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A COMPULSORY SMUGGLING MODEL OF INSPECTION GAME TAKING ACCOUNT OF FULFILLMENT PROBABILITIES OF PLAYERS' AIMS

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Abstract This paper deals with an inspection game of the customs and a smuggler. The customs can take two options of assigning a patrol or not. The smuggler has two strategies of shipping its cargo of contraband or not. Two players have several opportunities to take an action during a limited number of days but only the smuggler cannot discard any opportunity intentionally. When the smuggling coincides with the patrol, there are three possibilities that the customs captures the smuggler, the smuggler makes a success of the smuggling or none of them happens. If the smuggler is captured or there remains no day for playing the game, the game ends. There have been some studies so far on the inspection game. Some consider the cases that the smuggler has only one smuggling or the perfect-capture case that the customs can certainly arrest the smuggler on patrol, and others think of a recursive game without the probabilities of fulfilling players' purposes. However there has been little study in which they discussed the stochastic inspection. In this paper, we formulate the problem into a multi-stage two-person zero-sum stochastic game and investigate some characteristics of its equilibrium solution, some of which are given in closed forms in special cases.

Keywords: Game theory, inspection game, two-person zero-sum, multi-stage

1. Introduction

In this paper, we deal with an inspection game. We can find out an original of the inspection game in Dresher [2] and Maschler [6]. Maschler generalized the Dresher's modeling and tried to apply the game to an inspection problem of the treaty of arms reduction. He considered the game where a player called a violator wished to violate the treaty in secret and the other player called an inspector wanted to commit him to effective inspection. In his model, the violator must pay one of penalties if the violation is exposed by an inspection but he can escape the exposure by side payment of penalty q. Dresher discussed special cases of $q = 1/2$ and $q = 1$, and Maschler did a general case of $0 \leq q \leq 1$. Thomas and Nisgav [8] extended the Maschler's model to the game of the customs and a smuggler, in which the customs kept a watch on illegal actions of the smuggler by using one or two patrol boats and the number of boats affected reward obtained on the capture of the smuggler. They formulated the problem into a multi-stage recursive game and adopted a relatively simple numerical method to repeat solving a one-stage matrix game step by step.

Baston and Bostock [1] first gave a closed form of equilibrium. Furthermore they modified the perfect-capture assumption that the inspectors or the customs certainly capture the violator or the smuggler when both players meet, which had been adopted so far, to the non-perfect capture model. They succeeded to solve the game by introducing the capture probability depending on the number of patrol boats. But the opportunity for the smuggler to ship its cargo of contraband is still assumed to be once at most, as the preceding papers did. Garnaev [4] extended their work to a model of three patrol boats.

Compared with these studies, Sakaguchi [7] introduced an assumption that the smuggler possibly takes an action several times in the perfect-capture model. He considered two versions of the model. In the first version, the smuggler is forced to do an illegal action as many times as preplanned. In the second one, he may skip some of preplanned number of actions. He showed that the first model was easier to solve and derived again the formula Baston and Bostock found for the value of the game, from other point of view. His model is a kind of the repeated game with several opportunities of smuggling and then the value of the whole game is given by multiplying the value of the once-opportunity game by the number of the opportunities. Furthermore his discussion is based on the assumption that an optimal solution is always given by not a saddle point but an equilibrium point of mixed strategies, which is an obscure point about his study. Ferguson and Melolidakis [3] extended the Sakaguchi model. They assumed that the smuggler could get rid of the capture by means of paying some cost $q(\leq 1)$. If he is captured on his smuggling, he must pay unit cost 1 but if not captured, he loses nothing. Then an interesting point is which is the best: he should take illegal action or he should pay cost q to escape the capture by which he certainly loses cost 1.

In the history of studies on the inspection game, our model of this paper is located in the same position as the Sakaguchi's in terms of the number of opportunities of the smugglings. Unlike the Sakaguchi's, however, we deal with a non-perfect capture model, where the encounter of players stochastically results in one of three cases: the capture of the smuggler, the success of smuggling or none of them. In this sense, we take account of the fulfillment probabilities of players' aims in the game. From a similar point of view, we [5] already developed an inspection model, in which the smuggler is free to execute the smuggling or not at all times. However, in this paper, the smuggler is assumed to be compelled to smuggle as many times as preplanned by his financial or administrative constraints or other reasons. Because of another assumption in this model that the capture of the smuggler terminates the game, he could not exhaust all opportunity of smuggling before the end of the game. The assumption seems natural but never has been taken in other studies so far. In the previous models of non-perfect capture, the meeting of players was assumed to result in the capture or the successful smuggling. In the previous models with several opportunities of smugglings, the game proceeded to the next stage even though the smuggler was captured. Our game is a stochastic game in the meaning that the encounter of two players stochastically leads the game to different states at the next stage or a termination in a case.

In the next section, we describe assumptions of our model and formulate the problem into a two-person zero-sum stochastic game considering the transition of states. In Section 3, we investigate some characteristics of the game and optimal solutions to prove that the game has an equilibrium point of mixed strategies. Especially in some simple cases such as once-smuggling or once-patrol, we obtain closed form of formulas for the solution. In Section 4, we verify some properties of the solution stated in its previous sections by some numerical examples. They include the sensitivity analysis for the number of stages, patrols and smugglings.

2. Description of Assumptions and Formulation

We consider the following multi-stage two-person zero-sum game played by the customs and a smuggler, say Player A and Player B, respectively.

A1. A time horizon consists of N days and each player can take an action of patrol or smuggling once a day. A stage of the game is represented by the number of residual days.
A2. On the whole time horizon, Player A can patrol K times at most and Player B is compelled to smuggle L times. If the number of times of the actions is in excess of residual days, $K > N$ or $L > N$, the excess opportunities for actions are discarded in vain.
A3. On each opportunity of taking an action, Player A chooses one between two options {patrol, no-patrol} and Player B between options {smuggling, no-smuggling}.
A4. If Player A patrols and Player B smuggles on the same day, Player A captures B with probability p_1 but Player B succeeds in smuggling with probability p_2 or none of them possibly happens with residual probability $1 - p_1 - p_2$, where $p_1 + p_2 \leq 1$. For no patrol, Player B succeeds to smuggle for a certainty.
A5. The capture of Player B brings reward $\alpha > 0$ to Player A but the success of the smuggling rewards 1 to Player B. It is assumed to be

$$\alpha p_1 - p_2 > 0. \tag{1}$$

In this game, the reward of Player A is the same amount of loss for Player B, and vice versa. We define a payoff of the game by the reward of Player A.
A6. Unless Player B is captured, the game transfers to the next stage. The game ends on the capture of Player B or the exhaustion of N days of the time horizon.

In Assumption A5, the number α is set relatively to reward 1 of Player B on a successful smuggling. The condition (1) gives Player A the incentive to dispatch his patrol boats to the smuggling area. The condition also indicates $p_1 > 0$.

Let us suppose that the game is now at Stage n, where k opportunities of patrols are given to Player A and l smugglings depend on Player B for execution. We denote the game by $\Gamma(n, k, l)$. In the case of $n > l$, we can decompose it into sequent games, depending on players' strategies, as follows.

$$\Gamma(n,k,l) \equiv \begin{array}{c} \\ P \\ NP \end{array} \begin{array}{c} \begin{array}{cc} \qquad\qquad S \qquad\qquad & \qquad NS \end{array} \\ \left(\begin{array}{cc} \alpha p_1 - p_2 + (1-p_1)\Gamma(n-1,k-1,l-1) & \Gamma(n-1,k-1,l) \\ -1 + \Gamma(n-1,k,l-1) & \Gamma(n-1,k,l) \end{array} \right) \end{array}.$$

Two rows and two columns indicate two strategies {patrol (P), no-patrol (NP)} of Player A and {smuggling (S), no-smuggling (NS)} of Player B, respectively. Each element of the above matrix is self-explanatory but we dare to explain the derivation of the element in the first-row and the first-column when the patrol and the smuggling are put into action. Player A can obtain the expected reward $\alpha p_1 - p_2$ at the stage n and the game could transfer to the next stage with probability $1 - p_1$. The element of the 2nd-row and the 1st-column indicates that Player B gets reward 1 and the game moves to the next, unconditionally. However we must note that, in the case of $n = l$, Player B is forced to smuggle from Assumption A2 and does not have the strategy of no-smuggling (NS). Therefore we have the following 2×1 matrix for the transition in the case of $n = l$.

$$\Gamma(n,k,n) \equiv \begin{array}{c} \\ P \\ NP \end{array} \begin{array}{c} S \\ \left(\begin{array}{c} \alpha p_1 - p_2 + (1-p_1)\Gamma(n-1,k-1,n-1) \\ -1 + \Gamma(n-1,k,n-1) \end{array} \right) \end{array}. \tag{2}$$

Now we can calculate the value of $\Gamma(n,k,l)$, denoted by $v(n,k,l)$, by a recursive formulation in the case of $n > l$.

$$v(n,k,l) = val\begin{pmatrix} \alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1) & v(n-1,k-1,l) \\ -1 + v(n-1,k,l-1) & v(n-1,k,l) \end{pmatrix}, \quad (3)$$

where symbol val means the value of its sequent matrix game. In the case of $l = n$, we have the following formulation from the expression (2).

$$\begin{aligned} v(n,k,n) = & \ val\begin{pmatrix} \alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,n-1) \\ -1 + v(n-1,k,n-1) \end{pmatrix} \\ = & \ \max\{\alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,n-1), \ -1 + v(n-1,k,n-1)\}, \quad (4) \end{aligned}$$

where $n \geq 1$ is assumed. Furthermore, we have initial conditions for $n = 0, \ k = 0$ and $l = 0$ such as

$$v(0,k,l) = 0, \ v(n,k,0) = 0, \ v(n,0,l) = -l. \quad (5)$$

Indices k and l change within the region of $k \leq n$ and $l \leq n$ but, in special cases of $l > n$ or $k > n$, we may use the following setting

$$v(n,k,l) = \begin{cases} v(n,n,l), & for \ \ k > n , \\ v(n,k,n), & for \ \ l > n , \end{cases} \quad (6)$$

from Assumption A2. Now we have obtained a computational estimation for the value of the game. It starts from the initial conditions (5) and repeats the estimations (3) or (4) to reach specified parameters $n, \ k, \ l$ at the end. Now we state the numerical method proposed above to solve our inspection game as a theorem.

Theorem 1 *We can numerically solve our stochastic inspection game $\Gamma(n,k,l)$ by two recursive equations (3) and (4) with initial conditions (5).*

3. Properties of Game and Solution

For the game $\Gamma(n,k,l)$, we denote a mixed strategy of Players A by $\boldsymbol{x} = (x_1, x_2)$ at Stage n. It means that Player A takes strategy P or NP with probabilities x_1 or x_2, respectively, at the first stage n. Similarly, a mixed strategy $\boldsymbol{y} = (y_1, y_2)$ of Player B means that probabilities of taking strategies S or NS are y_1 or y_2, respectively, at Stage n. It follows that $x_1 + x_2 = 1$ and $y_1 + y_2 = 1$ for these strategies.

The discussion about a simple case of $l = n$ leads us to the following corollary.

Corollary 1 *In the case of $l = n$, an optimal strategy of Player A is $x_1^* = 1$ except for $k = 0$ and Player B must take only the strategy of the smuggling (S), of course. The value of the game $v(n,k,n)$ is given by the following function $f(n,k)$.*

$$f(n,k) \equiv \frac{\alpha p_1 - p_2}{p_1}\left\{1 - (1-p_1)^k\right\} - (n-k)(1-p_1)^k \ . \quad (7)$$

Proof: In the case of $n = 1$, $v(1,0,1) = -1$ for $k = 0$ and $v(1,1,1) = \max\{\alpha p_1 - p_2, \ -1\} = \alpha p_1 - p_2$ for $k = 1$ from Eq. (4). We can see the validity of the equation (7) in these initial cases. Now we suppose that equations $v(n-1,k,n-1) = f(n-1,k), \ k = 0, \cdots, n-1$ hold for Stage $n-1$. Using Eq. (7), the first term of expression (4) can be transformed into

$$\begin{aligned} & \alpha p_1 - p_2 + (1-p_1)f(n-1,k-1) \\ = & \ \alpha p_1 - p_2 + (1-p_1)\left[\frac{\alpha p_1 - p_2}{p_1}\left\{1 - (1-p_1)^{k-1}\right\} - (n-k)(1-p_1)^{k-1}\right] \\ = & \ \frac{\alpha p_1 - p_2}{p_1}\left\{1 - (1-p_1)^k\right\} - (n-k)(1-p_1)^k = f(n,k). \end{aligned}$$

For the second term, it follows that

$$\begin{aligned}-1+f(n-1,k) &= -1+\frac{\alpha p_1-p_2}{p_1}\left\{1-(1-p_1)^k\right\}-(n-1-k)(1-p_1)^k \\ &= -1+\frac{\alpha p_1-p_2}{p_1}\left\{1-(1-p_1)^k\right\}-(n-k)(1-p_1)^k+(1-p_1)^k \\ &= -1+f(n,k)+(1-p_1)^k<f(n,k)\end{aligned}$$

in the case of $k<n$ and

$$\begin{aligned}-1+v(n-1,n-1,n-1) &= -1+\frac{\alpha p_1-p_2}{p_1}\left\{1-(1-p_1)^{n-1}\right\} \\ &\leq -1+\frac{\alpha p_1-p_2}{p_1}\left\{1-(1-p_1)^n\right\}=-1+f(n,n)<f(n,n)\end{aligned}$$

in the case of $k=n$. Therefore, $v(n,k,n)=f(n,k)$ from Eq. (4). □

Corollary 1 points out an effective strategy of patrolling in the situation that the smuggler would take an action every day. It says that an optimal strategy of Player A is k successive patrols without any rest from the current stage. Any delay of the patrol brings the unconditional reward 1 to the smuggler. By the strategy, the game goes forward without the capture of Player B and reach the τth stage from the current with probability $(1-p_1)^{\tau-1}$ and Player A expects reward αp_1-p_2 at the stage. Once Player A fails to capture Player B all through the sequence of k patrols, which happens with probability $(1-p_1)^k$, Player B unconditionally gains reward $n-k$ after then. Considering these estimations, we can see that the expected payoff of the game is given by the function $f(n,k)$ as follows.

$$v(n,k,n)=\sum_{\tau=1}^{k}(\alpha p_1-p_2)(1-p_1)^{\tau-1}-(n-k)(1-p_1)^k=f(n,k)\ .$$

For a simple situation of $k=n$, where the customs can patrol every day, we have some properties of the game $\Gamma(n,n,l)$.

Corollary 2 *For game $\Gamma(n,n,l)$, an optimal strategy of Player A is $x_1^*=1$ at Stage n and one of Player B is arbitrary. The following function $g(l)$ gives the value of the game $v(n,n,l)$, which is independent of n.*

$$g(l)\equiv\frac{\alpha p_1-p_2}{p_1}\left\{1-(1-p_1)^l\right\}\ . \tag{8}$$

Proof: We can see the corollary holds for $k=n=1$ because of $v(1,1,0)=0=g(0)$ and $v(1,1,1)=\alpha p_1-p_2=g(1)$ from Eq. (7). Now let us assume the validity of $v(n-1,n-1,l)=g(l)$ for $l\leq n-1$. Player A can afford to patrol every day and $x_1^*=1$ is optimal, which is verified as follows. Applying $k=n$ to Eq. (3), we have

$$v(n,n,l)=val\begin{pmatrix}a & c\\ b & d\end{pmatrix}\ , \tag{9}$$

where

$$\begin{aligned}a &= \alpha p_1-p_2+(1-p_1)v(n-1,n-1,l-1),\ b=-1+v(n-1,n-1,l-1), \quad (10)\\ c &= v(n-1,n-1,l),\ d=v(n-1,n-1,l)\ . \quad (11)\end{aligned}$$

Noting that $a = \alpha p_1 - p_2 + (1-p_1)g(l-1) = g(l) \geq g(l-1) > -1 + g(l-1) = b$ and $c = d = g(l) = a$ in the above matrix, we can see that an optimal strategy is $x_1^* = 1$ and the value of the game is $g(l)$. In an additional case of $l = n$, we can see the function (8) gives the value of the game from Corollary 1 noting $f(n,n) = g(n)$. □

Under the situation that the customs can patrol every day, it happens with probability $(1-p_1)^{\tau-1}$ that the smuggler can execute the τth smuggling without any capture by the customs, even though Player B schedules l smugglings in any way. Because reward $\alpha p_1 - p_2$ is expected for the customs at the τth smuggling, the value of the game is given by $v(n,n,l) = \sum_{r=1}^{l}(\alpha p_1 - p_2)(1-p_1)^{r-1} = g(l)$. In the case of $n = k = l$, we can apply both Corollary 1 and 2 to calculate the value of the game noting that $f(n,n) = g(n)$.

Now we are going to discuss the relation between elements involved in the matrix (3) in the case of $n > l$. As the result, we have the following corollary.

Corollary 3 *There are some properties about the value of the game as follows:*

(i) Nondecreasingness for k: $v(n-1,k-1,l) \leq v(n-1,k,l)$.

(ii) Relation among elements of the expression (3):

$$\begin{aligned} \alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1) &\geq v(n-1,k-1,l), & (12)\\ -1 + v(n-1,k,l-1) &\leq v(n-1,k,l), & (13)\\ \alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1) &> -1 + v(n-1,k,l-1)\ . & (14) \end{aligned}$$

(iii) Nonincreasingness for n: $v(n-1,k,l) \geq v(n,k,l)$.

Proof: (i) Because the whole strategies of Player A for game $\Gamma(n-1,k,l)$ involve a strategy of intentionally discarding an opportunity of patrol, we can see the property (i).
(ii) In the case of $n = 1$, these conditions are valid. For indices from $(1,1,1)$ through current ones (n,k,l), inequalities (12), (13) and (14) are assumed to be satisfied. Because it usually holds the relation of "*a minimax value* $\geq$ *the value of the game* $\geq$ *a maxmin value*", we have the following inequalities from Property (i).

$$\begin{aligned} &\min\{\alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1),\ v(n-1,k,l)\} \\ &\quad \geq\ v(n,k,l)\ \geq\ \max\{v(n-1,k-1,l),\ -1 + v(n-1,k,l-1)\}. \qquad (15) \end{aligned}$$

We obtain an inequality $v(n,k-1,l) \geq -1 + v(n-1,k-1,l-1)$ by replacing index k with $k-1$ in the right inequality of the above expression (15) and another $\alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1) \geq v(n,k,l)$ from the left inequality. Using these inequalities, the following transformation is possible.

$$\begin{aligned} &v(n,k,l) - (1-p_1)v(n,k-1,l) \\ &\quad \leq\ \alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1) - (1-p_1)\{-1 + v(n-1,k-1,l-1)\} \\ &\quad =\ \alpha p_1 - p_2 + 1 - p_1 < \alpha p_1 - p_2 + 1\ . \end{aligned}$$

As a result, we have inequality $-1 + v(n,k,l) < \alpha p_1 - p_2 + (1-p_1)v(n,k-1,l)$ to make sure that inequality (14) is satisfied for $n+1$ and $l+1$. The replacement of l with $l-1$ in the left inequality of expression (15) brings $v(n-1,k,l-1) \geq v(n,k,l-1)$. From this inequality and the right inequality of (15), we have $v(n,k,l) \geq -1 + v(n-1,k,l-1) \geq -1 + v(n,k,l-1)$ to see the validity of inequality (13) for a new index $n+1$. Furthermore, we replace l with $l-1$ in the right inequality of (15) to obtain $v(n,k,l-1) \geq v(n-1,k-1,l-1)$. Using this inequality and the left inequality of (15), we have the following transformation.

$$\alpha p_1 - p_2 + (1-p_1)v(n,k,l-1) \geq \alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1) \geq v(n,k,l)\ .$$

It means that inequality (12) is valid for $n+1$.
(iii) The property is easily obtained from the left inequality of expression (15). □

Property (i) looks very reasonable from inherent properties or assumptions of the game. Property (iii) is also understandable. The increase of the number of stages brings less efficiency in terms of the coverage ratio of the number of patrols k over the total number of stages n and more advantage on the number of the smuggler's options about when he takes an action.

Corollary 3 makes us solve the matrix game (3) more easily. We rewrite the relation stated in the corollary.

$$v(n,k,l) = val \begin{pmatrix} a & \geq & c \\ \vee & & \wedge\!| \\ b & \leq & d \end{pmatrix} , \tag{16}$$

where $a,\ b,\ c$ and d are given by Eq. (10) and (11).

In the expression (16), some equality signs bring the domination among players' strategies or a saddle point for the game. (i) In the case of $c = d$, the strategy P dominates NP and the value of the game becomes $v(n,k,l) = c = d$. $x_1^* = 1$ and $y_1^* = 0$ is one of optimal solutions. (ii) In the case of $b = d$, $v(n,k,l) = d = b$ is the value of the game and ($x_1^* = 0,\ y_1^* = 0$) is one of optimal solutions. (iii) In the case of $a = c$, the value of the game and a solution are $v(n,k,l) = a = c$ and ($x_1^* = 1,\ y_1^* = 1$), respectively. Anyway, we can say that the solution of the game is given by a mixed strategy even in these special cases. In general, we can solve the matrix game by equilibrating the expected payoff using variables of mixed strategies and then obtain some recursive formulas for the solution of the game, as follows.

Theorem 2 *At any stage n, optimal strategies of players are given as mixed strategies. The optimal mixed strategy of the customs x_1^*, x_2^*, that of the smuggler y_1^*, y_2^* and the value of the game $v(n,k,l)$ are calculated as follows, using the value of the game $v(n-1,\cdot)$ at the next stage $n-1$.*

$$x_1^* = \frac{d-b}{a-b-c+d},\quad x_2^* = \frac{a-c}{a-b-c+d},\quad y_1^* = \frac{d-c}{a-b-c+d},\quad y_2^* = \frac{a-b}{a-b-c+d}, \tag{17}$$

$$\begin{aligned} v(n,k,l) &= \frac{ad-bc}{a-b-c+d} \\ &= \frac{\{\alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1)\}v(n-1,k,l) + \{1 - v(n-1,k,l-1)\}v(n-1,k-1,l)}{\alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1) + 1 - v(n-1,k,l-1) - v(n-1,k-1,l) + v(n-1,k,l)}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} a &= \alpha p_1 - p_2 + (1-p_1)v(n-1,k-1,l-1), \qquad & b &= -1 + v(n-1,k,l-1), \\ c &= v(n-1,k-1,l), & d &= v(n-1,k,l)\ . \end{aligned}$$

From the theorem, we have a closed form for the value of the game in the case of $l = 1$.

Corollary 4 *In the case of $l = 1$, the value of the game is given by*

$$v(n,k,1) = \frac{k}{n}(\alpha p_1 - p_2 + 1) - 1\ . \tag{19}$$

Optimal mixed strategy is

$$x_1^* = k/n,\ y_1^* = 1/n\ , \quad for\ k < n \tag{20}$$

and

$$x_1^* = 1,\ y_1^* = 0\ , \quad for\ k = n\ . \tag{21}$$

Proof: Let us abbreviate $v(n,k,1)$ as $v(n,k)$ since the number l is fixed to be one. The setting of $n=1$ bears the result of $v(1,1)=\alpha p_1-p_2,\ v(1,0)=-1$ and we can prove the rightness of Eq. (19) for $n=1$. Assume that the equation correctly gives $v(n-1,k)$ for $k=0,\cdots,n-1$. The recursive formula (18) becomes

$$v(n,k)=\frac{(\alpha p_1-p_2)v(n-1,k)+v(n-1,k-1)}{\alpha p_1-p_2+1-v(n-1,k-1)+v(n-1,k)} \tag{22}$$

by applying $l=1$. Furthermore, we can organize the above expression into

$$\frac{1}{v(n,k)+1}=\frac{1}{v(n-1,k)+1}\left(1-\frac{v(n-1,k-1)+1}{\alpha p_1-p_2+1}\right)+\frac{1}{\alpha p_1-p_2+1}\ .$$

Substituting Eq. (19) for $v(n-1,k)$ and $v(n-1,k-1)$, the right-hand side becomes

$$\frac{n-1}{k}\cdot\frac{1}{\alpha p_1-p_2+1}\left(1-\frac{k-1}{n-1}\right)+\frac{1}{\alpha p_1-p_2+1}=\frac{n}{k}\cdot\frac{1}{\alpha p_1-p_2+1}\ .$$

Now Eq. (19) is valid too for index (n,k). For $k=n$, we can verify the validity of the equation from $v(n,n)=g(1)=\alpha p_1-p_2$ given by expression (8). Optimal strategies (20) and (21) are derived by applying $l=1$ and Eq. (19) to formulas (17). But it should be noted that $d=v(n-1,k-1,l)=c$ for $k=n$. □

The optimality of the mixed strategy (20) is intuitively evident. Player B, who has one opportunity to smuggle, must select an execution date among n stages in a uniform manner not to make Player A anticipate the date accurately. Similarly, Player A has to choose k days of assigning patrols uniformly among n stages. The total number of the assignments is ${}_nC_k$ and among the number, there are ${}_{n-1}C_{k-1}$ combinations that the current stage n is selected as the first patrol day. We can see that $y_1^*=1/n$ and $x_1^*={}_{n-1}C_{k-1}/{}_nC_k=k/n$ are optimal mixed strategies at the current stage.

Since any patrol costs nothing in the inspection game, Player A is going to exhaust all opportunities of patrols assigned in advance. Player B faces to the same situation by other reason. He must do smuggle l times as assigned to him in advance. However some opportunity could be left unused because the game possibly terminates by the capture of Player B before the last stage 1. From now, we are going to discuss how many opportunities are used in practice. In this calculation, it is assumed that the customs exhausts his chances to patrol even though the smuggler has no chance to smuggle.

Let $N_P(n,k,l)$ or $N_S(n,k,l)$ be the expected number of patrols or smugglings executed actually in the game $\Gamma(n,k,l)$, respectively. We also denote optimal mixed strategies of players at the stage n by $x_1^*,\ y_1^*$. The state (n,k,l) can transfer to five states at the next stage $n-1$, that is, $(n-1,k,l)$, $(n-1,k-1,l)$, $(n-1,k,l-1)$, $(n-1,k-1,l-1)$ and a state of the end of the game. The transition probability of each state is $(1-x_1^*)(1-y_1^*)$, $x_1^*(1-y_1^*)$, $(1-x_1^*)y_1^*$, $x_1^*y_1^*(1-p_1)$ and $x_1^*y_1^*p_1$. When we denote the number of executed patrols and smugglings by random variables X and Y at the stage n, they must be $(X,Y)=(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ and $(1,1)$ in the above respective transition. Now we can derive the following recursive formulas to calculate $N_P(n,k,l)$ and $N_S(n,k,l)$, where x_1^* and y_1^* are optimal mixed strategies in the state (n,k,l).

$$\begin{aligned}N_P(n,k,l)&=(1-x_1^*)(1-y_1^*)N_P(n-1,k,l)+x_1^*(1-y_1^*)\{1+N_P(n-1,k-1,l)\}\\&\quad+(1-x_1^*)y_1^*N_P(n-1,k,l-1)+x_1^*y_1^*\{1+(1-p_1)N_P(n-1,k-1,l-1)\},\end{aligned} \tag{23}$$

$$Initial\ condition:\ N_P(0,k,l)=0,\ N_P(n,k,0)=k,\ N_P(n,0,l)=0. \quad (24)$$

$$N_S(n,k,l)=(1-x_1^*)(1-y_1^*)N_S(n-1,k,l)+x_1^*(1-y_1^*)N_S(n-1,k-1,l)$$
$$+(1-x_1^*)y_1^*\{1+N_S(n-1,k,l-1)\}+x_1^*y_1^*\{1+(1-p_1)N_S(n-1,k-1,l-1)\}, \quad (25)$$

$$Initial\ condition:\ N_S(0,k,l)=0,\ N_S(n,k,0)=0,\ N_S(n,0,l)=l. \quad (26)$$

4. Numerical Examples

Setting parameters as $\alpha = 2\mathrm{C}p_1 = 0.5$ and $p_2 = 0.3$, and varying the number of stages n from 1 through 7, we calculate the solution of the game $\Gamma(n,k,l)$ for all combinations of k and l. The values of the game are listed in Table 1. Table 2 shows optimal mixed strategies of players (x_1^*, y_1^*) at the first stage n. In Table 1, we can see that the values are not dependent on n in the case of $n = k$, as stated in Corollary 2, and they are nondecreasing for k and nonincreasing for n, as shown in Corollary 3. Furthermore, we can notice some tendencies for k or l. In the case that k is small or the customs is hardly able to patrol so many times, the value of the game decreases as l becomes larger. It is very natural. On the other hand, in the case of larger k, the increase of l gives more reward to Player A or more loss to Player B because the coincidence of the patrol and the smuggling happens more frequently and it brings positive expected reward $\alpha p_1 - p_2 > 0$ to Player A. For example, in the case of $n = 6$ and $k = 1$ in Table 1, the increase of l causes the decreasing of the value of the game for small l's but the decreasing turns round to the increasing between $l = 5$ and $l = 6$. Let us call the point, at which the decreasing/increasing tendency changes, "turning point". The turning point occurs among smaller l's as k increases. The point exists between $l = 2$ and 3 in the case of $k = 3$, which is ought to be compared with the above case of $k = 1$. This point affects the boundary with zero value of the game, from which we can judge which player is a winner or a loser. For a fixed k, there exists the boundary among larger l's than the turning point. Table 1 shows that the boundary exists between $l = 5$ and 6 in the case of $n = 6,\ k = 2$ and between $l = 3$ and 4 in the case of $n = 6,\ k = 3$.

In Table 2, we can verify what Corollary 1 and 2 say for some simple cases of $l = n$ and $k = n$. We can also check numbers in the column of $l = 1$ to see the uniform strategy $x_1^* = k/n,\ y_1^* = 1/n$, as stated in Corollary 4. But please note that the numbers involve the round-up errors. Provided that the strategy of Player A is uniform, the uniformity of Player B's strategy $y_1 = l/n$ would indicates that his capture in the early stage would happen equally likely as in the late stage. However the early capture is disadvantageous to Player B because it means that a large number of opportunities to smuggle are discarded without being used. That is why optimal strategy y_1^* is actually smaller than the uniform probability l/n and Player B tends to execute the smuggling comparatively later. For larger k, Player A would patrol with larger probability x_1 and it promotes the Player B's tendency of putting off smuggling. The reasoning seems to be true from the following observation in Table 2. For a specific (n,k), the number y_1^* in the column of $l = 1$ equals to $1/n$ as already known. The number in the column of a general l is always smaller than l times $1/n$ and its diminishing rate becomes more intensive as k increases. On the other hand, Player A wants to capture Player B earlier and x_1^* becomes larger than uniform probability k/n for any l. As l grows, Player A increases x_1^* anticipating that his patrol is more likely to run into the smuggling. The number x_1^* starts from the uniform probability k/n in the column of $l = 1$ and becomes larger as l grows.

Table 3 shows the expected numbers of patrols and smugglings executed actually in game $\Gamma(n,k,l)$ in the form of a pair $(N_P(n,k,l),\ N_S(n,k,l))$, which are calculated by recursive formulas (23)-(26). The game possibly ends before the last stage only when both of the

Table 1: Value of the game

n	k	l						
		1	2	3	4	5	6	7
1	0	-1						
	1	0.7						
	0	-1	-2					
	1	-0.15	0.20					
	2	0.70	1.05					
3	0	-1	-2	-3				
	1	-0.43	-0.64	-0.30				
	2	0.13	0.40	0.80				
	3	0.70	1.05	1.23				
4	0	-1	-2	-3	-4			
	1	-0.58	-1.00	-1.19	-0.80			
	2	-0.15	-0.15	0.06	0.55			
	3	0.28	0.54	0.83	1.10			
	4	0.70	1.05	1.23	1.31			
5	0	-1	-2	-3	-4	-5		
	1	-0.66	-1.21	-1.61	-1.77	-1.30		
	2	-0.32	-0.50	-0.52	-0.29	0.30		
	3	0.02	0.11	0.30	0.62	0.98		
	4	0.36	0.63	0.87	1.10	1.25		
	5	0.70	1.05	1.23	1.31	1.36		
6	0	-1	-2	-3	-4	-5	-6	
	1	-0.72	-1.34	-1.86	-2.23	-2.35	-1.80	
	2	-0.43	-0.74	-0.92	-0.92	-0.64	0.05	
	3	-0.15	-0.20	-0.15	0.03	0.41	0.85	
	4	0.13	0.28	0.45	0.69	0.98	1.19	
	5	0.42	0.70	0.91	1.11	1.25	1.33	
	6	0.70	1.05	1.23	1.31	1.36	1.38	
7	0	-1	-2	-3	-4	-5	-6	-7
	1	-0.76	-1.44	-2.03	-2.52	-2.86	-2.93	-2.30
	2	-0.51	-0.92	-1.20	-1.36	-1.32	-0.99	-0.20
	3	-0.27	-0.44	-0.50	-0.45	-0.23	0.20	0.73
	4	-0.03	0.00	0.08	0.25	0.52	0.86	1.13
	5	0.21	0.39	0.56	0.76	1.00	1.19	1.29
	6	0.46	0.74	0.94	1.12	1.25	1.33	1.36
	7	0.70	1.05	1.23	1.31	1.36	1.38	1.39

patrol and the smuggling are executed at the same stage. Therefore, if the number of stages n increases while remaining k and l fixed, the probability of the coincidence goes down so that both expected values $N_P(n,k,l)$ and $N_S(n,k,l)$ would be getting larger, as we see in the table. For fixed n and k, larger l lifts up the coincidence probability and causes the decrease of $N_P(n,k,l)$ for the patrol. For the smuggler, l is nothing but the planned number of the smugglings and then larger l affects $N_S(n,k,l)$ by the interaction of the coincidence probability and the planned number. That is why $N_S(n,k,l)$ decreases for larger l. In the case of $(n,k) = (6,3)$, as l increases, $N_S(n,k,l)$ becomes larger until $l = 4$ but becomes smaller for $l = 5, 6$. For fixed n and l, larger k pushes up $N_P(n,k,l)$ and pushes down $N_S(n,k,l)$.

Table 2: Optimal strategy

n	k	l=1	2	3	4	5	6	7
1	0	(0, 1)						
	1	(1, 1)						
2	0	(0, 1)	(0, 1)					
	1	(.50, .50)	(1, 1)					
	2	(1, 0)	(1, 1)					
3	0	(0, 1)	(0, 1)	(0, 1)				
	1	(.33, .33)	(.38, .62)	(1, 1)				
	2	(.67, .33)	(.76, .48)	(1, 1)				
	3	(1, 0)	(1, 0)	(1, 1)				
4	0	(0, 1)	(0, 1)	(0, 1)	(0, 1)			
	1	(.25, .25)	(.27, .45)	(.33, .67)	(1, 1)			
	2	(.50, .25)	(.53, .44)	(.67, .53)	(1, 1)			
	3	(.75, .25)	(.79, .38)	(.92, .33)	(1, 1)			
	4	(1, 0)	(1, 0)	(1, 0)	(1, 1)			
5	0	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)		
	1	(.20, .20)	(.21, .36)	(.23, .52)	(.30, .70)	(1, 1)		
	2	(.40, .20)	(.41, .35)	(.46, .48)	(.62, .56)	(1, 1)		
	3	(.60, .20)	(.62, .34)	(.70, .42)	(.88, .38)	(1, 1)		
	4	(.80, .20)	(.82, .31)	(.90, .30)	(.98, .19)	(1, 1)		
	5	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 1)		
6	0	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	
	1	(.17, .17)	(.17, .30)	(.18, .42)	(.21, .55)	(.28, .72)	(1, 1)	
	2	(.33, .17)	(.34, .29)	(.36, .40)	(.42, .51)	(.59, .59)	(1, 1)	
	3	(.50, .17)	(.51, .29)	(.55, .38)	(.64, .44)	(.84, .42)	(1, 1)	
	4	(.67, .17)	(.68, .28)	(.73, .34)	(.84, .33)	(.97, .23)	(1, 1)	
	5	(.83, .17)	(.85, .26)	(.89, .27)	(.97, .19)	(1, .10)	(1, 1)	
	6	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 1)	
7	0	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
	1	(.14, .14)	(.15, .26)	(.15, .36)	(.16, .46)	(.19, .58)	(.27, .73)	(1, 1)
	2	(.29, .14)	(.29, .25)	(.30, .35)	(.33, .44)	(.40, .53)	(.56, .61)	(1, 1)
	3	(.43, .14)	(.44, .25)	(.46, .33)	(.51, .41)	(.61, .46)	(.81, .45)	(1, 1)
	4	(.57, .14)	(.58, .24)	(.61, .31)	(.68, .36)	(.81, .36)	(.96, .27)	(1, 1)
	5	(.71, .14)	(.72, .24)	(.76, .29)	(.84, .29)	(.95, .22)	(1, .13)	(1, 1)
	6	(.86, .14)	(.86, .23)	(.90, .24)	(.96, .18)	(1, .10)	(1, .05)	(1, 1)
	7	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 1)

5. Conclusions

We deal with an inspection game of the customs vs. a smuggler, where two players have some opportunities to patrol or smuggle. But the smuggler is compelled to accomplish his task as many times as given. We take account of the probability of the capture of the smuggler or of successful smugglings and then our model involves the fulfillment probabilities of players' aims. It leads us to a multi-stage stochastic game, where a game at a stage can transfer to some states at the next stage or to the end of the game. After we formulate the stochastic game, we clarify that the solution is given as an optimal mixed strategy at any stage. In some special cases, such as once-patrol, we derive closed-formed formulas for optimal strategies or the value of the game. Because the patrol costs nothing, the customs is going to patrol as many times as possible. So is the smuggler because his task to smuggle is under compulsion. Because of additional property that the game is zero-sum, we can judge which player becomes a winner or a loser based on whether the value of the game becomes

Table 3: Expected number of executed patrols and smugglings

n	k	l						
		1	2	3	4	5	6	7
1	0	(0, 1)						
	1	(1, 1)						
2	0	(0, 1)	(0, 2)					
	1	(1, 1)	(1, 1.50)					
	2	(2, 1)	(1.50, 1.50)					
3	0	(0, 1)	(0, 2)	(0, 3)				
	1	(1, 1)	(1, 1.76)	(1, 2)				
	2	(1.89, 1)	(1.76, 1.56)	(1.50, 1.75)				
	3	(3, 1)	(2.50, 1.50)	(1.75, 1.75)				
4	0	(0, 1)	(0, 2)	(0, 3)	(0, 4)			
	1	(1, 1)	(1, 1.85)	(1, 2.45)	(1, 2.50)			
	2	(1.90, 1)	(1.80, 1.70)	(1.70, 2.02)	(1.50, 2)			
	3	(2.75, 1)	(2.50, 1.57)	(2.27, 1.78)	(1.75, 1.88)			
	4	(4, 1)	(3.50, 1.50)	(2.75, 1.75)	(1.88, 1.88)			
5	0	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)		
	1	(1, 1)	(1, 1.88)	(1, 2.62)	(1, 3.10)	(1, 3)		
	2	(1.91, 1)	(1.83, 1.77)	(1.75, 2.28)	(1.68, 2.43)	(1.50, 2.25)		
	3	(2.75, 1)	(2.54, 1.66)	(2.37, 2)	(2.22, 2.03)	(1.75, 2)		
	4	(3.60, 1)	(3.24, 1.57)	(2.97, 1.80)	(2.62, 1.88)	(1.88, 1.94)		
	5	(5, 1)	(4.50, 1.50)	(3.75, 1.75)	(2.88, 1.88)	(1.94, 1.94)		
6	0	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	
	1	(1, 1)	(1, 1.91)	(1, 2.70)	(1, 3.34)	(1, 3.72)	(1, 3.50)	
	2	(1.92, 1)	(1.85, 1.82)	(1.79, 2.43)	(1.73, 2.79)	(1.66, 2.82)	(1.50, 2.50)	
	3	(2.77, 1)	(2.59, 1.73)	(2.43, 2.18)	(2.31, 2.36)	(2.18, 2.27)	(1.75, 2.13)	
	4	(3.58, 1)	(3.26, 1.64)	(3.01, 1.97)	(2.85, 2.05)	(2.58, 2.01)	(1.88, 2)	
	5	(4.44, 1)	(3.98, 1.56)	(3.65, 1.81)	(3.40, 1.89)	(2.82, 1.94)	(1.94, 1.97)	
	6	(6, 1)	(5.50, 1.50)	(4.75, 1.75)	(3.88, 1.88)	(2.94, 1.94)	(1.97, 1.97)	
7	0	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, 7)
	1	(1, 1)	(1, 1.92)	(1, 2.75)	(1, 3.47)	(1, 4.04)	(1, 4.34)	(1, 4)
	2	(1.93, 1)	(1.87, 1.85)	(1.82, 2.53)	(1.76, 3.02)	(1.71, 3.27)	(1.65, 3.20)	(1.50, 2.75)
	3	(2.80, 1)	(2.63, 1.77)	(2.49, 2.31)	(2.37, 2.62)	(2.27, 2.68)	(2.15, 2.49)	(1.75, 2.25)
	4	(3.61, 1)	(3.31, 1.69)	(3.08, 2.12)	(2.91, 2.30)	(2.78, 2.26)	(2.55, 2.13)	(1.88, 2.06)
	5	(4.40, 1)	(3.96, 1.62)	(3.65, 1.95)	(3.46, 2.05)	(3.31, 2.02)	(2.80, 2)	(1.94, 2)
	6	(5.29, 1)	(4.72, 1.56)	(4.32, 1.82)	(4.11, 1.90)	(3.70, 1.94)	(2.91, 1.97)	(1.97, 1.98)
	7	(7, 1)	(6.50, 1.50)	(5.75, 1.75)	(4.88, 1.88)	(3.94, 1.94)	(2.97, 1.97)	(1.98, 1.98)

positive or not. In practice, we show that a parametric space of the number of stages, patrols or smugglings is almost partitioned into halves with the positive value of the game and the negative by numerical examples. In the paper, we also do the sensitivity analysis to numerically verify some properties of the game stated in some theorems and corollaries.

The above description about the characteristics of the game becomes a pointer to our future research. If the patrol spends some cost, the customs would take an action of the patrol in a moderate fashion. If the smuggling is not compulsory but the smuggler is penalized by the cancellation of the smuggling, he would be motivated to smuggle as many times as possible. If two players do not have the common criterion of the payoff between them, the game would become a bimatrix game and not the zero-sum game anymore. These are topics for our future research.

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