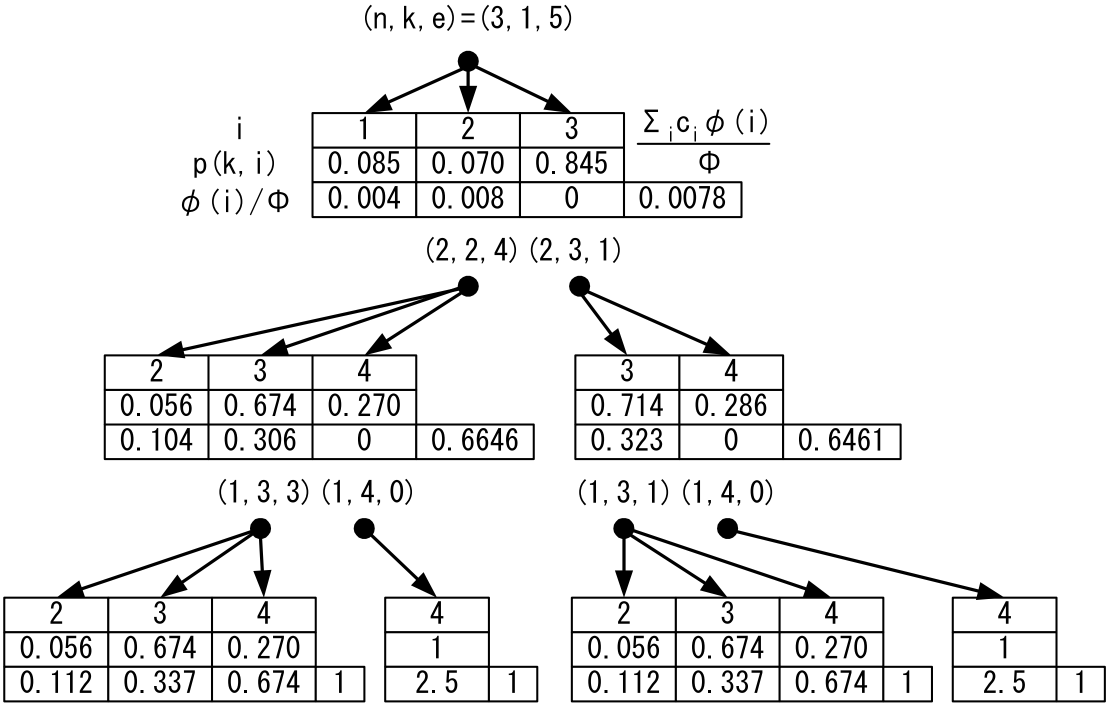

Figure 2: Transition of states

From Figure 2, we can calculate the probability that the evader follows a route with sequential cells $\{1, 2, 3, 3\}$. It might be $0.07 * 0.674 * 0.674 = 0.032$ and its whole non-detection probability is $\exp(-\alpha_2 * 0.008Q - \alpha_3 * 0.306 * (1 - 0.0078)Q - \alpha_3 * 0.337 * (1 - 0.6646) * (1 - 0.0078)Q) = \exp(-0.044Q)$. Another scenario of a route $\{1, 3, 4, 4\}$ has probability $0.845 * 0.286 * 1 = 0.242$ that the evader chooses the route and has the non-detection probability $\exp(-\alpha_4 * 2.5 * (1 - 0.6461) * (1 - 0.0078)Q) = \exp(-0.044Q)$. The above non-detection probabilities are the same. Now that we can verify the equivalence of the non-detection probabilities for all scenarios illustrated in Figure 2, we can say that the searcher's optimal plan of distributing searching resources are tough enough for all options of evader routes.

From now on, we turn our attention to value $\gamma_n(k, e)$. As seen from expression (31), $\gamma_n(k, e)$ is a direct pointer of the value of the game with the payoff of non-detection probability. We are going to examine the value of $\gamma_n(k, e)$. Let the initial energy of the evader be $e_0 = 9$.

(2) Effects of COD coefficient

Figure 3 illustrates $\gamma_n(k, e)$ for four combinations of energy $e = 9,\ 1$ and cells $k = 4,\ 7$. For cells 4 and 7, their COD coefficients are 8 and 3.33, respectively. In two cases of higher energy, their values are not so different because the high energy gives the flexibility for the evader to reach cells with high COD coefficients in the future, even though the coefficient of his present cell is low. On the other hand, in the case of lower energy, the γ-value depends mainly on the COD coefficient of the present cell. The value in case $(k, e) = (4, 1)$ is definitely larger than that of case $(7, 1)$. We can also verify property (i) of Corollary 1.

(3) Stationary value of the game

By the medium of $\gamma_n(k)$, we investigate the limiting value or the stationary value of the game with no energy constraint at an infinite number of stage. Values of $\gamma_1(k)$ and

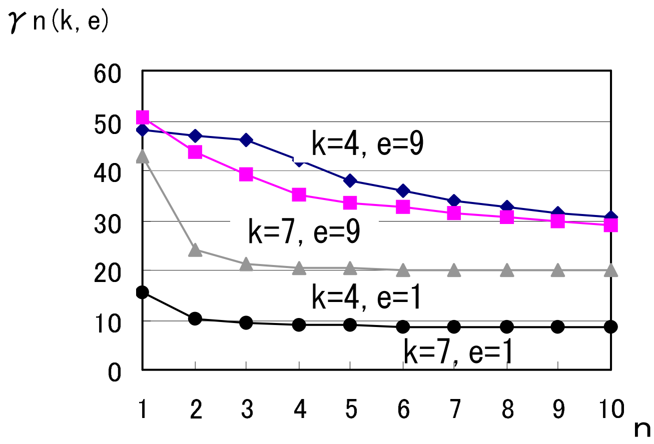

Figure 3: Effect of COD coefficient

$\lim_{n\to\infty} \gamma_n(k)$ are calculated for the parameter setting in Table 1 and for the two-neighbored-cell assumption. They are figured in Table 2.

Table 2: $\gamma_n(k)$

Cell k	1	2	3	4	5	6	7	8	9	10
$\gamma_1(k)$	23.67	31.67	46.67	48.24	49.90	38.65	50.65	42.32	38.75	35.42
Stationary points	23.67	31.67	43.00	43.00	43.00	38.65	39.67	37.57	37.27	35.42

The evader tends to run into cells with larger COD coefficient because the searcher cannot execute an effective search operation there compared to other places. However, when the searcher can make a certain guess on the hiding cell of the evader, the searcher concentrates his searching resources there, and he can make his search operation more effective as a result, even though the cell has a higher coefficient. That is why the COD coefficients of the neighborhood cells surrounding the cell are vital for the evader, too. Although Cell 3 and 9 have the highest coefficient 20, Cell 7 with a small coefficient 3.33 has a larger value than those cells in terms of $\gamma_1(\cdot)$. In this case, the whole cells $\boldsymbol{K}$ is only an equivalence class L_1 as well as the reachable-cell set for each cell. Cell i_1^* of Lemma 1(ii) is Cell 1 and therefore $\gamma_n(1)$ never changes from $\gamma_1(1) = 23.67$ at any stage n.

Now we modify only the neighborhood-cell assumption as shown in Table 3 and calculate initial value $\gamma_1(k)$ and stationary value when $n \to \infty$. These values are shown in Table 4. In this case, there are three equivalence classes of cells: $L_1 = \{1, 2, 3\}$, $L_2 = \{4, 5, 6, 7\}$ and $L_3 = \{8, 9, 10\}$. Because they have no reachable class, the γ-value is maintained unchanged for three cells $i_{u_1}^* = 2$, $i_{u_2}^* = 7$ and $i_{u_3}^* = 10$ given by Lemma 1(ii). We can observe that the γ-value of Cell 3 converges to its lower bound $c_3/\alpha_3 = 20$, as stated in Corollary 2. As the number of stages increases, the evader can go to any cell of reachable cells $R(k)$ of his current cell k, and especially he desires to go to the cells with higher $\gamma_n(\cdot)$. At the same time, larger n makes the searcher divide his available searching resources in his time horizon more efficiently. Consequently $\gamma_n(\cdot)$ goes down, as seen in Theorem 1(i). Under these general tendencies, $\gamma_n(\cdot)$ changes in the concrete way stated in Case (ii) of Theorem 2 while being affected by only the neighborhood cells. Then the γ-values end up with similarity in a set of cells, especially when the cells belong to their neighborhood cells. However, Lemma 1

(ii) says that $\gamma_n(\cdot)$ happens to be kept being as constant as $\gamma_1(\cdot)$. Each of $L_1,\ L_2,\ L_3$ is a small size of equivalence class of cells such that almost all of their members are in their neighborhood cells. In each class, the γ-values happen to converge to the same value with exceptional cells 2, 7 or 10 given by Lemma 1 (ii), as Table 4 shows.

Table 3: Neighborhood-cell $N(k)$

N(1)	N(2)	N(3)	N(4)	N(5)	N(6)	N(7)	N(8)	N(9)	N(10)
{1,2,3}	{1,2}	{2,3}	{4,5,6}	{4,5,6,7}	{4,5,6,7}	{6,7}	{8,9}	{8,9,10}	{9,10}

Table 4: $\gamma_n(k)$

Cell k	1	2	3	4	5	6	7	8	9	10
$\gamma_1(k)$	23.67	3.67	21.67	26.57	29.90	29.90	6.90	28.75	35.42	26.67
Stationary points	20.00	3.67	20.00	26.57	26.57	26.57	6.90	28.75	28.75	26.67

To the above case, we add the modification of $N(3) = \{2, 3, 4\}$, which gives the evader more ability to move from Cell 3 to 4. The results are shown in Table 5. The equivalence classes of cells are the same as $L_1,\ L_2,\ L_3$ of the previous case but L_2 becomes reachable from L_1. Now Lemma 1(ii) brings us two cells $i^*_{u_1} = 7$ and $i^*_{u_2} = 10$ with constant γ-values. Because $\gamma_1(2) = \min_{j\in R(2)} \gamma_1(j)$, an equal sign becomes active in inequality (55) for $k = 2$. That is why $\gamma_n(2)$ is kept constant at any stage.

Table 5: $\gamma_n(k)$

Cell k	1	2	3	4	5	6	7	8	9	10
$\gamma_1(k)$	23.67	3.67	29.67	26.57	29.90	29.90	6.90	28.75	35.42	26.67
Stationary points	22.58	3.67	26.57	26.57	26.57	26.57	6.90	28.75	28.75	26.67

The additional mobility from Cell 3 to 4 raises γ-value for not only Cell 3 but also Cell 1 compared with Table 4. In general, higher mobility of the evader pulls up γ-values of cells connected to each other by their reachability because the evader has the propensity to go to cells with larger COD coefficients. Larger γ-value indicates the poor effectiveness of searcher's resources Φ for the detection of the evader, as seen in Equation (31). In Table 2, in the case of two-neighbored-cell constraint, figures are larger than in Table 4 and 5.

5. Conclusions

In this paper, we discuss a multi-stage stochastic game of the so-called search allocation problem, which has not been studied so far. In the game, a searcher has a strategy of distributing his searching resources to detect an evader, who is trying to evade the searcher by his moving strategy. As results, we present formulae for the value of the game and its equilibrium solution, and clarify some properties of them. As many studies have pointed out, we can hardly propose elegant methods but often find numerical solution methods for optimal solutions, even for the one-sided game of optimizing the searcher's strategy. Finding ρ by Equation (22) in Section 3.1 is representative of those methods. However, the principle of Nash equilibrium, saying that the optimal strategy of one player must be the best response to the optimal strategy of the other one, enable us to give explicit formulae for the value of the game and its solutions, as seen in Theorem 2. In the formulae, two terms are separately involved. One term depends on the parameters of the number of the stage, COD coefficients of the cells, the evader's position and his energy. The other depends only on the searcher's resources.

Because the evader's position is exposed to the searcher at every stage, there is no space, in the long run, for the evader to manipulate his moving strategy and effectively conceal his distribution from the searcher. On the other hand, as seen in Theorem 2, even though the evader knows the searcher's remaining resources at every stage, this is not useful information for the evader at the moment that he has to choose a cell as his hiding point. Our problem is modeled on the disadvantages of the evader. In the so-called datum search game, the evader's position is revealed to the searcher only at initial time, but it is usually kept in secret after that. For this reason, the datum search game has been formulated as a single-stage game in almost all studies, and we can say that the game is proper for us to discuss the effectiveness of the short-term strategies of players. However, if we desire to discuss the long-term strategies, we must not ignore our model of a sequence of the datum search games repeated over the long term. We use the detection probability of the evader as the payoff of the game. How far our results in this paper can be extended to other payoff will be the next problem in the future.

Acknowledgement

The author acknowledges anonymous referees for their comments, which are helpful to improve my paper.

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