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INEFFICIENCY EVALUATION WITH AN ADDITIVE DEA MODEL UNDER IMPRECISE DATA, AN APPLICATION ON IAUK* DEPARTMENTS

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Abstract In imprecise data envelopment analysis (IDEA) (Cooper et al. [4]), the corresponding DEA models become non-linear and an important problem is to transform them into a linear programming one. In most of the current approaches to this problem, the number of decision variables increases dramatically, and usually the favorable results of these models are taken in several occasions. In this paper an additive DEA model is employed to evaluate the technical inefficiency of decision making units (DMUs) under imprecise data. The non-linear DEA model is transformed into an equivalent linear one, then the *translation invariant* property is used and a *one-stage* approach is introduced in this inefficiency evaluation. The approach rectifies the computational burden of previous methods in applications.

Keywords: DEA, efficiency, additive model, imprecise data

1. Introduction

Data envelopment analysis (DEA) is a non-parametric approach to measuring and evaluating the relative efficiencies of decision making units (DMUs) that utilize multiple inputs to produce multiple outputs. The original DEA method [2] requires that the values of all data must be known exactly without any variations. However, this assumption may not be always true and the need to deal with categorical and ordinal data has been reported in DEA literature (Banker and Morey [1], Kamakura [11]; Rousseau and Semple [15]; among others). Recently, Cooper et al. [4] addressed the problem of imprecise data in DEA in some more general cases including interval data, ordinal data and ratio bounded data. In dealing with these data, the obtained DEA models are usually non-linear. To rectify this drawback, Cooper et al. [4] proposed some methods to convert the non-linear model to a linear one through scale and variable transformations. Moreover, Cooper et al. [6] extended [4] to a general case. In addition to it, in this context, one can read Kim et al. [12], and Cooper et al. [5]. Recently Lee et al. [14], Entani et al. [8], Zhu [16, 17], Despotis and Smirlis [7] and Jahanshahloo et al. [9, 10] have investigated to IDEA from some different point of views.

The approach of most of these authors involves too many data and variable transformations making the measurement process unnecessarily complicated. The variable transformations alone increase dramatically the number of decision variables from $(m+s)$ to $(m+s)\times n$, where m, s, and n represent the number of inputs, outputs and DMUs, respectively. These transformations convert both the exact and imprecise data, including preference data and interval numbers, into constraints. But, during the process, the number of iterations and computation times increase rapidly and in some cases the favorable results of IDEA models

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are taken in several occasions (for example see [14]).

There is also another problem, reduction of the ability of IDEA linear models in comparison with standard DEA models. This drawback is a result of some linearizing processes which cause some difficulty in the interpretation of new variables. For example, in efficiency evaluation problem, most of IDEA models lead only to an efficiency score, and one cannot obtain any information on the important aspects of efficiency measurement such as the inefficiency resources, peer set, slacks variables and so on. These difficulties have been removed by introducing a *one-stage* approach in some important cases of imprecise data.

The rest of the paper is organized as follows. Section 2 presents an additive DEA model under interval data and based on this formulation, we can define upper and lower bounds of technical inefficiencies for each DMU, as the one done by Despotis and Smirlis [7]. We take these results from envelopment form of DEA models but this is not the main purpose of this paper. Section 3 is devoted to an extension of our interval ADD model in order to incorporate the ordinal data, thus dealing with the more general case of imprecise data. Our approach in this inefficiency evaluation is based on converting ordinal data into interval data by employing the translation invariant property of ADD model. Section 4 concludes.

2. Additive Model under Interval Data

Suppose that we have n DMUs which utilize inputs x_{ij} for $i = 1, \cdots, m$ to produce output y_{rj} for $r = 1, \cdots, s$ and $j = 1, \cdots n$. Let $\mathbf{X} \in \mathbb{R}^{m \times n}$ and $\mathbf{Y} \in \mathbb{R}^{s \times n}$ be the input and output matrix respectively. Now consider the variable return to scale (VRS) version of additive model [3] in technical inefficiency evaluation when DMU_k is under evaluation:

$$\begin{array}{lll} max & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ & \\ s.t. & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{ik} & i = 1, \cdots, m \\ & -\sum_{j=1}^{n} y_{rj}\lambda_j + s_r^+ = -y_{rk} & r = 1, \cdots, s \\ & \sum_{j=1}^{n} \lambda_j = 1 & \\ & s_i^- \geq 0,\ s_r^- \geq 0,\ \lambda_j \geq 0 & \forall i,\ \forall r,\ \forall j \end{array} \tag{2.1}$$

Now the following definition of efficiency for an efficient DMU in the above additive model is to be considered.

Definition 1. *DMU_k is ADD-efficient if the optimal value of the model 2.1 is equal to zero.*

If we take the dual of model 2.1, we have

$$\begin{array}{lll} min & \sum_{i=1}^{m} x_{ik}v_i - \sum_{r=1}^{s} y_{rk}u_r + u_0 & \\ s.t. & \sum_{i=1}^{m} x_{ij}v_i - \sum_{r=1}^{s} y_{rj}u_r + u_0 \geq 0 & j = 1, \cdots, n \\ & v_i \geq 1,\ u_r \geq 1 & \forall i,\ \forall r \\ & u_0 & free \end{array} \tag{2.2}$$

The above model is useful for our purpose in dealing with imprecise data.

2.1. Formulation as a linear model

Now assume that input and output levels of each DMU are not known exactly, so the above models are non-linear since output/input levels are also variables whose exact values have to be estimated. First we assume that the data have interval form. Let $x_{ij} \in [\underline{x}_{ij}, \overline{x}_{ij}]$ and $y_{rj} \in [\underline{y}_{rj}, \overline{y}_{rj}]$, where lower and upper bounds are known, i.e.

$$\begin{array}{ll} \forall i,\ \forall j, & \underline{x}_{ij} \leq x_{ij} \leq \overline{x}_{ij} \\ \forall r,\ \forall j, & \underline{y}_{rj} \leq y_{rj} \leq \overline{y}_{rj} \end{array}$$

According to [7] let

$$\begin{array}{ll} x_{ij} = \underline{x}_{ij} + (\overline{x}_{ij} - \underline{x}_{ij}) s_{ij}\ ; & \forall i,\ \forall j,\quad 0 \leq s_{ij} \leq 1 \\ y_{rj} = \underline{y}_{rj} + (\overline{y}_{rj} - \underline{y}_{rj}) t_{rj}\ ; & \forall r,\ \forall j,\quad 0 \leq t_{rj} \leq 1 \end{array}$$

With these transformations the new variables s_{ij} and t_{rj} are added to the problem and the weighted sum of inputs and outputs takes the form

$$\begin{array}{l} \sum_i x_{ij} v_i + \sum_r y_{rj} u_r + u_0 = \\ \sum_i \{\underline{x}_{ij} v_i + (\overline{x}_{ij} - \underline{x}_{ij}) s_{ij} v_i\} - \sum_r \{\underline{y}_{rj} u_r + (\overline{y}_{rj} - \underline{y}_{rj}) t_{rj} u_r\} + u_0 \end{array}$$

Now set

$$\begin{array}{lcl} q_{ij} := s_{ij} v_i & \text{and} & p_{rj} := t_{rj} u_r \\ 0 \leq q_{ij} \leq v_i & \text{and} & 0 \leq p_{rj} \leq u_r \end{array}$$

With the above substitutions, the model 2.2 is finally transformed into the following linear programming

$$\begin{array}{lll} min & z_k = \sum_{i=1}^{m} \{\underline{x}_{ik} v_i + (\overline{x}_{ik} - \underline{x}_{ik}) q_{ik}\} - \sum_{r=1}^{s} \{\underline{y}_{rk} u_r + (\overline{y}_{rk} - \underline{y}_{rk}) p_{rk}\} + u_0 & \\ s.t. & \sum_{i=1}^{m} \{\underline{x}_{ij} v_i + (\overline{x}_{ij} - \underline{x}_{ij}) q_{ij}\} - \sum_{r=1}^{s} \{\underline{y}_{rj} u_r + (\overline{y}_{rj} - \underline{y}_{rj}) p_{rj}\} + u_0 \geq 0 & \forall j \\ & 0 \leq q_{ij} \leq v_i & \forall i,\ \forall j \\ & 0 \leq p_{rj} \leq u_r & \forall r,\ \forall j \\ & v_i \geq 1,\ u_r \geq 1 & \forall i,\ \forall r \\ & u_0\ \ free & \end{array} \tag{2.3}$$

The additive DEA model with exact data is derived as a special case of the model 2.3. The above model adjusts the weights and the levels of inputs and outputs in favor of DMU_k, but with too many variables and constraints.

In the rest of this section we relax these requirements and introduce a procedure to extract a matrix of exact data from the bounded data and to identify the inefficiencies of DMUs in a *one-stage* method.

2.2. Measuring a technical inefficiency under interval data

Consider the following model:

$$\begin{array}{lll} min & z'_k = \sum_{i=1}^{m} \underline{x}_{ik} v_i - \sum_{r=1}^{s} \overline{y}_{rk} u_r + u_0 & \\ s.t. & \sum_{i=1}^{m} \overline{x}_{ij} v_i - \sum_{r=1}^{s} \underline{y}_{rj} u_r + u_0 \geq 0 & j = 1, \cdots, n,\ j \neq k \\ & \sum_{i=1}^{m} \underline{x}_{ik} v_i - \sum_{r=1}^{s} \overline{y}_{rk} u_r + u_0 \geq 0 & \\ & v_i \geq 1,\ u_r \geq 1 & \forall i,\ \forall r \\ & u_0 & free \end{array} \tag{2.4}$$

The model 2.4 is a DEA model with exact data, where the levels of inputs and outputs are adjusted in favor of DMU_k against the other DMUs. Now we have the following theorem

Theorem 1. *If z_k^* and $z_k'^*$ are the optimal values of the models 2.3 and 2.4, respectively, then $z_k^* = z_k'^*$.*

Proof. Assume first that $(V^*, U^*, Q^*, P^*, u_0^*)$ is an optimal solution of the model 2.3 with the optimal value z_k^*, where $V^* = (v_i^*\ ;\ i = 1,\cdots,m)$, $U^* = (u_r^*\ ;\ r = 1,\cdots,s)$, $Q^* = (q_{ij}^*\ ;\ i = 1,\cdots,m,\ j = 1,\cdots,n)$ and $P^* = (p_{rj}^*\ ;\ r = 1,\cdots,s,\ j = 1,\cdots,n)$.
According to the objective function, z_k increases monotonically with the increase of q_{ik} and decreases with the increase of p_{rk}. Therefore, in optimality we have $q_{ik}^* = 0$ and $p_{rk}^* = u_r$ and the kth constraints convert to $\sum_i \underline{x}_{ik} v_i^* - \sum_r \overline{y}_{rk} u_r^* + u_o^* \geq 0$.
For $j \neq k$, according to $0 \leq q_{ij}^* \leq v_i^*$ and $0 \leq p_{rj}^* \leq u_r^*$, we have
$0 \leq \sum_i \{\underline{x}_{ij} v_i^* + (\overline{x}_{ij} - \underline{x}_{ij}) q_{ij}^*\} - \sum_i \{\underline{y}_{rj} u_r^* + (\overline{y}_{rj} - \underline{y}_{rj}) p_{rj}^*\} + u_0^*$
$\leq \sum_i \{\underline{x}_{ij} v_i^* + (\overline{x}_{ij} - \underline{x}_{ij}) v_i^*\} - \sum_i \{\underline{y}_{rj} u_r^* + (\overline{y}_{rj} - \underline{y}_{rj}) 0\} + u_0^*$
$= \sum_i \overline{x}_{ij} v_i^* - \sum_r \underline{y}_{rj} u_r^* + u_0^*$
that is, (V^*, U^*, u_0^*) is a feasible solution of the model 2.4. Thus, $z_k'^* \leq z_k^*$.
Conversely, assume that (V^*, U^*, u_0^*) is an optimal solution of the model 2.4 with $z_k'^*$ as optimal value. Let

$$\overline{q}_{ij} = \begin{cases} 0 & j = k \\ v_i^* & j \neq k \end{cases} \qquad \overline{p}_{rj} = \begin{cases} u_r^* & j = k \\ 0 & j \neq k \end{cases}$$

so, if $\overline{Q} = (\overline{q}_{ij}\ ;\ i = 1,\cdots,m,\ j = 1,\cdots,n)$ and $\overline{P} = (\overline{p}_{rj}\ ;\ r = 1,\cdots,s,\ j = 1,\cdots,n)$, then $(V^*, U^*, \overline{Q}, \overline{P}, u_0^*)$ is a feasible solution of 2.3. Hence $z_k^* \leq z_k'^*$, and this completes the proof. □

This theorem implies that the additive DEA model with interval data can be solved by a sequence of linear programming with exact data of the form 2.4. If we take the dual of the model 2.4, we have

$$\begin{array}{lll} max & \displaystyle\sum_i s_i^- + \sum_r s_r^+ & \\ s.t. & \displaystyle\sum_{j \neq k} \overline{x}_{ij} \lambda_j + \underline{x}_{ik} \lambda_k + s_i^- = \underline{x}_{ik} & i = 1,\cdots,m \\ & \displaystyle\sum_{j \neq k} \underline{y}_{rj} \lambda_j + \overline{y}_{rk} \lambda_k - s_r^+ = \overline{y}_{rk} & r = 1,\cdots,s \\ & \displaystyle\sum_j \lambda_j = 1 & \\ & s_i^- \geq 0,\ s_r^- \geq 0,\ \lambda_j \geq 0 & \forall i,\ \forall r,\ \forall j \end{array} \tag{2.5}$$

The above linear programming is in the envelopment side and based on the theorem 2, and helps us to obtain individual slacks and determine the source of inefficiencies, efficient projections and return to scale (RTS) classifications of units under interval data assumption.

2.3. Upper and lower bounds of technical inefficiency using exact data and classification of the units

Consider the following model that, contrary to the model 2.5, provides the worst possible position for the unit k against the other units.

$$
\begin{array}{lll}
max & \sum_i s_i^- + \sum_r s_r^+ & \\
s.t. & \sum\limits_{j\neq k} \underline{x}_{ij}\lambda_j + \overline{x}_{ik}\lambda_k + s_i^- = \overline{x}_{ik} & i = 1,\cdots,m \\
& \sum\limits_{j\neq k} \overline{y}_{rj}\lambda_j + \underline{y}_{rk}\lambda_k - s_r^+ = \underline{y}_{rk} & r = 1,\cdots,s \\
& \sum\limits_{j} \lambda_j = 1 & \\
& s_i^- \geq 0,\ s_r^- \geq 0,\ \lambda_j \geq 0 & \forall i,\ \forall r,\ \forall j
\end{array}
\tag{2.6}
$$

We can now state the following theorem:
Theorem 2. *If $\underline{z}_k$ and $\overline{z}_k$ are the optimal values of the models 2.5 and 2.6, respectively, then $\underline{z}_k \leq \overline{z}_k$.*

Proof. Assume that $(\tilde{\lambda}_j, \tilde{s}_i^-, \tilde{s}_r^+ \ ;\ \forall i, \forall r, \forall j)$ is a feasible solution of 2.5, so
$\tilde{s}_i^- = \underline{x}_{ik} - \sum\limits_{j\neq k} \overline{x}_{ij}\tilde{\lambda}_j - \underline{x}_{ik}\tilde{\lambda}_k = (1-\tilde{\lambda}_k)\underline{x}_{ik} - \sum\limits_{j\neq k} \overline{x}_{ij}\tilde{\lambda}_j$ for all i, similarly
$\tilde{s}_r^+ = \sum\limits_{j\neq k} \overline{y}_{rj}\tilde{\lambda}_j + \overline{y}_{rk}\tilde{\lambda}_k - y_{rk} = \sum\limits_{j\neq k} \overline{y}_{rj}\tilde{\lambda}_j - (1-\tilde{\lambda}_k)y_{rk}$ for all r. Since $1-\tilde{\lambda}_k \geq 0$, we have

$$
\begin{array}{lll}
\forall i, & \tilde{s}_i^- \leq (1-\tilde{\lambda}_k)\overline{x}_{ik} - \sum\limits_{j\neq k} \underline{x}_{ij}\tilde{\lambda}_j & (\dagger) \\
\forall r, & \tilde{s}_r^+ \leq \sum\limits_{j\neq k} \overline{y}_{rj}\tilde{\lambda}_j - (1-\tilde{\lambda}_k)\underline{y}_{rk} & (\ddagger)
\end{array}
$$

Now let

$$
\begin{array}{ll}
\forall i, & \hat{s}_i^- =: (1-\tilde{\lambda}_k)\overline{x}_{ik} - \sum\limits_{j\neq k} \underline{x}_{ij}\tilde{\lambda}_j \\
\forall r, & \hat{s}_r^+ =: \sum\limits_{j\neq k} \overline{y}_{rj}\tilde{\lambda}_j - (1-\tilde{\lambda}_k)\underline{y}_{rk}
\end{array}
$$

It is easy to verify that $(\tilde{\lambda}_j, \hat{s}_i^-, \hat{s}_r^+ \ ;\ \forall i, \forall r, \forall j)$ is a feasible solution of the model 2.6, and also according to (†) and (‡) we have

$$\sum_i \tilde{s}_i^- + \sum_r \tilde{s}_r^+ \leq \sum_i \hat{s}_i^- + \sum_r \hat{s}_r^+$$

This shows that any feasible solution of the model 2.5 corresponds to a feasible solution of the model 2.6 whose objective function value is at most equal to that of the later and this completes the proof. □

The results of the models 2.5 and 2.6 provide lower and upper bounds respectively, for possible inefficiencies in terms of slacks for each DMUs. On the basis of the bounded interval

Table 1: Interval data

DMU	Inputs				Outputs			
j	$\underline{x}_{1j}$	$\overline{x}_{1j}$	$\underline{x}_{2j}$	$\overline{x}_{2j}$	$\underline{y}_{1j}$	$\overline{y}_{1j}$	$\underline{y}_{2j}$	$\overline{y}_{2j}$
1	18	20	148	151	100	105	90	92
2	25	27	160	162	154	160	54	55
3	19	23	145	152	150	151	51	52
4	26	27	175	178	135	138	72	75
5	20	22	156	158	194	195	66	69
6	56	58	253	255	131	133	72	74

$[\underline{z}_j, \overline{z}_j]$, the units can be classified in three subsets as follows*

$$\begin{aligned} E^{++} &= \{j | \overline{z}_j = 0\} \\ E^{+} &= \{j | \underline{z}_j = 0 \ and \ \overline{z}_j > 0\} \\ E^{-} &= \{j | \underline{z}_j > 0\} \end{aligned}$$

The set E^{++} contains the units that are technically efficient in each of the data levels and these units always lie on the frontier of the production possibility set (PPS). The set E^{+} contains units that are efficient in the maximal case, but there exist data levels under which they lose their efficiency and finally, E^{-} contains the units that are always inefficient.

2.4. Numerical example

As an illustration, we applied the above procedure for the interval data setting of Table 1 (6 units with 2 inputs and 2 outputs). The inefficiencies and classifications of the units obtained by applying the models 2.5 and 2.6 are presented in Table 2.
Also, the last two columns of this table give the values of individual slacks and lambda variables by evaluating each unit with the model 2.5.

Table 2: Inefficiency evaluation and classification of 6 units[a]

DMU	Inefficiency		DMUs	Slacks	Lambdas
j	$\underline{z}_j$	$\overline{z}_j$	classes	(>0)	(>0)
1	0	0	E^{++}	None	$\lambda_1^* = 1$
2	50	69	E^{-}	$s_1^{-*} = 3$	$\lambda_5^* = 1$
				$s_2^{-*} = 2$	
				$s_1^{+*} = 34$	
				$s_2^{+*} = 11$	
3	0	33.5	E^{+}	None	$\lambda_3^* = 1$
4	45.125	78.565	E^{-}	$s_1^{-*} = 4.75$	$\lambda_1^* = 0.375$
				$s_2^{-*} = 19.625$	$\lambda_3^* = 0.625$
				$s_1^{+*} = 20.75$	
5	0	0	E^{++}	None	$\lambda_5^* = 1$
6	161.668	190.565	E^{-}	$s_1^{-*} = 34.667$	$\lambda_1^* = 0.334$
				$s_2^{-*} = 97.334$	$\lambda_3^* = 0.666$
				$s_1^{+*} = 29.667$	

[a] LINDO solver used for these computations.

*For similar classifications in terms of efficiency score intervals, see [7].

3. An Extension of Inefficiency Evaluation for Dealing with Imprecise Data

To formulate the general case, which is the main purpose of this paper, let us introduce the following sets by which one can distinguish the inputs and the outputs into cardinal (exact and/or interval) and ordinal data:
$I = \{1, ..., m\}$: the index set for inputs.
$O = \{1, ..., s\}$: the index set for outputs.
CI: the subset of indices for cardinal inputs ($CI \subseteq I$)
OI: the subset of indices for ordinal inputs ($OI \subseteq I, \ \ CI \cup OI = I$)
CO: the subset of indices for cardinal outputs ($CO \subseteq O$)
OO: the subset of indices for ordinal outputs ($OO \subseteq O, \ \ CO \cup OO = O$)
So, the ADD model with imprecise data can be demonstrated as follows:

$$\begin{array}{lll} min & \sum_{i\in CI} x_{ik}v_i + \sum_{i\in OI} x_{ik}v_i - \sum_{r\in CO} y_{rk}u_r - \sum_{r\in OO} y_{rk}u_r + u_0 & \\ s.t. & \sum_{i\in CI} x_{ij}v_i + \sum_{i\in OI} x_{ij}v_i - \sum_{r\in CO} y_{rj}u_r - \sum_{r\in OO} y_{rj}u_r + u_0 \geq 0 & \forall j \\ & ordinal\ relations\ among\ \{x_{ij} \mid i \in OI\ and\ j = 1, \cdots, n\} & \\ & ordinal\ relations\ among\ \{y_{rj} \mid r \in OO\ and\ j = 1, \cdots, n\} & \\ & v_i \geq 1\ ,\ u_r \geq 1 & \forall i,\ \forall r \\ & u_0 \quad free & \end{array} \tag{3.1}$$

Obviously the above model is non-linear. This drawback is rectified by converting ordinal data into interval data, employing the translation invariant property of the ADD model. We deal with the weak ordinal relation case. For the sake of illustration and to simplify the presentation, following [16], let us assume that the weak ordinal relations represented as follows

$$x_{i1} \leq x_{i2} \leq \cdots \leq x_{ik} \leq \cdots \leq x_{in}$$
$$y_{r1} \leq y_{r2} \leq \cdots \leq y_{rk} \leq \cdots \leq y_{rn}$$

where $i \in OI$ and $r \in OO$. Now, according to the *"translation invariant"* property of the model 2.1(see [13]),

$$x^*_{i1} \leq x^*_{i2} \leq \cdots \leq x^*_{ik} = 1 \leq \cdots \leq x^*_{in}$$
$$y^*_{r1} \leq y^*_{r2} \leq \cdots \leq y^*_{rk} = 1 \leq \cdots \leq y^*_{rn}.$$

Then, we can have a set of optimal solutions for the weak ordinal data such that

$$-M'_i \leq x_{i1} \leq x_{i2} \leq \cdots \leq x_{ik} = 1 \leq \cdots \leq x_{in} \leq M_i$$
$$-M'_r \leq y_{r1} \leq y_{r2} \leq \cdots \leq y_{rk} = 1 \leq \cdots \leq y_{rn} \leq M_r$$

where M'_i, M_i, M'_r and M_r, for $i \in OI$ and $r \in OO$, are sufficiently large positive numbers. Now, for the inputs and outputs in the weak ordinal relations, we can set up the following intervals

$$if\ i \in OI,\ set: \begin{cases} x_{ij} \in [-M'_i, 1] & j = 1, \cdots, k-1 \\ x_{ik} = 1 & \\ x_{ij} \in [1, M_i] & j = k+1, \cdots, n \end{cases} \tag{3.2}$$

$$and\ if\ r \in OO,\ set \begin{cases} y_{rj} \in [-M'_r, 1] & j = 1, \cdots, k-1 \\ y_{rk} = 1 & \\ y_{rj} \in [1, M_r] & j = k+1, \cdots, n. \end{cases} \tag{3.3}$$

3.1. Formulation as a linear model

Based on 3.2 and 3.3, we can write

$$\begin{aligned} & x_{ij} = -M'_i + (1+M'_i)s_{ij} \ \ ; \ \ 0 \le s_{ij} \le 1 \ for \ j = 1, \cdots, k-1 \\ \text{and} \quad & x_{ij} = 1 + (M_i - 1)s_{ij} \ \ ; \ \ 0 \le s_{ij} \le 1 \ for \ j = k+1, \cdots, n. \end{aligned}$$

Similarly

$$\begin{aligned} & y_{rj} = -M'_r + (1+M'_r)t_{rj} \ \ ; \ \ 0 \le t_{rj} \le 1 \ for \ j = 1, \cdots, k-1 \\ \text{and} \quad & y_{rj} = 1 + (M_r - 1)t_{rj} \ \ ; \ \ 0 \le t_{rj} \le 1 \ for \ j = k+1, \cdots, n. \end{aligned}$$

Under this setting, the ordinal relations can be established among the variables s_{ij} and t_{rj}. In the case of ordinal inputs, consider the following relations
i) If $l \le k < l'$ (or $l < k \le l'$), then we have $x_{il} \le x_{il'}$ and these relations are satisfied automatically.
ii) For $l < l' \le k$ or $k \le l < l'$, the ordinal relation $x_{il} \le x_{il'}$, takes the form $s_{il} \le s_{il'}$. The same relations can be considered for the ordinal outputs case. We now use the above transformations to convert the obtained nonlinear model into a linear model as follows:

$$\begin{aligned} min \ \check{z}_k = & \sum_{i \in CI} \{\underline{x}_{ik} v_i + (\overline{x}_{ik} - \underline{x}_{ik}) q_{ik}\} + \sum_{i \in OI} v_i - \sum_{r \in CO} \{\underline{y}_{rk} u_r + (\overline{y}_{rk} - \underline{y}_{rk}) p_{rk}\} - \sum_{r \in OO} u_r + u_0 \\ s.t. \ \ & \sum_{i \in CI} \{\underline{x}_{ij} v_i + (\overline{x}_{ij} - \underline{x}_{ij}) q_{ij}\} + \sum_{i \in OI} \{(-M'_i) v_i + (1 + M'_i) q_{ij}\} + \\ & - \sum_{r \in CO} \{\underline{y}_{rj} u_r + (\overline{y}_{rj} - \underline{y}_{rj}) p_{rj}\} - \sum_{r \in OO} \{(-M'_r) u_r + (1 + M'_r) p_{rj}\} + u_0 \ge 0 \qquad j = 1, \cdots, k-1 \\ & \sum_{i \in CI} \{\underline{x}_{ik} v_i + (\overline{x}_{ik} - \underline{x}_{ik}) q_{ik}\} + \sum_{i \in OI} v_i - \sum_{r \in CO} \{\underline{y}_{rk} u_r + (\overline{y}_{rk} - \underline{y}_{rk}) p_{rk}\} - \sum_{r \in OO} u_r + u_0 \ge 0 \\ & \sum_{i \in CI} \{\underline{x}_{ij} v_i + (\overline{x}_{ij} - \underline{x}_{ij}) q_{ij}\} + \sum_{i \in OI} \{v_i + (M_i - 1) q_{ij}\} + \\ & - \sum_{r \in CO} \{\underline{y}_{rj} u_r + (\overline{y}_{rj} - \underline{y}_{rj}) p_{rj}\} - \sum_{r \in OO} \{u_r + (M_r - 1) p_{rj}\} + u_0 \ge 0 \qquad j = k+1, \cdots, n \\ & \quad ordinal \ relations \ on \ the \ set \ \{q_{ij} \mid i \in OI \ and \ j = 1, \cdots, k-1\} \\ & \quad ordinal \ relations \ on \ the \ set \ \{q_{ij} \mid i \in OI \ and \ j = k+1, \cdots, n\} \\ & \quad ordinal \ relations \ on \ the \ set \ \{p_{rj} \mid r \in OO \ and \ j = 1, \cdots, k-1\} \\ & \quad ordinal \ relations \ on \ the \ set \ \{p_{rj} \mid r \in OO \ and \ j = k+1, \cdots, n\} \\ & \quad 0 \le q_{ij} \le v_i \ \ , \ \ 0 \le p_{rj} \le u_r \qquad \forall i, \ \forall r, \ \forall j \\ & \quad v_i \ge 1 \ , \ u_r \ge 1 \qquad \forall i, \ \forall r \end{aligned} \tag{3.4}$$

As can be seen, the above model is linear and the ordinal relations represented among the variables q_{ij} and p_{rj}, that were defined in the same way as interval data case. Also this model is a natural extension of the ordinal ADD model, but it has too many variables and constraints. So similar to the bounded data case, we introduce an equivalent linear model that helps us to form a matrix of exact data in determining the efficiency or inefficiency of the units and also the individual slacks, under the normal size of variables and constraints.

3.2. An exact data model for inefficiency evaluation

We use the inherent property of the model 3.4 in the inefficiency evaluation of DMU_k, to introduce the following model:

$$
\begin{aligned}
min \quad & \tilde{z}'_k = \sum_{i\in CI} \underline{x}_{ik} v_i + \sum_{i\in OI} v_i - \sum_{r\in CO} \overline{y}_{rk} u_r - \sum_{r\in OO} u_r + u_0 \\
s.t. \quad & \sum_{i\in CI} \overline{x}_{ij} v_i + \sum_{i\in OI} v_i - \sum_{r\in CO} \underline{y}_{rj} u_r - \sum_{r\in OO} (-M'_r) u_r + u_0 \geq 0 \quad j = 1,\cdots,k-1 \\
& \sum_{i\in CI} \underline{x}_{ik} v_i + \sum_{i\in OI} v_i - \sum_{r\in CO} \overline{y}_{rk} u_r - \sum_{r\in OO} u_r + u_0 \geq 0 \\
& \sum_{i\in CI} \overline{x}_{ij} v_i + \sum_{i\in OI} M_i v_i - \sum_{r\in CO} \underline{y}_{rj} u_r - \sum_{r\in OO} u_r + u_0 \geq 0 \qquad j = k+1,\cdots,n \\
& v_i \geq 1\ ,\ u_r \geq 1 \qquad i = 1,\cdots,m, \quad r = 1,\cdots,s \\
& u_0 \quad free
\end{aligned}
\tag{3.5}
$$

Before introducing a *one-stage* approach in the inefficiency evaluation of units under imprecise data with an envelope form of the ADD model, we prove the following theorem which establishes the relationship between models 3.4 and 3.5, justifying our use of 3.5 in the subsequent development.

Theorem 3. *If $\tilde{z}_k$ and $\tilde{z}'_k$ are the optimal values of the models 3.4 and 3.5 respectively, then $\tilde{z}_k = \tilde{z}'_k$.*

Proof. Let $(V^*, U^*, Q^*, P^*, u_0^*)$ be an optimal solution of the model 3.4 with $\tilde{z}_k$ as an optimal value, where $V^* = (v_i^*\ ; i = 1,\cdots,m)$, $U^* = (u_r^*\ ; r = 1,\cdots,s)$, $Q^* = (q_{ij}^*\ ; i = 1,\cdots,m,\ j = 1,\cdots,n)$ and $P^* = (p_{rj}^*\ ; r = 1,\cdots,s,\ j = 1,\cdots,n)$.
It is easy to verify that for $i \in CI$ and $r \in CO$ we have $q_{ik}^* = 0$ and $p_{rk}^* = u_r^*$, so the kth constraint of 3.4 takes the form

$$\sum_{i\in CI} \underline{x}_{ik} v_i^* + \sum_{i\in OI} v_i^* - \sum_{r\in CO} \overline{y}_{rk} u_r^* - \sum_{r\in OO} u_r^* + u_0^* \geq 0; \tag{†}$$

Now for $j \neq k$ we have two cases:
i) $j < k$: In this case:

$$0 \leq \sum_{i\in CI} \{\underline{x}_{ij} v_i^* + (\overline{x}_{ij} - \underline{x}_{ij}) q_{ij}^*\} + \sum_{i\in OI} \{(-M'_i) v_i^* + (1+M'_i) q_{ij}^*\} - \sum_{r\in CO} \{\underline{y}_{rj} u_r^* + (\overline{y}_{rj} - \underline{y}_{rj}) p_{rj}^*\} - \sum_{r\in OO} \{(-M'_r) u_r^* + (1 + M'_r) p_{rj}^*\} + u_0^*$$

Now using $0 \leq q_{ij} \leq v_i$, $0 \leq p_{rj} \leq u_r$, $-1 \leq M'_i, M'_r$ and $1 \leq M_i, M_r\ \forall i, \forall r, \forall j$, we obtain

$$\sum_{i\in CI} \overline{x}_{ij} v_i^* + \sum_{i\in OI} v_i^* - \sum_{r\in CO} \underline{y}_{rj} u_r^* - \sum_{r\in OO} (-M'_r) u_r^* + u_0^* \geq 0 \tag{‡}$$

ii) $j > k$: Similar to the first case we obtain

$$\sum_{i\in CI} \overline{x}_{ij} v_i^* + \sum_{i\in OI} M_i v_i^* - \sum_{r\in CO} \underline{y}_{rj} u_r^* - \sum_{r\in OO} u_r^* + u_0^* \geq 0 \tag{♯}$$

Now according to (†), (‡) and (♯) we conclude that (V^*, U^*, u_0^*) is a feasible solution of the model 3.5, so $\tilde{z}'_k \leq \tilde{z}_k$.
Conversely, assume that (V^*, U^*, u_0^*) is an optimal solution of the model 3.5 with $\tilde{z}'_k$ as the optimal value. Now let

$$\bar{q}_{ij} = \begin{cases} v_i^* & \forall i,\ \forall j\ (j \neq k) \\ 0 & i \in CI,\ j = k \end{cases} \qquad \bar{p}_{rj} = \begin{cases} 0 & \forall r,\ \forall j\ (j \neq k) \\ u_r^* & r \in CO,\ j = k \end{cases}$$

If we define $\bar{Q} := (\bar{q}_{ij};\ i = 1, \cdots, m,\ j = 1, \cdots, n)$ and $\bar{P} := (\bar{p}_{rj};\ r = 1, \cdots, s,\ j = 1, \cdots, n)$, then $(V^*, U^*, \bar{Q}, \bar{P}, u_0^*)$ is a feasible solution of the model 3.4. This implies $\tilde{z}_k \leq \tilde{z}'_k$, and thus, completes the proof. □

Consider now the dual of the model 3.5:

$$\begin{array}{lll}
max & \sum_i s_i^- + \sum_r s_r^+ & \\
s.t. & \sum_{j \neq k} \overline{x}_{ij}\lambda_j + \underline{x}_{ik}\lambda_k + s_i^- = \underline{x}_{ik} & i \in CI \\
& \sum_{j \leq k} \lambda_j + \sum_{j > k} M_i \lambda_j + s_i^- = 1 & i \in OI \\
& \sum_{j \neq k} \underline{y}_{rj}\lambda_j + \overline{y}_{rk}\lambda_k - s_r^+ = \overline{y}_{rk} & r \in CO \\
& -\sum_{j < k} M'_r \lambda_j + \sum_{j \geq k} \lambda_j - s_r^+ = 1 & r \in OO \\
& \sum_j \lambda_j = 1 & \\
& s_i^- \geq 0,\ s_r^+ \geq 0,\ \lambda_j \geq 0 & \forall i,\ \forall r,\ \forall j
\end{array} \tag{3.6}$$

Based on the theorem 3, we can apply the model 3.6 for inefficiency evaluation of DMUs with imprecise data in one stage, under the normal size of constraints and variables. Also from the optimal solutions of this model, we get some information about the inefficiency resources, peer set, slacks variables and so on.
This approach can be used for strong ordinal case. Note that for the application of the model 3.6, we need only set M_i and M'_r for some $i \in OI$ and $r \in OO$. Indeed upper and lower bounds for inefficiencies, as formulated in the models 2.5 and 2.6, and the classification of the units are also applicable in the imprecise data setting provided that the models be expanded for this case.

3.3. Numerical examples

To illustrate the above procedure and for the comparison purposes, we apply the model 3.6 to the example given in [4] and presented in Table 3.
Five units are considered with two inputs (one exact and one interval) and two outputs (one exact and one ordinal). For comparison purposes, we use Lee et al. [14] method by employing the VRS version of the ADD model in stage two. In computation, M_i and M'_r do not have to be set equal to very large numbers, and the results are not very sensitive to them. In this example, we use M'_2 equal to 30,10,40,1 and 20, for inefficiency evaluation of the units 1, 2, 3, 4 and 5 respectively, but this is not the unique choice to get these results.

Table 3: Exact and imprecise data

DMU	Inputs			Outputs	
	Exact	Interval		Exact	Ordinal
	x_1	$\underline{x}_2$	$\overline{x}_2$	y_1	y_2^a
1	100	0.6	0.7	2000	4
2	150	0.8	0.9	1000	2
3	150	1	1	1200	5
4	200	0.7	0.8	900	1
5	200	1	1	600	3

Source: Cooper et al. (1999), [a] Ordinal rank (5=the best; 1=the worst).

Table 4: Inefficiency evaluation results with two methods[a]

DMU	Ineff with Lee et al. method[b]			Ineff with model 3.6		
j	Ineff $z_k^{'*}$	Slacks (>0)	Lambdas (>0)	Ineff $\underline{z}_k^*$	Slacks (>0)	Lambdas (>0)
1	0	None	$\lambda_1^* = 1$	0	None	$\lambda_1^* = 1$
2	1321.43	$s_1^{-*} = 33.33$ $s_2^{-*} = 733.34$ $s_2^{+*} = 554.76$	$\lambda_1^* = 0.666$ $\lambda_3^* = 0.334$	1050.1	$s_1^{-*} = 50$ $s_2^{-*} = 0.1$ $s_1^{+*} = 1000$	$\lambda_1^* = 1$
3	0	None	$\lambda_3^* = 1$	0	None	$\lambda_3^* = 1$
4	1200	$s_1^{-*} = 100$ $s_1^{+*} = 1100$	$\lambda_1^* = 1$	1200	$s_1^{-*} = 100$ $s_1^{+*} = 1100$	$\lambda_1^* = 1$
5	2314.29	$s_1^{-*} = 50$ $s_1^{+*} = 600$ $s_2^{+*} = 1664.29$	$\lambda_1^* = 1$	1500.3	$s_1^{-*} = 100$ $s_2^{-*} = 0.3$ $s_1^{+*} = 1400$	$\lambda_1^* = 1$

[a] LINDO solver used for these computations.

[b] Here, we used Lee et al. [14] model with VRS version of ADD model in stage two.

As can be seen, the results of our one-stage approach in classifying the units into technically efficient or inefficient, is the same as the results obtained using Lee et al. [14] method. The differences in amounts of individual slacks may have been caused by different methods in achieving the matrix of exact data in these methods.

We now apply this IDEA approach to the 42 departments of IAUK. Table 5 reports the data with post graduate students (x_1), bachelor students (x_2), masters students (x_3), graduated students (y_1), members for scholarship (y_2), research products (y_3) and manager satisfaction (y_4). Note that y_4 is an ordinal data.

With respect to Kim et al.(1999), the current paper reports y_4 differently with "1" for the worst and "4" for the best, since larger output values are preferred in DEA. The result of the application 3.6 and Lee et al. method for this data setting is presented in Table 6.

In addition to inefficiency scores, our method provides benchmarks with magnitudes and can reflect information on return to scale classification. It can be seen that our approach yields the same efficient DMUs and larger efficiency scores for inefficient DMUs compared to those in Lee et al. [14] are assumed.

We finally note that one advantage of converting imprecise data and using standard DEA models is that the additional information can be obtained in the normal size of variables and constraints.

Table 5: Data for the 42 departments

DMU	x_1	x_2	x_3	y_1	y_2	y_3	y_4^a
1	0	261	0	225	1	1	3
2	0	170	56	213	2	0	3
3	0	281	70	326	2	0	3
4	0	138	33	159	1	0	2
5	164	0	0	52	1	0	3
6	291	815	0	1014	2	2	2
7	0	0	61	50	0	0	4
8	113	95	0	73	0	0	2
9	0	727	0	675	3	0	3
10	0	773	0	697	2	0	3
11	0	0	66	46	0	0	3
12	346	197	0	132	0	0	1
13	0	988	0	812	8	10	2
14	0	0	34	32	0	0	2
15	0	795	0	601	6	2	2
16	0	672	0	591	6	12	2
17	0	166	0	166	7	0	4
18	0	761	0	761	0	3	2
19	193	124	0	293	0	0	3
20	484	0	0	361	0	0	1
21	0	517	0	434	0	4	2
22	0	584	0	492	1	4	2
23	0	682	0	565	2	3	2
24	0	565	0	423	1	2	2
25	0	603	0	433	1	3	2
26	0	373	0	332	1	1	1
27	0	347	0	328	2	3	3
28	0	0	70	51	0	3	4
29	0	328	0	170	0	1	3
30	0	267	0	123	0	0	3
31	262	0	0	219	3	0	3
32	0	1023	0	794	2	0	4
33	366	995	0	1111	2	2	3
34	0	266	15	238	3	4	3
35	172	375	0	547	4	3	3
36	0	460	0	385	4	8	3
37	223	0	535	232	14	6	4
38	0	1202	58	1158	12	0	3
39	0	1025	61	394	4	1	3
40	0	0	69	50	0	2	4
41	314	0	0	204	0	0	1
42	371	0	0	226	0	0	1

[a] Four ordinal scales (4=the best; 1=the worst)

Table 6: Inefficiency evaluation results[a]

DMU	Model 3.6				Lee et al. method			
	Ineff	Slacks		Lambdas	Ineff	Slacks		Lambdas
	$\underline{z}_k^*$	(>0)		(>0)	$z_k^{\prime *}$	(>0)		(>0)
1	33.45	s_2^{-*}=29.16	s_2^{+*}=4.29	λ_9^*=0.01 λ_{17}^*=0.66 λ_{27}^*=0.33	37.35		s_1^{+*} = 31.53 s_2^{+*} = 5.16 s_4^{+*} = 0.66	λ_{16}^* = 0.06 λ_{17}^* = 0.83 λ_{18}^* = 0.11
2	0	None		λ_2^*=1	0	None		λ_2^*=1
3	0	None		λ_3^*=1	0	None		λ_3^*=1
4	11.55	s_3^{-*}=10.69	s_2^{+*}=0.45 s_3^{+*}=0.41	λ_{14}^*=0.65 λ_{17}^*=0.21 λ_{18}^*=0.14	12.2	s_3^{-*}=7.38	s_2^{+*} = 2.57 s_3^{+*} = 0.21 s_4^{+*} = 1.86	λ_7^*=0.42 λ_{17}^*=0.51 λ_{18}^*=0.07
5	0	None		λ_5^*=1	0	None		λ_5^*=1
6	0	None		λ_6^*=1	0	None		λ_6^*=1
7	0	None		λ_7^*=1	0	None		λ_7^*=1
8	121.90	s_1^{-*}=0.94	s_1^{+*}=115.67 s_2^{+*}=5.29	λ_{17}^*=0.57 λ_{31}^*=0.43	123.47	s_1^{-*} = 0.94	s_1^{+*} = 115.67 s_2^{+*} = 5.29 s_4^{+*} = 1.57	λ_{17}^* = 0.57 λ_{31}^* = 0.43
9	0	None		λ_9^*=1	0	None		λ_9^*=1
10	0	None		λ_{10}^*=1	0	None		λ_{10}^*=1
11	9	s_3^{-*}=5	s_1^{+*}=4	λ_7^*=1	11.56	s_3^{-*} = 11	s_4^{+*} = 0.56	λ_7^* = 0.78 λ_{14}^* = 0.22
12	418	s_1^{-*}=40.80 s_2^{-*}=24.68	s_1^{+*}=45.51 s_2^{+*}=6.91 s_3^{+*}=0.09	λ_{17}^*=0.97 λ_{35}^*=0.03	421	s_1^{-*} = 46 s_2^{-*} = 31	s_1^{+*} = 34 s_2^{+*} = 7 s_3^{+*}=3	λ_{17}^* = 1
13	0	None		λ_{13}^*=1	0	None		λ_{13}^*=1
14	0	None		λ_{14}^*=1	0	None		λ_{14}^*=1
15	120.37	s_2^{-*}=110.5	s_3^{+*}=9.82	λ_{13}^*=0.04 λ_{16}^*=0.95 λ_{18}^*= 0.01	120.37	s_2^{-*} = 110.55	s_3^{+*} = 9.82	λ_{13}^* = 0.04 λ_{16}^*=0.95 λ_{18}^*= 0.01
16	0	None		λ_{16}^*=1	0	None		λ_{16}^*=1
17	0	None		λ_{17}^*=1	0	None		λ_{17}^*=1
18	0	None		λ_{18}^*=1	0	None		λ_{18}^*=1
19	0	None		λ_{19}^*=1	0	None		λ_{19}^*=1
20	0	None		λ_{20}^*=1	0	None		λ_{20}^*=1
21	67.92		s_1^{+*}=63.88 s_2^{+*}=4.04	λ_{16}^*=0.24 λ_{17}^*=0.37 λ_{18}^*=0.39	68.67		s_1^{+*} = 63.88 s_2^{+*} = 4.04 s_4^{+*} = 0.75	λ_{16}^* = 0.24 λ_{17}^* = 0.37 λ_{18}^* = 0.39
22	77.85		s_1^{+*}=75.78 s_2^{+*}=2.07	λ_{16}^*=0.20 λ_{17}^*=0.27 λ_{18}^*=0.53	78.39		s_1^{+*} = 75.78 s_2^{+*} = 2.07 s_4^{+*} = 0.54	λ_{16}^* = 0.20 λ_{17}^* = 0.27 λ_{18}^* = 0.53
23	111	s_2^{-*}=59.81	s_1^{+*}=51.19	λ_{16}^*=0.07 λ_{17}^*=0.22 λ_{18}^*=0.70	111.44	s_2^{-*} = 59.81	s_1^{+*}=51.19 s_4^{+*}=0.44	λ_{16}^* = 0.07 λ_{17}^*=0.22 λ_{18}^*=0.70
24	143.33	s_2^{-*}=2.33	s_1^{+*}=139.67 s_2^{+*}=1.33	λ_{17}^*=0.33 λ_{18}^*=0.67	144	s_1^{-*} = 2.33	s_1^{+*} = 139.67 s_2^{+*} = 1.33 s_4^{+*} = 0.67	λ_{17}^*=0.33 λ_{18}^*=0.67
25	164.45		s_1^{+*}=163.17 s_2^{+*}=1.28	λ_{16}^*=0.08 λ_{17}^*=0.25 λ_{18}^*=0.66	164.95		s_1^{+*}=163.17 s_2^{+*}=1.28 s_4^{+*}=0.51	λ_{16}^*=0.08 λ_{17}^*=0.25 λ_{18}^*=0.66
26	44.67	s_2^{-*}=8.67	s_1^{+*}=32.33 s_2^{+*}=3.67	λ_{17}^*=0.67 λ_{18}^*=0.33	47	s_2^{-*}=8.67	s_1^{+*}=32.33 s_2^{+*}=3.67 s_3^{+*}=2.33	λ_{17}^*=0.67 λ_{18}^*=0.33
27	5.07		s_1^{+*} = 1.11 s_2^{+*}=3.96	λ_{16}^*=0.22 λ_{17}^*=0.66 λ_{18}^*=0.12	5.40		s_1^{+*} = 1.11 s_2^{+*}=3.96 s_4^{+*}=0.33	λ_{16}^*=0.22 λ_{17}^*=0.66 λ_{18}^*=0.12
28	0	None		λ_{28}^*=1	0	None		λ_{28}^*=1
29	157	s_2^{-*}=101.67	s_1^{+*}=50 s_2^{+*}=5.33	λ_{17}^*=0.67 λ_{18}^*=0.33	162.07		s_1^{+*}=156.43 s_2^{+*}=5.19 s_4^{+*}=0.45	λ_{16}^*=0.02 λ_{17}^*=0.72 λ_{18}^*=0.26
30	151	s_2^{-*}=101	s_1^{+*}=43 s_2^{+*}=7	λ_{17}^*=1	152	s_2^{-*}=101	s_1^{+*}=43 s_2^{+*}=7 s_4^{+*}=1	λ_{17}^*=1
31	0	None		λ_{31}^*=1	0	None		λ_{31}^*=1
32	0	None		λ_{32}^*=1	0	None		λ_{32}^*=1
33	0	None		λ_{33}^*=1	0	None		λ_{33}^*=1
34	9.60		s_1^{+*}=9.03 s_2^{+*}=0.57	λ_{17}=0.26 λ_{18}=0.17 λ_{28}=0.21 λ_{36}= 0.35	16.58		s_1^{+*}=15.17 s_2^{+*}=1.41	λ_{14}=0.20 λ_{16}=0.29 λ_{17}=0.37 λ_{28}=0.12 λ_{36}=0.03
35	0	None		λ_{35}^*=1	0	None		λ_{35}^*=1
36	0	None		λ_{36}^*=1	0	None		λ_{36}^*=1
37	0	None		λ_{37}^*=1	0	None		λ_{37}^*=1
38	0	None		λ_{38}^*=1	0	None		λ_{38}^*=1
39	669.99	s_2^{-*}=616.95 s_3^{-*}=50.83	s_2^{+*}=2.21	λ_{17}^*=0.49 λ_{27}^*=0.33 λ_{38}^*=0.18	692.70	s_2^{-*}=631 s_3^{-*}=61	s_2^{+*}=0.32 s_3^{+*}=0.15 s_4^{+*}=0.23	λ_{17}^*=0.62 λ_{18}^*=0.38
40	2.67	s_3^{-*}=2	s_1^{+*}=0.67	λ_7^*=0.33 λ_{28}^*=0.67	2.67	s_3^{-*}=2	s_1^{+*}=0.67	λ_7^*=0.33 λ_{28}^*=0.67
41	70	s_1^{-*}=52	s_1^{+*}=15 s_2^{+*}=3	λ_{31}^*=1	72	s_1^{-*}=52	s_1^{+*}=15 s_2^{+*}=3 s_4^{+*}=2	λ_{31}^*=1
42	100.91	s_1^{-*}=98.06	s_2^{+*}=2.85	λ_{20}^*=0.05 λ_{31}^*=0.95	102.81	s_1^{-*}=98.06	s_2^{+*}=2.85 s_4^{+*}=1.90	λ_{20}^*=0.05 λ_{31}^*=0.95

[a] LINDO solver used for these computations.

4. Conclusion

In this paper an alternative approach in evaluating the technical inefficiencies of decision making units under imprecise data setting is developed. The translation invariant property of an additive DEA model is used for converting the obtained non-linear models into equivalent linear ones. Then an envelopment form model is introduced for inefficiency evaluation of DMUs with imprecise data in one stage, under the normal size of constraints and variables. Also, based on the optimal solutions of this model, one can find some information about the inefficiency resources, peer set, slacks variables and the other important aspects of efficiency measurement. The proposed approach is classification invariant with the other methods in this case, can be easily implemented and can lighten in the application the computational burden of the previous methods.

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References

[1] R.J. Banker and R.C. Morey: The use of categorical variables in data envelopment analysis. *Management Science*, **32** (1986), 1613–1627.

[2] A. Charnes, W.W. Cooper, and E. Rhodes: Measuring the efficiency of decision making unit. *European Journal of Operational Research*, **2** (1978), 429–444.

[3] W.W. Cooper, K.S. Park, and J.T. Pastor: RAM: a range adjusted measure of efficiency for use with additive models, and relations to other models and measures in DEA. *Journal of Productivity Analysis*, **11** (1999), 5–42.

[4] W.W. Cooper, K.S. Park, and G. Yu: IDEA and AR-IDEA: models for dealing with imprecise data in DEA. *Management Science*, **45** (1999), 597–607.

[5] W.W. Cooper, K.S. Park, and G. Yu: IDEA (imprecise data envelopment analysis) with CMDs (column maximum decision making units). *Journal of Operational Research Society*, **52** (2001), 176–181.

[6] W.W. Cooper, K.S. Park, and G. Yu: An illustrative application of IDEA to a Korean mobile telecommunication company. *Operations Research*, **46** (2001), 807–820.

[7] D.K. Despotis and Y.G. Smirlis: Data envelopment analysis with imprecise data. *European Journal of Operational Research*, **140** (2002), 24–36.

[8] T. Entani, Y. Maeda, and H. Tanaka: Dual models of interval DEA and its extension to interval data. *European Journal of Operational Research*, **166** (2002), 32–45.

[9] G.R. Jahanshahloo, R. Kazemi Matin, and A. Hadi Vencheh: On return to scale of fully efficient DMUs in data envelopment analysis under interval data. *Applied Mathematics and Computation*, **154** (2004), 31–40.

[10] G.R. Jahanshahloo, R. Kazemi Matin, and A. Hadi Vencheh: On FDH efficiency analysis with interval data. *Applied Mathematics and Computation*, **159** (2004), 47–55.

[11] W.A. Kamakura: A note on the use of categorical variables in data envelopment analysis. *Management Science*, **34** (1988), 1273–1276.

[12] S.H. Kim, C.G. Park, and K.S. Park: An application of data envelopment analysis in telephone offices evaluation with partial data. *Computers & Operations Research*, **26** (1999), 59–72.

[13] C.A. Knox Lovell and J.T. Pastor: Units invariant and translation invariant DEA models. *Operations Research Letters*, **18** (1995), 147–151.

[14] Y.K. Lee, K.S. Park, and S.H. Kim: Identification of inefficiencies in additive model based IDEA. *Computers & Operations Research*, **29** (2002), 1661–1676.

[15] J.J. Rousseau and J.H. Semple: Categorical outputs in data envelopment analysis. *Management Science*, **39** (1993), 384–386.

[16] J. Zhu: Imprecise data envelopment analysis (IDEA): a review and improvement with an application. *European Journal of Operational Research*, **144** (2003), 513–529.

[17] J. Zhu: Efficiency evaluation with strong ordinal input and output measures. *European Journal of Operational Research*, **146** (2003), 477–485.

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