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THE PRICING OF OPTIONS WITH STOCHASTIC BOUNDARIES IN A GAUSSIAN ECONOMY

Masaaki Kijima
Tokyo Metropolitan University and
Daiwa Securities Group Chair, Kyoto University

Teruyoshi Suzuki
Hokkaido University

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Abstract This article considers the pricing of options with stochastic boundaries in a Gaussian economy. More specifically, prices of corporate discount bonds and knock-out exchange options are obtained in closed form. The key tools for doing this are the change of measure and the reflection principle of a driftless Gaussian process with a deterministic diffusion coefficient.

Keywords: Finance, reflection principle, arcsine law, change of measure, exchange option, knock-out option

1. Introduction

In this article, we consider the pricing of options with stochastic boundaries in a Gaussian economy. Knock-out and knock-in options with some security as a boundary price are such examples. More specifically, let $N(t)$ be the time t price of a risky security, and consider a knock-out option written on a security $S(t)$. Suppose that the option expires when $S(t)$ ever hits a fraction of the risky security, $\alpha N(t)$ say. Such options are a natural extension of the boundary options previously studied in the finance literature.

Options with deterministic boundaries have been extensively studied in the finance literature. Merton [15], Rubinstein and Reiner [17] and Kunitomo and Ikeda [13] are such examples. See also, e.g., Andersen and Brotherton-Ratcliffe [1], Beaglehole, Dybvig and Zhou [3] and Baldi, Caramellino and Iovino [2] for numerical techniques to evaluate boundary options. This article is a continuation of such studies and an extension to the case of stochastic boundaries.

Our model is not restricted to such boundary options only, but can also be applied to the pricing of corporate bonds in the structural approach. Namely, as in Black and Cox [4], let $A(t)$ be the time t total value of a firm's asset, and suppose that the firm issues two classes of claims, a single homogeneous class of debt with maturity T and the residual claim (equity). In the structural approach, the firm is assumed to default when the firm's asset value $A(t)$ hits the boundary $d(t)$ at the first time. Briys and de Varenne [5] claimed that it is more natural to assume $d(t) = \alpha v(t,T)$, $0 < \alpha \leq 1$, for valuing corporate bonds in a stochastic interest-rate economy than simply setting $d(t)$ to be a deterministic function of time t. Here, $v(t,T)$ denotes the default-free discount bond with maturity T. Note that, in this model, the firm's default is equivalent to a knock-out of the asset value $A(t)$ with stochastic boundary $d(t)$.

The key tool to obtain pricing formulas in closed form is the change of measure. Consider a frictionless market in which three assets are traded continuously, the money-market account $M(t)$ and two risky securities $S(t)$ and $N(t)$. We take one of the risky security, $N(t)$ say, as the numeraire. The change of measure is determined so that the relative price processes $\{M(t)/N(t); 0 \le t \le T\}$ and $\{S(t)/N(t); 0 \le t \le T\}$ are both martingales under the new probability measure Q^N. However, under some assumption on the payoff function, the pricing of contingent claims can be reduced to the evaluation of one-dimensional expectation. Closed-form solutions for the pricing formulas are then obtained in a Gaussian framework. For this purpose, the reflection principle of a driftless Gaussian process with a deterministic diffusion coefficient plays an important role. As a byproduct, an extension of the arcsine law for the standard Brownian motion is also derived.

This article is organized as follows. In the next section, we provide the information necessary for later developments. Namely, the formulas of the change of measure and the reflection principle of a driftless Gaussian process are stated explicitly. These results are illustrated using the extended Vasicek model. Based on the results, Section 3 obtains the prices of corporate discount bonds and knock-out exchange options in closed form. Section 4 concludes this article.

Throughout the article, we consider a continuous-time economy with a positive finite horizon T in which the three assets are traded continuously in the frictionless market. A European contingent claim with terminal payoff function $h(S, N)$ is considered, whose time t price is denoted by $C(t)$. It is assumed throughout that there exists a risk-neutral probability measure P, as given, in the economy and the claim can be priced by the risk-neutral pricing paradigm.

2. The Setup and Preliminaries

In this section, we provide the information necessary for the following development. The change of measure and the reflection principle of a driftless Gaussian process with a deterministic diffusion coefficient are the key tools for this study. These results are illustrated using the extended Vasicek model.

2.1. The change of measure

Let $\{(B_0(t), B_1(t)); 0 \le t \le T\}$ be a two-dimensional standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, P)$ and assume that the filtration $\mathcal{F} = \{\mathcal{F}_t; 0 \le t \le T\}$ is generated by the Brownian motion.

Let $r(t)$ be the instantaneous interest rate at time t, adapted to the filtration $\mathcal{F}$, and assume that it satisfies the regularity conditions. The money-market account $M(t)$ is defined by

$$\mathrm{d}M(t) = r(t)M(t)\mathrm{d}t, \quad 0 \le t \le T; \quad M(0) = 1.$$

On the other hand, the risky securities are assumed to follow the stochastic differential equations (SDE's for short)

$$\frac{\mathrm{d}N(t)}{N(t)} = r(t)\mathrm{d}t - \sigma_0(t)\mathrm{d}B_0(t), \quad 0 \le t \le T, \tag{2.1}$$

and

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \rho(t)\sigma_1(t)\mathrm{d}B_0(t) + \sqrt{1-\rho^2(t)}\sigma_1(t)\mathrm{d}B_1(t), \quad 0 \le t \le T, \tag{2.2}$$

respectively, under the risk-neutral probability measure P. Here, $\mathrm{d}B_0(t)\mathrm{d}B_1(t) = 0$ and $\sigma_i(t)$, $i = 0, 1$, and $\rho(t)$ are adapted to the filtration $\mathcal{F}$, and satisfy the regularity conditions.

The instantaneous covariance between the two assets is given by

$$\frac{\mathrm{d}N(t)}{N(t)}\frac{\mathrm{d}S(t)}{S(t)} = -\rho(t)\sigma_0(t)\sigma_1(t)\mathrm{d}t$$

and the volatility of the risky security $S(t)$ is equal to $\sigma_1(t)$. Note that the continuous-time economy we consider is *complete.*

Let $S^N(t) = S(t)/N(t)$ be the *relative price* of $S(t)$ to $N(t)$. By Ito's formula, the relative price process $\{S^N(t); 0 \le t \le T\}$ satisfies the SDE

$$\frac{\mathrm{d}S^N(t)}{S^N(t)} = \Big(\rho(t)\sigma_1(t) + \sigma_0(t)\Big)\Big(\sigma_0(t)\mathrm{d}t + \mathrm{d}B_0(t)\Big) + \sqrt{1-\rho^2(t)}\sigma_1(t)\mathrm{d}B_1(t).$$

Similarly, let $M^N(t) = M(t)/N(t)$. Then, we have

$$\frac{\mathrm{d}M^N(t)}{M^N(t)} = \sigma_0(t)\Big(\sigma_0(t)\mathrm{d}t + \mathrm{d}B_0(t)\Big).$$

Now, define

$$\mathrm{d}B_0^N(t) = \sigma_0(t)\mathrm{d}t + \mathrm{d}B_0(t), \qquad \mathrm{d}B_1^N(t) = \mathrm{d}B_1(t). \tag{2.3}$$

Accordingly, we obtain

$$\begin{aligned}&\Big(\rho(t)\sigma_1(t) + \sigma_0(t)\Big)\mathrm{d}B_0^N(t) + \sqrt{1-\rho^2(t)}\sigma_1(t)\mathrm{d}B_1^N(t)\\ &\quad = \Big(\rho(t)\sigma_1(t) + \sigma_0(t)\Big)\Big(\sigma_0(t)\mathrm{d}t + \mathrm{d}B_0(t)\Big) + \sqrt{1-\rho^2(t)}\sigma_1(t)\mathrm{d}B_1(t).\end{aligned}$$

Let Q^N be a new probability measure, equivalent to the risk-neutral measure P, under which the process $\{(B_0^N(t), B_1^N(t));\ 0 \le t \le T\}$ is a two-dimensional standard Brownian motion. The existence of such a measure is guaranteed by the Girsanov theorem (see, e.g., Karatzas and Shreve [9]). It follows that

$$\frac{\mathrm{d}M^N(t)}{M^N(t)} = \sigma_0(t)\mathrm{d}B_0^N(t) \tag{2.4}$$

and

$$\frac{\mathrm{d}S^N(t)}{S^N(t)} = \Big(\rho(t)\sigma_1(t) + \sigma_0(t)\Big)\mathrm{d}B_0^N(t) + \sqrt{1-\rho^2(t)}\sigma_1(t)\mathrm{d}B_1^N(t), \tag{2.5}$$

whence the relative price processes $\{M^N(t); 0 \le t \le T\}$ and $\{S^N(t); 0 \le t \le T\}$ are both martingales under Q^N, provided that the processes are square integrable.

Since the continuous-time economy is complete, any contingent claim can be replicated by the three assets, $S(t)$, $N(t)$ and $M(t)$. That is, let $C(t)$ be the time t price of a contingent claim. Then, there exists a portfolio process $\{(\alpha(t), \beta(t), \gamma(t)); 0 \le t \le T\}$, adapted to the filtration $\mathcal{F}$, such that

$$C(t) = \alpha(t)S(t) + \beta(t)N(t) + \gamma(t)M(t), \quad 0 \le t \le T.$$

Under the self-financing assumption, we have

$$\mathrm{d}C(t) = \alpha(t)\mathrm{d}S(t) + \beta(t)\mathrm{d}N(t) + \gamma(t)\mathrm{d}M(t), \quad 0 \le t \le T.$$

Taking the relative price, we then obtain

$$C^N(t) = C^N(0) + \int_0^t \alpha(u)\mathrm{d}S^N(u) + \int_0^t \gamma(u)\mathrm{d}M^N(u), \quad 0 \le t \le T,$$

where $C^N(t) = C(t)/N(t)$. Since $\{M^N(t); 0 \le t \le T\}$ as well as $\{S^N(t); 0 \le t \le T\}$ is a martingale under Q^N, it follows that

$$C(0) = N(0)E^N\left[\frac{C(t)}{N(t)}\right], \quad 0 \le t \le T,$$

where E^N denotes the expectation operator under Q^N.

At this point, we consider the following two cases. First, suppose that the payoff function of the claim is given by $h(S, N, M)$. Then, the price of the contingent claim is calculated as

$$C(0) = N(0)E^N\left[\frac{h(S(T), N(T), M(T))}{N(T)}\right].$$

In particular, when the payoff function at the maturity is given by

$$\frac{h(S(T), N(T), M(T))}{N(T)} = g(S^N(T), M^N(T))$$

for some function $g(x, y)$, we obtain

$$C(0) = N(0)E^N\left[g(S^N(T), M^N(T))\right].$$

Hence, we need to consider the bivariate process $\{(S^N(t), M^N(t)); 0 \le t \le T\}$ defined by (2.5) and (2.4), respectively, under Q^N. See Geman, El Karoui and Rochet [6] for details of the change of measure.

Next, if the claim matures at time T with payoff function $h(S, N)$, the price of the contingent claim is given by

$$C(0) = N(0)E^N\left[\frac{h(S(T), N(T))}{N(T)}\right].$$

When in particular the payoff function at the maturity is such that

$$\frac{h(S(T), N(T))}{N(T)} = f(S^N(t)) \tag{2.6}$$

for some function $f(x)$, we have

$$C(0) = N(0)E^N\left[f(S^N(T))\right]. \tag{2.7}$$

In this case, we only need the SDE (2.5) to calculate the expectation (2.7).

In the latter case (2.6), let $\sigma(t)$ be defined by

$$\begin{aligned} \sigma^2(t) &= \Big(\rho(t)\sigma_1(t) + \sigma_0(t)\Big)^2 + \Big(1 - \rho^2(t)\Big)\sigma_1^2(t) \\ &= \sigma_0^2(t) + 2\rho(t)\sigma_0(t)\sigma_1(t) + \sigma_1^2(t), \end{aligned} \tag{2.8}$$

and let $\{B^N(t); 0 \le t \le T\}$ denote another standard (one-dimensional) Brownian motion under Q^N. Then, from (2.5), the relative price process $\{S^N(t); 0 \le t \le T\}$ is equal in law to the solution to the SDE

$$\frac{\mathrm{d}S^N(t)}{S^N(t)} = \sigma(t)\mathrm{d}B^N(t), \quad 0 \le t \le T. \tag{2.9}$$

In order to evaluate the expectation (2.7), it is enough to assume that there exists a unique solution to the SDE (2.9) and use the solution as the relative price $\{S^N(t); 0 \le t \le T\}$.

Remark 1 In the latter case, define $\mathrm{d}B_0^*(t) = \mathrm{d}B_0(t)$ and

$$\sqrt{1-\rho^2(t)}\sigma_1(t)dB_1^*(t) = \mu^*(t)dt + \sqrt{1-\rho^2(t)}\sigma_1(t)dB_1(t)$$

instead of (2.3), where $\mu^*(t)$ is defined appropriately. Let Q^* be a probability measure, equivalent to the risk-neutral measure P, under which the process $\{(B_0^*(t), B_1^*(t));\ 0 \le t \le T\}$ is a two-dimensional standard Brownian motion. It is readily seen that (cf. (2.5))

$$\frac{\mathrm{d}S^N(t)}{S^N(t)} = \Big(\rho(t)\sigma_1(t) + \sigma_0(t)\Big)\mathrm{d}B_0^*(t) + \sqrt{1-\rho^2(t)}\sigma_1(t)\mathrm{d}B_1^*(t),$$

whence the relative price process $\{S^N(t); 0 \le t \le T\}$ is a martingale under Q^*, provided that the process is square integrable. Using the volatility process defined by (2.8), the relative price process $\{S^N(t); 0 \le t \le T\}$ can be constructed according to (2.9). Hence, under the measure Q^*, although the process $\{M^N(t); 0 \le t \le T\}$ is not a martingale, the contingent claim can be evaluated by (2.7). In fact, any measure that makes the relative price process $\{S^N(t); 0 \le t \le T\}$ a martingale plays the same role in this situation. See Kijima and Muromachi [12] for related results.

2.2. The reflection principle

Let us consider a process defined by

$$X(t) = \int_0^t \sigma(s)\mathrm{d}B(s), \quad t \ge 0, \tag{2.10}$$

and assume that $\sigma(t)$ is a deterministic function of time t. Define $M_X(t) = \max_{0 \le s \le t} X(s)$. Since $\sigma(t)$ is deterministic, a slight extension of the ordinary reflection principle (see, e.g., Karlin and Taylor [10]) can be applied to obtain

$$\begin{aligned} P(X(t) \le x) &= P(X(t) \le x,\ M_X(t) \le y) + P(X(t) \le x,\ M_X(t) \ge y) \\ &= P(X(t) \le x,\ M_X(t) \le y) + P(X(t) \le 2y - x), \quad x \le y,\ y \ge 0. \end{aligned}$$

Denoting

$$f(x,y) = \frac{\partial}{\partial x} P(X(t) \le x,\ M_X(t) \le y),$$

it follows that

$$f(x,y) = \frac{1}{\Sigma(t)}\left\{\phi\left(\frac{x}{\Sigma(t)}\right) - \phi\left(\frac{x-2y}{\Sigma(t)}\right)\right\}, \quad x \le y,\ y \ge 0, \tag{2.11}$$

where

$$\Sigma(t) = \sqrt{\int_0^t \sigma^2(s)\mathrm{d}s}, \tag{2.12}$$

and where $\phi(z) = \exp(-z^2/2)/\sqrt{2\pi}$ is the standard normal density function.

Next, consider a process $\{Y(t); t \geq 0\}$ defined by

$$Y(t) = \mu \int_0^t \sigma^2(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B(s) \tag{2.13}$$

for some constant μ. Let

$$G(x, y; \mu) = P(Y(t) \leq x,\ M_Y(t) \leq y), \quad x \leq y,\ y \geq 0,$$

where $M_Y(t) = \max_{0 \leq s \leq t} Y(s)$, and denote

$$g(x, y; \mu) = \frac{\partial}{\partial x} G(x, y; \mu).$$

It is readily seen that[1]

$$P(Y(t) \geq x,\ m_Y(t) \geq y; \mu) = G(-x, -y; -\mu), \quad x \geq y,\ y \leq 0, \tag{2.14}$$

for $m_Y(t) = \min_{0 \leq s \leq t} Y(s)$.

Proposition 1 *Suppose that the diffusion parameter $\sigma(t)$ is a deterministic function of time t and that μ is a constant. Then, for the process $\{Y(t); t \geq 0\}$ defined by (2.13), we have*

$$G(x, y; \mu) = \Phi\left(\frac{x - \mu\Sigma^2(t)}{\Sigma(t)}\right) - \mathrm{e}^{2\mu y}\Phi\left(\frac{x - 2y - \mu\Sigma^2(t)}{\Sigma(t)}\right),$$

where $\Phi(\cdot)$ is the standard normal distribution function. The density function is given by

$$g(x, y; \mu) = \frac{1}{\Sigma(t)}\left\{\phi\left(\frac{x - \mu\Sigma^2(t)}{\Sigma(t)}\right) - \mathrm{e}^{2\mu y}\phi\left(\frac{x - 2y - \mu\Sigma^2(t)}{\Sigma(t)}\right)\right\}.$$

Proof: See Appendix.

As an application of Proposition 1, we state an extension of the *arcsine law* (see, e.g., Protter [16]) for the standard Brownian motion. For this purpose, consider the Gaussian process $X(t)$ defined by (2.10), and let

$$g_t = \sup\{s \leq t : X(s) = 0\}.$$

The random quantity g_t represents the *last exit time* of $X(t)$ from state 0 before time t.

Corollary 1 *For the Gaussian process $X(t)$ defined by (2.10), we have*

$$P(g_t > s) = 1 - \frac{2}{\pi}\arcsin\sqrt{\frac{\int_0^s \sigma^2(u)\mathrm{d}u}{\int_0^t \sigma^2(u)\mathrm{d}u}}.$$

Proof: See Appendix.

[1]It is noted that the third argument of $P(\cdot, \cdot, ; \mu)$ represents the drift parameter of the process $\{Y(t); t \geq 0\}$. Throughout the paper, we use this convention.

2.3. An illustrative example

We illustrate the results using the Hull–White extended Vasicek model [7]. Suppose that the spot rate process $r(t)$ follows the SDE

$$\mathrm{d}r(t) = \left\{\kappa(t)[f(0,t) - r(t)] + \frac{\partial}{\partial t} f(0,t) + \phi(t)\right\} \mathrm{d}t + \gamma \mathrm{d}B_0(t),$$

where

$$\phi(t) = \gamma^2 \int_0^t \exp\left\{-2\int_s^t \kappa(u)\mathrm{d}u\right\}\mathrm{d}s.$$

The significance of the extended Vasicek model is that the model is consistent with the current term structure of interest rates (see, e.g., Inui and Kijima [8]). Moreover, the time t price of the discount bond maturing at time T, denoted by $v(t,T)$, follows the SDE (2.1). That is, under the risk-neutral measure P, we have

$$\frac{\mathrm{d}v(t,T)}{v(t,T)} = r(t)\mathrm{d}t - \gamma b_T(t)\mathrm{d}B_0(t), \tag{2.15}$$

where

$$b_T(t) = \int_t^T \exp\left\{-\int_t^s \kappa(u)\mathrm{d}u\right\}\mathrm{d}s.$$

Hence, we take $N(t) = v(t,T)$, $t \le T$, with $\sigma_0(t) = \gamma b_T(t)$ in (2.1). Note that the volatility $\sigma_0(t)$ is a deterministic function of time t. In this section, we assume that the stock price $S(t)$ follows the SDE (2.2) with $\sigma_1(t)$ and $\rho(t)$ being both deterministic functions of time t.

In this setting, the relative price defined by

$$S^N(t) = \frac{S(t)}{v(t,T)}, \quad 0 \le t \le T,$$

is in fact the *forward price* of stock $S(t)$. The probability measure Q^N defined through (2.3) is called the *forward measure*. Note that the relative price processes $\{M^N(t); 0 \le t \le T\}$ and $\{S^N(t); 0 \le t \le T\}$ are both martingales under the forward measure Q^N. Also, the relative price processes are Gaussian since all the coefficients in (2.4) and (2.5) are deterministic functions of time t. It is important to notice that the current forward-rate curve $f(0,t)$ does not appear in the SDE (2.15). Hence, it has no impact on the price of any contingent claim in this setting.

Consider a European call option written on $S(t)$ with exercise price K and maturity T. The payoff function is given by $h(S,N) = \max\{S-K,\, 0\}$, whence

$$\frac{h(S(T),N(T))}{N(T)} = \max\{S^N(T) - K,\, 0\},$$

since $N(T) = v(T,T) = 1$. It follows from (2.7) that the option price is evaluated as

$$C(0) = v(0,T)E^N\left[\max\{S^N(T) - K,\, 0\}\right]. \tag{2.16}$$

The forward price $S^N(t)$ follows the SDE (2.9) with the volatility $\sigma(t)$ defined by (2.8). Note that the volatility $\sigma(t)$ is a deterministic function of time t. Hence, the forward price process $\{S^N(t); 0 \le t \le T\}$ is not only a martingale but also a Gaussian process. It follows that the expectation term in (2.16) is given by the Black–Scholes formula, $\mathrm{BS}(S,K,T;\sigma,r)$ say, with volatility σ being replaced by $\sqrt{\gamma^2 b_T^2(t) + 2\gamma\rho(t)b_T(t)\sigma_1(t) + \sigma_1^2(t)}$ and interest rate r being 0. See, e.g., Kijima [11] for details.

For the forward price process $\{S^N(t); 0 \le t \le T\}$, define $Y(t) = \log S^N(t)$. Then, we have

$$Y(t) = Y(0) - \frac{1}{2}\int_0^t \sigma^2(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B^N(s).$$

Hence, taking $\mu = -1/2$, we can obtain the distributions of $M_Y(t)$ and $m_Y(t)$ from Proposition 1. We shall use these results in the next section.

Remark 2 When evaluating bond options written on $v(t,\tau)$ with maturity $T < \tau$, we will take $S(t) = v(t,\tau)$. Since the bond price $v(t,\tau)$ follows the SDE (2.15) with T being replaced by τ, the coefficients in the SDE (2.2) are given by $\sigma_1(t) = -\gamma b_\tau(t)$ and $\rho(t) = 1$. Hence, in this case, we have a single source, i.e. $B_0(t)$, for uncertainty and the measure Q^* that makes the relative price $S^N(t) = v(t,\tau)/v(t,T)$ a martingale is *unique*, in contrast to the case stated in Remark 1. In other words, the measure Q^* is necessarily the forward measure, so that not only the forward bond price $v(t,\tau)/v(t,T)$ but also the relative price $M(t)/v(t,T)$ must be martingales.

3. The Pricing of Stochastic Boundary Options

This section applies the results obtained in the previous section to derive the no-arbitrage prices of some stochastic boundary options. Throughout this section, we assume that the volatility $\sigma(t)$ defined in (2.8) is deterministic in order to derive closed-form solutions. Otherwise, we need to employ the Monte Carlo simulation using the SDE's (2.4) and (2.5) under Q^N to evaluate the expectation in (2.7).

3.1. Knock-out exchange options

We first consider an exotic option, called a *knock-out exchange option* [2], with underlying risky securities $N(t)$ and $S(t)$. The knock-out time of the knock-out exchange option is defined by

$$\tau = \inf\{t > 0 : S(t) = \alpha N(t)\}, \quad 0 < \alpha \le 1, \tag{3.1}$$

where $S(0) > \alpha N(0)$. That is, the option is knocked out (down and out) when the value of the security $S(t)$ hits a fraction of the value of the security $N(t)$. Knock-in exchange options can be considered similarly.

Suppose that the payoff function of the option at maturity T is given by

$$h(S,N) = 1_{\{\tau>T\}} \max\{S - N,\, 0\},$$

see (2.6). That is, the option holder has the right to exchange the security $S(t)$ by the security $N(t)$ at the maturity T, if the option has not been knocked out until the maturity date T.

From (2.7), the value of the knock-out exchange option is given by

$$C(0) = N(0)E^N\left[1_{\{\tau>T\}} \max\left\{S^N(T) - 1,\, 0\right\}\right], \tag{3.2}$$

where the process $\{S^N(t); 0 \le t \le T\}$ follows the SDE (2.9) under the measure Q^N. It should be noted that the knock-out epoch (3.1) can be rewritten as

$$\tau = \inf\{t : S^N(t) = \alpha\}, \quad 0 < \alpha \le 1, \tag{3.3}$$

[2]Margrabe [14] shows a closed form solution for a plain exchange option. In fact when $\alpha \to 0$ our result coincides with his result.

where $S^N(0) > 1$. Hence, the knock-out epoch τ is determined by the constant boundary under the probability measure Q^N. Also,

$$\{\tau > T\} = \left\{\min_{0\le s\le t} S^N(s) > \alpha\right\} = \left\{\min_{0\le s\le t} \log S^N(s) > \log\alpha\right\}.$$

Hence, the probability density function of $S^N(t)$ conditional on the event $\{\tau > T\}$ is calculated using Proposition 1.

Proposition 2 *The value of the knock-out exchange option maturing at time T with knock-out level α is given by*

$$\begin{aligned} C(0) &= S(0)\left\{\Phi\left(\frac{\log \ell_0 + \frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right) - \frac{1}{q_0}\Phi\left(\frac{\log \ell_0/q_0^2 + \frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)\right\} \\ &\quad - N(0)\left\{\Phi\left(\frac{\log \ell_0 - \frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right) - q_0\Phi\left(\frac{\log \ell_0/q_0^2 - \frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)\right\}, \end{aligned}$$

where

$$\ell_0 = \frac{S(0)}{N(0)}, \quad q_0 = \frac{S(0)}{\alpha N(0)}$$

and where $\Sigma(t)$ is defined by (2.12).

Proof: From (3.2) and (3.3), we obtain

$$\begin{aligned} C(0) &= N(0)E^N\left[1_{\{\tau>T\}}\max\left\{S^N(T) - 1,\, 0\right\}\right] \\ &= N(0)E^N\left[1_{\{\tau>T,\, S^N(T)>1\}}S^N(T) - 1_{\{\tau>T,\, S^N(T)>1\}}\right] \\ &= N(0)E^N\left[1_{\{\tau>T,\, S^N(T)>1\}}S^N(T)\right] - N(0)P^N\left(\tau > T,\, S^N(T) > 1\right). \end{aligned} \tag{3.4}$$

First, from Proposition 1 and Equation (2.14), the second term in the above equation is calculated as

$$\begin{aligned} P^N\left(\tau > T, S^N(T) > 1\right) &= P^N\left(\min_{0\le s\le T} S^N(s) > \alpha,\, S^N(T) > 1\right) \\ &= P^N\left(\min_{0\le s\le T} \log S^N(s) > \log\alpha,\, \log S^N(T) > 0\right) \\ &= P^N\left(m_Y(T) > -\log q_0,\, Y(T) > -\log \ell_0;\, \mu = -\frac{1}{2}\right) \\ &= G\left(\log \ell_0, \log q_0, \frac{1}{2}\right). \end{aligned} \tag{3.5}$$

Second, in order to calculate the first term in (3.4), we set $\mu = -1/2$ in (2.13). It is well known that the process $\{S^N(t); 0 \le t \le T\}$ defined by $S^N(t)/S^N(0) = \exp\{Y(t)\}$ is an exponential martingale and

$$\tilde{P}(A) = E\left[\frac{S^N(T)}{S^N(0)}1_A\right], \quad A \in \mathcal{F},$$

defines a new probability measure. By the Girsanov theorem, the process $\{\tilde{B}(t); 0 \le t \le T\}$ defined by

$$\tilde{B}(t) = B(t) - \int_0^t \sigma(s)\mathrm{d}s$$

is a standard Brownian motion under $\tilde{P}$. Also,

$$Y(t) = \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)\mathrm{d}\tilde{B}(s),$$

which is the same as (2.13) with $\mu = 1/2$. Therefore,

$$E^N\left[1_{\{\tau>T, S^N(T)>1\}}\frac{S^N(T)}{S^N(0)}; \mu = -\frac{1}{2}\right] = \tilde{P}\left(\tau > T, S^N(T) > 1; \mu = \frac{1}{2}\right).$$

By a similar calculation leading to (3.5), we obtain

$$\tilde{P}\left(\tau > T, S^N(T) > 1; \mu = \frac{1}{2}\right) = G\left(\log \ell_0, \log q_0, -\frac{1}{2}\right). \tag{3.6}$$

The proposition is now proved by combining Equations (3.4), (3.5) and (3.6).

3.2. Corporate discount bonds

We next consider the structural model of Briys and de Varenne [5] to obtain a risky discount bond price. That is, let $S(t)$ denote the time t value of a corporate firm's total asset and let $N(t)$ be the time t price of the default-free discount bond with maturity T.

Suppose that the firm issues two classes of claims, a single homogeneous class of debt with face value F and the residual claim (equity). Suppose also that the firm defaults at the first hitting time defined by

$$\tau = \inf\{t : S(t) = \alpha F N(t)\}, \quad 0 < \alpha \le 1,$$

if any, where $S(0) > F$. Default occurs when the firm cannot prepare a proportion of the discounted, promised value that the firm repays at the maturity of the debt.

When default occurs prior to the debt maturity T, debt holders receive a fraction of the firm's asset value, $f_1 \alpha F N(\tau)$ say, at the default time epoch τ. If, on the other hand, no default occurs before the maturity, they receive the face value F at time T in the case that the firm's asset value $S(T)$ is greater than F, while they will receive a fraction of the firm's asset value, $f_2 S(T)$ say, at time T in the case that $S(T)$ is not greater than F.

Summarizing, the payoff function of the debt holders at the maturity is given by

$$D(T) = 1_{\{\tau<T\}} f_1 \alpha F + 1_{\{\tau>T, S(T)>F\}} F + 1_{\{\tau>T, S(T)<F\}} f_2 S(T). \tag{3.7}$$

Here, note that $N(T) = 1$ since $N(t) = v(t, T)$, the default-free discount bond with maturity T, and receiving $f_1 \alpha F N(\tau)$ at the default time epoch τ is equivalent to receiving $f_1 \alpha F$ at the maturity T.

Taking the relative prices in (3.7) with respect to $N(t)$, it follows that

$$D^N(T) = 1_{\{\tau<T\}} f_1 \alpha F + 1_{\{\tau>T, S^N(T)>F\}} F + 1_{\{\tau>T, S^N(T)<F\}} f_2 S^N(T).$$

Also, the default time epoch τ is defined by

$$\tau = \inf\{t : S^N(t) = \alpha F\}, \quad 0 < \alpha \le 1,$$

where $S^N(0) > F$. Hence,

$$P(\tau > t) = P(m_Y(t) > \log(\alpha F)), \quad t > 0,$$

where $Y(t) = \log S^N(t)$ and $m_Y(t) = \min_{0 \le s \le t} Y(s)$ as before.

Using Proposition 1, we can recover the result obtained in Briys and de Varenne [5] as follows. The proof is similar to that of Proposition 2 and omitted.

Proposition 3 *The price of the corporate discount bond with maturity T is given by*

$$\begin{aligned} D(0) &= N(0)\alpha f_1 F\left\{1-\Phi\left(\frac{\log q_0-\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)+q_0\Phi\left(\frac{-\log q_0-\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)\right\} \\ &\quad + N(0)F\left\{\Phi\left(\frac{\log \ell_0-\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)-q_0\Phi\left(\frac{\log \ell_0/q_0^2-\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)\right\} \\ &\quad + f_2 S(0)\left\{\Phi\left(\frac{\log q_0+\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)-\frac{1}{q_0}\Phi\left(\frac{-\log q_0+\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)\right. \\ &\qquad \left. -\Phi\left(\frac{\log \ell_0+\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)+\frac{1}{q_0}\Phi\left(\frac{\log \ell_0/q_0^2+\frac{1}{2}\Sigma^2(T)}{\Sigma(T)}\right)\right\}, \end{aligned}$$

where

$$q_0=\frac{S(0)}{\alpha F N(0)}, \quad \ell_0=\frac{S(0)}{FN(0)},$$

and where $\Sigma(t)$ is defined by (2.12).

4. Concluding Remarks

In this article, we derive the prices in closed form of corporate discount bonds and knock-out exchange options in a Gaussian economy. The key tool for doing this is the change of measure formula. Since the market we consider consists of the money market $M(t)$ and two risky securities $S(t)$ and $N(t)$, we take one of the risky security, $N(t)$ say, as the numeraire. The change of measure is determined so that the relative price processes $\{M(t)/N(t); 0 \le t \le T\}$ and $\{S(t)/N(t); 0 \le t \le T\}$ are both martingales under the new probability measure Q^N. However, under the assumption (2.6) on the payoff function, the pricing of contingent claims can be reduced to the evaluation of one-dimensional expectation; see (2.7). The reflection principle of a driftless Gaussian process with a deterministic diffusion coefficient plays an important role for the purpose. As a byproduct, an extension of the arcsine law for the standard Brownian motion is also derived.

As a future work, consider a contingent claim that can be replicated by $N(t)$ and $S(t)$ only. That is, the time t price of the claim is given by

$$C(t)=\alpha(t)S(t)+\beta(t)N(t), \quad 0 \le t \le T,$$

without using the money-market account $M(t)$. Under the self-financing assumption, we obtain

$$C^N(t)=C^N(0)+\int_0^t \alpha(u)\mathrm{d}S^N(u), \quad 0 \le t \le T.$$

Let Q^* be any measure, equivalent to the risk-neutral measure P, that makes the relative price process $\{S^N(t); 0 \le t \le T\}$ a martingale. It then follows that

$$C(0)=N(0)E^*\left[\frac{C(t)}{N(t)}\right], \quad 0 \le t \le T.$$

Note that, in this situation, the relative price $M^N(t)$ needs not be a martingale under Q^*, and the same conclusion holds as in Remark 1. The problem of interest is therefore to recognize what class of contingent claims can be replicated without the money-market account. This will be an important future work.

Appendix

Proof of Proposition 1. From (2.13), we have

$$\mathrm{d}Y(t) = \mu\sigma^2(t)\mathrm{d}t + \sigma(t)\mathrm{d}B(t), \quad Y(0) = 0,$$

where $\{B(t); 0 \le t \le T\}$ is a standard Brownian motion. Define the process $\{\tilde{B}(t); 0 \le t \le T\}$ by

$$\tilde{B}(t) = B(t) - \mu \int_0^t \sigma(s)\mathrm{d}s,$$

and consider a probability measure $\tilde{P}$ under which the process $\{\tilde{B}(t); 0 \le t \le T\}$ is a standard Brownian motion. It follows that

$$\mathrm{d}Y(t) = \sigma(t)\mathrm{d}\tilde{B}(t), \quad 0 \le t \le T.$$

Hence, from the Girsanov theorem, we obtain

$$\begin{aligned} G(x, y; \mu) &= \tilde{P}\left(Y(t) < x,\, M_Y(t) < y\right) \\ &= E\left[\exp\left\{-\frac{1}{2}\mu^2 \int_0^t \sigma^2(s)\mathrm{d}s + \mu \int_0^t \sigma(s)\mathrm{d}B(s)\right\} 1_{\{Y(t)<x, M_Y<y\}}\right] \\ &= \int_{-\infty}^x \mathrm{e}^{-\frac{1}{2}\mu^2\Sigma(t)^2 + \mu z} f(z, y)\mathrm{d}z, \end{aligned}$$

where $f(x, y)$ is defined by (2.11). The proposition is now proved at once.

Proof of Corollary 1. Let $X(0) = 0$, and assume $0 < s \le t$. Then, from the symmetry in law of $X(t)$, we obtain

$$P\left(\sup_{s<u\le t} X(u) > 0,\, X(s) < 0\right) = 2P(X(t) > 0,\, X(s) < 0).$$

It follows that, for $s < t$,

$$\begin{aligned} P(g_t > s) &= P(g_t > s,\, X(s) < 0) + P(g_t > s,\, X(s) > 0) \\ &= 2P(g_t > s,\, X(s) < 0) \\ &= 2P(\sup_{s<u\le t} X(u) > 0,\, X(s) < 0) \\ &= 4P(X(t) > 0,\, X(s) < 0). \end{aligned}$$

In order to calculate the distribution of g_t, let

$$\eta(s, t) = \sqrt{\int_s^t \sigma^2(u)\mathrm{d}u}.$$

Since $X(s)$ and $X(t)$ are jointly Gaussian, there exist standard normal random variables Z_1 and Z_2, independent of each other, such that

$$X(s) = v(0, s)Z_1, \quad X(t) = v(0, s)Z_1 + v(s, t)Z_2,$$

where equality stands for equality in law. It follows that $\{X(s) < 0\} = \{Z_1 < 0\}$ and

$$\{X(t) > 0\} = \left\{Z_2 > -\frac{\eta(0, s)}{\eta(t, s)} Z_1\right\}.$$

Now, in order to calculate the joint probability $P\left(Z_2 > -\frac{v(0,s)}{v(t,s)}Z_1,\ Z_1 < 0\right)$, we use the polar coordinates:

$$\begin{aligned} P\left(Z_2 > -\frac{\eta(0,s)}{\eta(t,s)}Z_1,\ Z_1 < 0\right) &= \int_{-\infty}^{0}\int_{-\frac{\eta(0,s)}{\eta(s,t)}}^{\infty}\frac{1}{2\pi}\exp\left\{-\frac{x^2+y^2}{2}\right\}\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{\pi/2}^{\pi-\arcsin\frac{\eta(0,t)}{\eta(0,s)}}\int_0^{\infty}\exp\left\{\frac{-r^2}{2}\right\}r\mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{4}-\frac{1}{2\pi}\arcsin\frac{\eta(0,s)}{\eta(0,t)}. \end{aligned}$$

It follows that

$$P(g_t > s) = 4P\left(Z_2 > -\frac{\eta(0,s)}{\eta(t,s)}Z_1,\ Z_1 < 0\right) = 1-\frac{2}{\pi}\arcsin\sqrt{\frac{\int_0^s\sigma^2(u)\mathrm{d}u}{\int_0^t\sigma^2(u)\mathrm{d}u}},$$

completing the proof.

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Teruyoshi Suzuki,
Graduate School of Economics,
Hokkaido University, Nishi 7, Kita 9,
Kita-ku, Sapporo 060-0809, Japan.
Email: suzuki@econ.hokudai.ac.jp