Journal of the Operations Research
Society of Japan
2006, Vol. 49, No. 4, 290-305

GENERATION OF SEMI-PH PROCESS

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(Received February 8, 2005; Revised June 1, 2006)

Abstract A class of semi-Markov process was introduced by Latouche and referred as the semi-Poisson process in which the intervals of time between events are identically, exponentially distributed random variables and are not independent but dependent random variables. In this paper, we extend the class of semi-Poisson process to the class in which the random variables are identically distributed with a given PH-distribution and are not independent but dependent. We refer it as the semi-PH process.

Keywords: Markov process, phase-type distribution, semi-Markov process, semi-PH process

1. Introduction

In conventional queueing theory, it has been well established assumption that a sequence of random variables for input and/or service process of the queue is a sequence of independently, identically, distributed random variables. Although it is sometimes a realistic representation of actual input and/or service process, it sometimes misleads the results when the process has correlation. There may be many ways for the study of queues with correlation characteristics. One of approaches toward the study is to compare a well-known standard queueing system to another one such that the only difference lies in correlations between interarrival times and/or service times. For the study along that way, a class of semi-Markov process was introduced and referred as the semi-Poisson process by Latouche[6]. In the semi-Poisson process, the intervals of time between events are identically, exponentially distributed random variables and are not independent but dependent random variables. Using the semi-Poisson process as the input process, the M/M/1 queue with correlated input was studied in the paper. Related works are listed in [5, 7, 8].

In this paper, we extend the class of semi-Poisson process to the class in which the random variables are identically distributed with a given PH-distribution and are not independent but dependent random variables. We are concerned with the method to generate the sequence of random variables in such a way that those random variables have a common marginal distribution given in a form of PH-distribution and those are not independent but dependent each other. For the sake of generation of the sequence of random variables, we use a framework of semi-Markov process and a number of exponentially distributed random variables. We refer the semi-Markov process as the semi-PH process. The semi-PH process is discussed in Kishi and Kino [1–4]. As applications, the semi-PH process can be used as an input and/or service process for variety of queueing systems in order to study the effect of correlations on the behaviour of the queue comparing with the well-known standard queues.

This paper is organized as follows. In Section 2, we study the method to generate the semi-PH process. The outline of the method and basic idea for the study are described

in subsection 2.1. In Section 3, we show three examples of semi-PH processes. In the first example, we rewrite the original semi-Poisson process by our LST(Laplace-Stieltjes Transform) representation. Second example shows that we need at least one additional exponentially distributed random variable to generate the semi-PH process with two-phase Erlang distribution, while the last example shows that we need no additional random variable to generate the semi-PH process with hyper-exponential distribution. In Section 4, we study the method to generate a sequence of random variables which satisfies a given correlation function and a given common marginal distribution at the same time. We study this problem by a heuristic approach. A numerical example is given in this section. We conclude the paper in Section 5.

2. Semi-PH Process

2.1. Preliminary

Let τ_n be the time epoch of n-th event and $X_n(=\tau_n-\tau_{n-1})$ be the time interval between events for $n=1,2,\ldots,$ where $\tau_0=0$. We are concerned with the method by which to generate the sequence of positive random variables $\{X_n\}_{n=1}^{\infty}$ in such a way that these random variables have a common marginal distribution given in the form of PH-distribution and are not independent, but instead are dependent each other, i.e., the positive random variables $\{X_n\}_{n=1}^{\infty}$ have an identically distributed common marginal PH-distribution and have correlation. We refer to the given PH-distribution as the target distribution. Let X be a random variable distributed by a target PH-distribution and the LST(Laplace-Stieltjes Transform) be $E(e^{-\theta X})$. We assume that all of eigenvalues of the subgenerator for the target distribution are real. In other words, all of zeros of the denominator of the LST $E(e^{-\theta X})$ are real. That is our start point. Our goal is to generate the sequence of $\{X_n\}_{n=1}^{\infty}$ stated above. For the sake of the generation of a sequence of positive random variables $\{X_n\}$, we use a framework of the semi-Markov process $\{S(t), 0\le t\}$ on state space $\{1,2,\ldots,N\}$ with a transition probability matrix $\boldsymbol{P}$ of the embedded Markov chain. Subsection 2.3 describes the semi-Markov process. The semi-Markov process is defined such that the LST of X_n can be written in the form $E(e^{-\theta X_n})=\sum_{\ell=1}^{N}\pi_\ell E(e^{-\theta U_\ell})$, where $\boldsymbol{\pi}(=(\pi_\ell)_{\ell=1}^{N})$ is a stationary probability vector of $\boldsymbol{P}$, i.e., $\boldsymbol{\pi}=\boldsymbol{\pi}\boldsymbol{P}$ and the positive random variable U_ℓ is the time interval that the process is in state ℓ. We refer the $\{U_\ell\}_{\ell=1}^{N}$ as kernel variables.

The basic idea is as follows. If we can find both $\boldsymbol{\pi}$ and $\{U_\ell\}_{\ell=1}^{N}$ which satisfy the relation $E(e^{-\theta X})=\sum_{\ell=1}^{N}\pi_\ell E(e^{-\theta U_\ell})$ consistently, then $E(e^{-\theta X})=E(e^{-\theta X_n})$ is true. That implies our goal. To find those variables, we apply a technique of partial fraction expantion to the LST. Outline of the way to our goal is as follows:

(1) Expand $E(e^{-\theta X})$ into partial fraction form and find coefficient for each partial fraction. This procedure is described in subsection 2.2.
(2) Find kernel variables of $\{U_\ell\}_{\ell=1}^{N}$. The idea to find them is described in subsection 2.4.
(3) Expand the LST of $E(e^{-\theta U_\ell})$ for $\ell=1,2,\ldots,N$ into partial fraction form and find coefficient for each partial fraction. This procedure is described in subsection 2.5.
(4) Derive the stationary probability vector $\boldsymbol{\pi}$ solving a set of linear equations. This procedure is written in subsection 2.7.
(5) Find a transition matrix $\boldsymbol{P}$ which satisfies $\boldsymbol{\pi}=\boldsymbol{\pi}\boldsymbol{P}$. We apply hueristic approach to find it. The approach is described in Section 4.

2.2. Target PH-distribution

Let $\boldsymbol{Q}$ be a $(T\times T)$ matrix with negative diagonal, non-negative off-diagonal elements, non-positive row sums, and at least one negative row sum.

Based on $\boldsymbol{Q}$, we define a continuous time Markov process with $(T+1)$ states $(0, 1, 2, \ldots, T)$ and with an infinitesimal generator $\boldsymbol{Q}^*$ in the form

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{q} & \boldsymbol{Q} \end{pmatrix} \tag{1}$$

where the state 0 is an absorption state, the states $(1, 2, \ldots, T)$ are transient, $\boldsymbol{0}$ is the row vector with all elements being 0, $\boldsymbol{q} = -\boldsymbol{Q}\boldsymbol{1}$ and $\boldsymbol{1}$ is the column vector with all elements being 1. We write $\boldsymbol{\alpha}^* = (\alpha_0, \boldsymbol{\alpha})$ to denote the initial probability vector of $\boldsymbol{Q}^*$. The PH-distribution represented by $(\boldsymbol{\alpha}, \boldsymbol{Q})$ is defined as the distribution of the absorption time to the absorption state 0 with the initial probability $\boldsymbol{\alpha}^*$ in the Markov process. We call $\boldsymbol{Q}$ a PH-subgenerator of the Markov process and $(\boldsymbol{\alpha}, \boldsymbol{Q})$ a PH-representation of the PH-distribution. The probability distribution function $F(t)$ and the density function $f(t)$ for a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ are given as

$$F(t) = 1 - \boldsymbol{\alpha} \exp(\boldsymbol{Q}t)\boldsymbol{1} \qquad \text{and} \qquad f(t) = \boldsymbol{\alpha} \exp(\boldsymbol{Q}t)\boldsymbol{q}$$

for $t \in [0, \infty)$, respectively. If $\alpha_0 \neq 0$, then the PH-distribution has a mass at time 0. In the case that $\alpha_0 \neq 0$, we can simply add $f(0) = \alpha_0$ to absolutely continuous part of $f(t)$, $t > 0$ to obtain $f(t)$ for $t \in [0, \infty)$. In this paper, we therefore assume that $\alpha_0 = 0$ or equivalently $\boldsymbol{\alpha}\boldsymbol{1} = 1$. For a positive random variable X with the PH-representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$, the LST is written in the form

$$E(e^{-\theta X}) = \boldsymbol{\alpha}(\theta \boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{q}.$$

We assume that the LST can be written as

$$E(e^{-\theta X}) = \frac{N(\theta)}{D(\theta)} \tag{2}$$

where the denominator $D(\theta)$ is a polynomial of θ of order T in the form

$$D(\theta) = (\theta + \lambda_1)^{m_1}(\theta + \lambda_2)^{m_2} \cdots (\theta + \lambda_K)^{m_K}, \quad m_1 + m_2 + \cdots + m_K = T \tag{3}$$

and the numerator $N(\theta)$ is also a polynomial of θ of at most $T - 1$ order. Moreover, we assume that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K. \tag{4}$$

Note that $D(\theta) = \det(\theta \boldsymbol{I} - \boldsymbol{Q})$ and that each $-\lambda_i$ is a distinct eigenvalue of the PH-subgenerator $\boldsymbol{Q}$ with multiplicity $m_i (> 0)$ for $i = 1, 2, \ldots, K$. Under these assumptions, we can decompose the LST $E(e^{-\theta X})$ to the form of partial fraction expansion

$$E(e^{-\theta X}) = \sum_{i=1}^{K} \sum_{j=1}^{m_i} \frac{B(i, j)}{(\theta + \lambda_i)^{m_i - j + 1}} \tag{5}$$

where

$$B(i, j) = \frac{1}{(j-1)!} \frac{d^{j-1}}{d\theta^{j-1}} (\theta + \lambda_i)^{m_i} E(e^{-\theta X}) \Big|_{\theta = -\lambda_i}. \tag{6}$$

The coefficient given in (6) is rearranged into row vector form as

$$\boldsymbol{g}^* = (\boldsymbol{g}_1^*, \boldsymbol{g}_2^*, \ldots, \boldsymbol{g}_K^*) \tag{7}$$

where $\boldsymbol{g}_k^* = (B(k,1), B(k,2), \ldots, B(k,m_k))$ for $k = 1, 2, \ldots, K$. Let

$$\boldsymbol{\varphi}_k^*(\theta)^t = \Big(\frac{1}{(\theta+\lambda_k)^{m_k}}, \frac{1}{(\theta+\lambda_k)^{m_k-1}}, \ldots, \frac{1}{(\theta+\lambda_k)}\Big) \quad \text{for} \quad k = 1, 2, \ldots, K$$

and

$$\boldsymbol{\varphi}^*(\theta)^t = \big(\boldsymbol{\varphi}_1^*(\theta)^t, \boldsymbol{\varphi}_2^*(\theta)^t, \ldots, \boldsymbol{\varphi}_K^*(\theta)^t\big) \tag{8}$$

then the LST (5) can be written in the inner product form

$$E(e^{-\theta X}) = \boldsymbol{g}^* \boldsymbol{\varphi}^*(\theta) \tag{9}$$

where we use the notation $\boldsymbol{a}^t$ to denote the transpose of vector $\boldsymbol{a}$.

Remark 2.1 We assume that all of eigenvalues of the PH-subgenerator are real. The class of PH-distribution that satisfies such an assumption consists of a linear combination of Erlang distributions, as shown in (5). Here, we use the term "linear combination" in place of "mixture " because the coefficient (6) may be either positive or negative. ∎

2.3. Semi-Markov process

Consider a semi-Markov process $\{S(t), 0 \le t\}$ on the state space $\{1, 2, \ldots, N\}$, where $T \le N$. The semi-Markov process $S(t)$ changes its state at the time epoch $0 < \tau_0 < \tau_1 < \cdots$. The transitions between states follow an irreducible N-states Markov chain with a transition probability matrix $\boldsymbol{P} = (p_{ij})$, $i, j = 1, 2, \ldots, N$. The stationary probability vector for the Markov chain is denoted by $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$, i.e.,

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}.$$

For $n = 1, 2, \ldots$, we denote $X_n = \tau_n - \tau_{n-1}$ and $S(t) = S_n$ for $\tau_{n-1} \le t < \tau_n$. The semi-Markov kernel of $S(t)$ is defined as

$$P(S_{n+1} = j, X_n \le t | S_n = i) = p_{ij} G_i(t) \quad \text{for} \quad i, j = 1, 2, \ldots, N \tag{10}$$

where $G_i(t)$ is a distribution function of a positive random variable U_i, i.e., $G_i(t) = P(U_i \le t)$ for $i = 1, 2, \ldots, N$. We refer the positive random variables $\{U_\ell\}_{\ell=1}^N$ as kernel variables. Note that $G_i(t)$ can be interpreted as the distribution function of the time interval that the process is in state i, i.e., $G_i(t) = P(X_n \le t | S_n = i)$ for $i = 1, 2, \ldots, n$ and for $n = 1, 2, \ldots$ We assume that the embedded Markov chain start with the stationary state, i.e., the initial state of the process is

$$P(S_1 = i) = \pi_i \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{11}$$

See Fig.1 for a sample path of the semi-Markov process.

Proposition 2.1 For $n = 1, 2, \ldots,$

$$E(e^{-\theta X_n}) = \sum_{i=1}^{N} \pi_i E(e^{-\theta U_i}). \tag{12}$$

Proof. The LST form of the relation (10) is $E(e^{-\theta X_n}, S_{n+1} = j | S_n = i) = p_{ij} E(e^{-\theta U_i})$. Using this form and the initial condition (11), we have the relation

$$E(e^{-\theta X_n}) = \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} E(e^{-\theta X_n}, S_{n+1} = j | S_n = i) P(S_n = i | S_1 = k) P(S_1 = k)$$

for $n = 1, 2, \ldots$ The proof is brief obtained after calculation. ∎

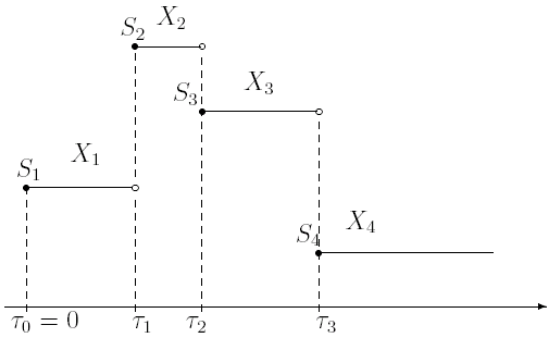

Figure 1: A sample path of the semi-Markov process

2.4. Kernel variables

Proposition 2.1 suggests that if we can find π_i and U_i, $i = 1, 2, \ldots, N$, which satisfy the relation

$$E(e^{-\theta X}) = \sum_{i=1}^{N} \pi_i E(e^{-\theta U_i})$$

then the relation $E(e^{-\theta X_n}) = E(e^{-\theta X})$ holds for $n = 1, 2, \ldots$, which implies that each marginal distribution of X_n in the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ agrees with the target distribution of X. It is clear, however, that random variables $\{X_n\}_{n=1}^{\infty}$ are not independent but dependent random variables.

To compose kernel variables $\{U_i\}_{i=1}^{N}$, we introduce $K + M$ exponentially distributed random variables $\{Z_i\}_{i=1}^{K+M}$. Let Z_i be an exponentially distributed random variable with parameter λ_i for $i = 1, 2, \ldots, K + M$, where we assume that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K < \lambda_{K+1} < \cdots < \lambda_{K+M} \tag{13}$$

and $M = N - T$, i.e.,

$$E(e^{-\theta Z_i}) = \frac{\lambda_i}{(\theta + \lambda_i)} \quad \text{for} \quad i = 1, 2, \ldots, K + M.$$

Note that parameters $\{\lambda_i\}_{i=1}^{K}$ are the same values which appear in the denominator (3) of the LST of the target distribution, while the values of parameters $\{\lambda_i\}_{i=K+1}^{K+M}$ of additional M random variables are arbitrary according to assumption (13).

To take into account of the multiplicity m_i of each λ_i, $i = 1, 2, \ldots, K$, in the denominator $D(\theta)$ of the target LST, we denote $s(i) = m_1 + m_2 + \cdots + m_i$ for $i = 1, 2, \ldots, K$ and define summation over n independent random variables $\{Z_{ik}\}_{k=1}^{n}$ as

$$V_i(n) = Z_{i1} + Z_{i2} + \cdots + Z_{in} \quad \text{for} \quad i = 1, 2, \ldots, K$$

where

$$E(e^{-\theta Z_i}) = E(e^{-\theta Z_{ik}}) \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Using this notation, kernel variable U_ℓ is defined as follows. For the index ℓ in the range $s(i-1) < \ell \leq s(i)$,

$$U_\ell = V_i(s(i) - \ell + 1) + V_{i+1}(m_{i+1}) + \cdots + V_K(m_K) + Z_{K+1} + \cdots + Z_{K+M} \tag{14}$$

for $i = 1, 2, \ldots, K$. For the index ℓ in the range $s(K)(= T) < \ell \leq T + M(= N)$,

$$U_\ell = Z_{\ell+K-T} + Z_{\ell+K-T+1} + \cdots + Z_{K+M}. \tag{15}$$

To better understand of the ordering of the index for U_ℓ, a simple example is given in the following.

Example 2.1 Consider the denominator of LST of the target distribution given in the form

$$D(\theta) = (\theta + \lambda_1)^2(\theta + \lambda_2)^2$$

and add two additional exponentially distributed random variables Z_3 and Z_4 with parameters λ_3 and λ_4, respectively, where

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4.$$

In this case, $m_1 = 2$, $m_2 = 2$, $T = m_1 + m_2 = 4$, $K = 2$, $M = 2$ and $N = K + M = 6$. Kernel variables $\{U_\ell\}_{\ell=1}^6$ are composed as

$$\begin{array}{lcccccc}
U_1 = & Z_1+ & Z_1+ & Z_2+ & Z_2+ & Z_3+ & Z_4 \\
U_2 = & & Z_1+ & Z_2+ & Z_2+ & Z_3+ & Z_4 \\
U_3 = & & & Z_2+ & Z_2+ & Z_3+ & Z_4 \\
U_4 = & & & & Z_2+ & Z_3+ & Z_4 \\
U_5 = & & & & & Z_3+ & Z_4 \\
U_6 = & & & & & & Z_4
\end{array}$$

where, for the sake of brevity of notation, we reduce the double index of random variable Z_{ik} to Z_i. ∎

2.5. Partial fraction expansion

The LST form for random variables defined by (14) and (15) are given in the form

$$E(e^{-\theta U_\ell}) = (\frac{\lambda_i}{\theta + \lambda_i})^{s(i)-\ell+1} \prod_{k=i+1}^{K} (\frac{\lambda_k}{\theta + \lambda_k})^{m_k} \prod_{k=K+1}^{K+M} \frac{\lambda_k}{\theta + \lambda_k} \tag{16}$$

and

$$E(e^{-\theta U_\ell}) = \prod_{k=\ell+K-T}^{K+M} (\frac{\lambda_k}{\theta + \lambda_k}) \tag{17}$$

respectively. The LST given by (16) can be expanded to the partial fraction form

$$\begin{aligned} E(e^{-\theta U_\ell}) &= \sum_{j=0}^{s(i)-\ell} \frac{\beta(\ell, \ell + j)}{(\theta + \lambda_i)^{s(i)-\ell+1-j}} + \sum_{k=i+1}^{K} \sum_{j=0}^{m_k-1} \frac{\beta(\ell, s(k-1) + j + 1)}{(\theta + \lambda_k)^{m_k-j}} \\ &+ \sum_{k=K+1}^{K+M} \frac{\beta(\ell, s(K) + k - K))}{\theta + \lambda_k} \end{aligned} \tag{18}$$

where for $0 \leq m \leq s(i) - 1$

$$\beta(\ell, \ell + m) = \frac{1}{m!} \frac{d^m}{d\theta^m} (\theta + \lambda_i)^{s(i)-\ell+1} E(e^{-\theta U_\ell}) \Big|_{\theta=-\lambda_i}, \tag{19}$$

for $0 \leq m \leq m_k - 1$, $i+1 \leq k \leq K$

$$\beta(\ell, s(k-1)+m+1) = \frac{1}{m!}\frac{d^m}{d\theta^m}(\theta+\lambda_k)^{m_k} E(e^{-\theta U_\ell})\Big|_{\theta=-\lambda_k}, \tag{20}$$

and for $K+1 \leq k \leq K+M$

$$\beta(\ell, s(K)+k-K) = (\theta+\lambda_{K+k}) E(e^{-\theta U_\ell})\Big|_{\theta=-\lambda_{K+k}}. \tag{21}$$

Similarly, the LST given by (17) can be expanded to the partial fraction form

$$E(e^{-\theta U_\ell}) = \sum_{m=\ell+K-T}^{K+M} \frac{\beta(\ell, m)}{(\theta+\lambda_m)} \tag{22}$$

where for $\ell + K - T \leq m \leq K+M$

$$\beta(\ell, m) = (\theta+\lambda_m) E(e^{-\theta U_\ell})\Big|_{\theta=-\lambda_m}. \tag{23}$$

2.6. Stationary probability vector

We define an $(N \times N)$ upper triangular matrix $\boldsymbol{C} = (\beta(i,j))$ in the form

$$\boldsymbol{C} = \begin{pmatrix} \beta(1,1) & \beta(1,2) & \cdots & \beta(1,N) \\ & \beta(2,2) & \cdots & \beta(2,N) \\ & & \ddots & \vdots \\ & & & \beta(N,N) \end{pmatrix} \tag{24}$$

where $\beta(i,j) = 0$ for $i > j$.

Proposition 2.2 $\boldsymbol{C}$ is a regular matrix.

Proof. Using assumption (13) and the definition of $\beta(\ell,\ell)$ given by (19),(20),(21),(23), we can confirm that each diagonal element $\beta(\ell,\ell)$ is strictly positive for $\ell = 1, 2, \ldots, N$, so that $\det(\boldsymbol{C})$ is also strictly positive, because $\det(\boldsymbol{C}) = \prod_{\ell=1}^{N} \beta(\ell,\ell)$. ∎

We compose vector $\boldsymbol{g}$ using vector $\boldsymbol{g}^*$ defined by (7), adding M zeros, in the form

$$\boldsymbol{g} = (\boldsymbol{g}^*, \overbrace{0, 0, \ldots, 0}^{M}) \tag{25}$$

and vector $\boldsymbol{\varphi}(\theta)$ using $\boldsymbol{\varphi}(\theta)^*$ defined by (8) in the form

$$\boldsymbol{\varphi}(\theta)^t = (\boldsymbol{\varphi}^*(\theta)^t, \overbrace{\varphi_{K+1}(\theta), \varphi_{K+2}(\theta), \ldots, \varphi_N(\theta)}^{M}) \tag{26}$$

where

$$\varphi_\ell(\theta) = \frac{1}{\theta+\lambda_\ell} \quad \text{for} \quad \ell = K+1, \ldots, N.$$

Using (25) and (26), one can see that the LST of X_n can be written in the form

$$E(e^{-\theta X_n}) = \boldsymbol{\pi} \boldsymbol{C} \boldsymbol{\varphi}(\theta). \tag{27}$$

Proposition 2.3 Linear equation $\boldsymbol{\pi}\boldsymbol{C} = \boldsymbol{g}$ has unique solution and $\boldsymbol{\pi}\boldsymbol{1} = 1$.

Proof. Uniqueness of $\boldsymbol{\pi}$ is clear from Proposition 2.2. Short calculation yields $\boldsymbol{\pi}\mathbf{1} = 1$. ∎

The main results of the present paper are summarized in the following proposition.

Proposition 2.4 If the solution of lenear equation

$$\boldsymbol{\pi}\boldsymbol{C} = \boldsymbol{g} \tag{28}$$

is non-negative, then

(a) for $n = 1, 2, \ldots$

$$E(e^{-\theta X_n}) = E(e^{-\theta X}) \tag{29}$$

and

(b)

$$E(X_1 X_n) = \boldsymbol{\nu}\ \mathrm{diag}(\boldsymbol{\pi})\boldsymbol{P}^{n-1}\boldsymbol{\nu}^t \tag{30}$$

where $\boldsymbol{\nu} = (\nu_\ell)_{\ell=1}^N$, $\nu_\ell = E(U_\ell)$ for $\ell = 1, 2, \ldots, N$.

Proof. From Porosition 2.1 and 2.3, if the solution $\boldsymbol{\pi}$ is non-negative, then the $\boldsymbol{\pi}$ is a probability vector which satisfies $E(e^{-\theta X_n}) = \sum_{\ell=1}^N \pi_\ell E(e^{-\theta U_\ell})$ for $n = 1, 2, \ldots$ On the other hand, using relations (18), (22) and (28), identifying the coefficients of each term on both sides, the relation $\sum_{\ell=1}^N \pi_\ell E(e^{-\theta U_\ell}) = E(e^{-\theta X})$ is shown to be true. In other words, $\boldsymbol{g}^*\boldsymbol{\varphi}^*(\theta) = \boldsymbol{\pi}\boldsymbol{C}\boldsymbol{\varphi}(\theta)$. This proves the statement (a).

From the definition of a covariance function, for $n = 1, 2, \ldots$,

$$E(X_1 X_n) = \int_0^\infty \int_0^\infty x_1 x_n dP(X_1 \le x_1, X_n \le x_n). \tag{31}$$

Brief calculation yields the relation

$$\begin{aligned} P(X_1 \le x_1, X_n \le x_n) &= \sum_{i=1}^N \sum_{j=1}^N P(X_1 \le x_1, S_1 = i, X_n \le x_n, S_n = j) \\ &= \sum_{i=1}^N \sum_{j=1}^N G_j(x_n) p_{ij}^{(n-1)} G_i(x_1) \pi_i. \end{aligned} \tag{32}$$

Substitution (32) into (31) yields the statement (b). ∎

Remark 2.2 As far as the authors know, there is no case that an negative element appeares in the solution of (28).

The correlation function of n is defined in the form

$$\rho_n = \frac{E(X_1 X_n) - E(X)^2}{E(X^2) - E(X)^2} \quad \text{for} \quad n = 1, 2, \ldots \tag{33}$$

Note that in this subsection we have shown the method of generating the elements of the stationary probability vector $\boldsymbol{\pi}$, while the problem of generating the elements of its transition probability matrix mentioned in subsection 2.3 remains to be solved.

2.7. Transition probability matrix

There are several possible transition probability matrices $\boldsymbol{P}$ which satisfy $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$. Here, we show two examples.

Example 2.2 Let

$$\boldsymbol{P} = \begin{pmatrix} a_1 & 1-a_1 & 0 & \dots & 0 \\ 0 & a_2 & 1-a_2 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{N-1} & 1-a_{N-1} \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} \tag{34}$$

and

$$a_i = 1 - \frac{\pi_N}{\pi_i} \quad \text{for} \quad i = 1, 2, \dots, N-1.$$

The relation $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$ is true, and $\boldsymbol{P}$ is a probability matrix if $\pi_N \leq \pi_i$ for $i = 1, 2, \dots, N-1$.

Example 2.3 Let

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & & \dots & 0 & 1 \\ a_1 & a_2 & \dots & \dots & a_{N-1} & a_N \end{pmatrix} \tag{35}$$

and

$$a_1 = \frac{\pi_1}{\pi_N},\ a_2 = \frac{\pi_2 - \pi_1}{\pi_N},\ \cdots,\ a_N = \frac{\pi_N - \pi_{N-1}}{\pi_N}.$$

The relation $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$ is true and $\boldsymbol{P}$ is a probability matrix if $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_N$.

Note that we can rearrange the order of states according to the order of $(\pi_i)_{i=1}^N$ to satisfy the probability condition.

3. Examples

3.1. Semi-Poisson process

We introduce a simple example of a semi-Poisson process proposed by Latouche[6]. This example shows an LST representation for the semi-Poisson process rather than the distribution function representation used in the original paper.

Let Z_1 be a random variable distributed according to the target exponential distribution with parameter λ_1 and let Z_2 and Z_3 be two additional random variables distributed according to exponential distribution with parameter λ_2 and λ_3, respectively, where $0 < \lambda_1 < \lambda_2 < \lambda_3$. Thus, we have

$$E(e^{-\theta Z_1}) = \frac{\lambda_1}{\theta + \lambda_1}, \qquad E(e^{-\theta Z_2}) = \frac{\lambda_2}{\theta + \lambda_2}, \qquad E(e^{-\theta Z_3}) = \frac{\lambda_3}{\theta + \lambda_3}.$$

Here, $B(1,1) = \lambda_1$ and $\boldsymbol{g}^* = (\lambda_1)$, so that the coefficient vector becomes $\boldsymbol{g} = (\lambda_1, 0, 0)$.

Kernel variables are composed such that

$$\begin{cases} U_1 = & Z_1 + & Z_2 + & Z_3 \\ U_2 = & & Z_2 + & Z_3 \\ U_3 = & & & Z_3. \end{cases}$$

The LST and its partial fraction expansion forms for U_1 and U_2 are given as

$$\begin{cases} E(e^{-\theta U_1}) = \displaystyle\prod_{i=1}^{3}\left(\frac{\lambda_i}{\theta+\lambda_i}\right) = \sum_{i=1}^{3}\frac{\beta(1,i)}{\theta+\lambda_i} \\ E(e^{-\theta U_2}) = \displaystyle\prod_{i=2}^{3}\left(\frac{\lambda_i}{\theta+\lambda_i}\right) = \sum_{i=2}^{3}\frac{\beta(2,i)}{\theta+\lambda_i}. \end{cases}$$

Calculating each coefficient $\beta(i,j)$ and arranging the coefficients into matrix form, we obtain the coefficient matrix $\boldsymbol{C}$ in the form

$$\boldsymbol{C} = (\beta(i,j)) = \begin{pmatrix} \dfrac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} & \dfrac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} & \dfrac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \\ & \dfrac{\lambda_2\lambda_3}{\lambda_3-\lambda_2} & \dfrac{\lambda_2\lambda_3}{\lambda_2-\lambda_3} \\ & & \lambda_3 \end{pmatrix}$$

where $\beta(i,j)=0$ for $i>j$. Solving the linear equation $\boldsymbol{\pi C}=\boldsymbol{g}$, the stationary probabilities $\boldsymbol{\pi}=(\pi_i)_{i=1}^3$ are obtained in the form

$$\pi_1 = (1-\frac{\lambda_1}{\lambda_2})(1-\frac{\lambda_1}{\lambda_3}), \quad \pi_2 = \frac{\lambda_1}{\lambda_2}(1-\frac{\lambda_1}{\lambda_3}), \quad \pi_3 = \frac{\lambda_1}{\lambda_3}.$$

Consequently,

$$E(e^{-\theta X_n}) = \sum_{\ell=1}^{3}\pi_\ell E(e^{-\theta U_\ell}) = E(e^{-\theta Z_1}) \quad \text{for} \quad n=1,2,\ldots$$

If $\frac{1}{2} \le \frac{\lambda_1}{\lambda_2}$ and $1 \le \frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}$, then $\pi_1 \le \pi_2 \le \pi_3$. Therefore, the transition probability matrix $\boldsymbol{P}$ can be written in the form

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{pmatrix} \quad \text{where} \quad a_1 = \frac{\pi_1}{\pi_3},\ a_2 = \frac{\pi_2-\pi_1}{\pi_3},\ a_3 = \frac{\pi_3-\pi_2}{\pi_3}. \tag{36}$$

For another case, if $\frac{1}{2} \ge \frac{\lambda_1}{\lambda_2}$ and $1 \ge \frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}$, then $\pi_1 \le \pi_3$ and $\pi_2 \le \pi_3$. In this case, the transition probability matrix $\boldsymbol{P}$ can be written in the form

$$\boldsymbol{P} = \begin{pmatrix} a_1 & 1-a_1 & 0 \\ 0 & a_2 & 1-a_2 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{where} \quad a_1 = 1-\frac{\pi_3}{\pi_1} \quad \text{and} \quad a_2 = 1-\frac{\pi_3}{\pi_2}. \tag{37}$$

3.2. Semi-PH process with Erlang distribution

Consider a random variable X distributed according to a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ where

$$\boldsymbol{\alpha} = (1,0), \quad \boldsymbol{Q} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_1 \end{pmatrix} \quad \boldsymbol{q}^t = (0,\lambda_1).$$

Equivalently, the X is distributed according to a two-phase Erlang distribution with parameter λ_1 and the LST is written in the form

$$E(e^{-\theta X}) = \frac{N(\theta)}{D(\theta)} = \left(\frac{\lambda_1}{\theta+\lambda_1}\right)^2.$$

Coefficient vector $\boldsymbol{g}^*$ of the target distribution is thus $\boldsymbol{g}^* = (\lambda_1^2, 0)$.

We use one additional random variable to generate the semi-PH process. Let Z_1 and Z_2 be random variables distributed according to exponential distribution with parameters λ_1 and λ_2, respectively, where $0 < \lambda_1 < \lambda_2$. Thus

$$E(e^{-\theta Z_1}) = \frac{\lambda_1}{\theta + \lambda_1}, \qquad E(e^{-\theta Z_2}) = \frac{\lambda_2}{\theta + \lambda_2}.$$

We compose kernel variables such that

$$\begin{cases} U_1 = & Z_1 + & Z_1 + & Z_2 \\ U_2 = & & Z_1 + & Z_2 \\ U_3 = & & & Z_2. \end{cases}$$

The LST and its partial fraction expansion form for U_1 and U_2 are given as

$$\begin{cases} E(e^{-\theta U_1}) = \left(\dfrac{\lambda_1}{\theta + \lambda_1}\right)^2 \left(\dfrac{\lambda_2}{\theta + \lambda_2}\right) & = \dfrac{\beta(1,1)}{(\theta + \lambda_1)^2} + \dfrac{\beta(1,2)}{\theta + \lambda_1} + \dfrac{\beta(1,3)}{\theta + \lambda_2} \\ E(e^{-\theta U_2}) = \left(\dfrac{\lambda_1}{\theta + \lambda_1}\right) \left(\dfrac{\lambda_2}{\theta + \lambda_2}\right) & = \dfrac{\beta(2,2)}{\theta + \lambda_1} + \dfrac{\beta(2,3)}{\theta + \lambda_2}. \end{cases}$$

Setting coefficient vector $\boldsymbol{g} = (\lambda_1^2, 0, 0)$ and calculating each coefficient $\beta(i,j)$ and arranging each coefficient $\beta(i,j)$ into matrix form, we obtain the coefficient matrix $\boldsymbol{C}$ in the form

$$\boldsymbol{C} = (\beta(i,j)) = \begin{pmatrix} \dfrac{\lambda_1^2\lambda_2}{\lambda_2 - \lambda_1} & \dfrac{\lambda_1^2\lambda_2}{(\lambda_1 - \lambda_2)^2} & \dfrac{\lambda_1^2\lambda_2}{(\lambda_2 - \lambda_1)^2} \\ & \dfrac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1} & \dfrac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \\ & & \lambda_2 \end{pmatrix}$$

where $\beta(i,j) = 0$ for $i > j$. Solving the linear equation $\boldsymbol{\pi C} = \boldsymbol{g}$, the stationary probabilities $\boldsymbol{\pi} = (\pi_i)_{i=1}^3$ are obtained in the form

$$\pi_1 = 1 - \frac{\lambda_1}{\lambda_2}, \quad \pi_2 = \frac{\lambda_1}{\lambda_2}, \quad \pi_3 = 0.$$

Consequently,

$$E(e^{-\theta X_n}) = (1 - \frac{\lambda_1}{\lambda_2})E(e^{-\theta U_1}) + \frac{\lambda_1}{\lambda_2}E(e^{-\theta U_2}) = E(e^{-\theta X}) \quad \text{for} \quad n = 1, 2, \ldots$$

As the result, the generic random variable U_3 is not needed in order to realize the semi-PH process. In this example, we have to add at least one additional variable, otherwise we can not generate the objective semi-PH process.

If $\frac{1}{2} \leq \frac{\lambda_1}{\lambda_2}$, then $\pi_1 \leq \pi_2$, else $\pi_1 \geq \pi_2$. Therefore, for the case in which $\pi_1 \leq \pi_2$, the transition probability matrix $\boldsymbol{P}$ can be written in the form

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 \\ a_1 & 1 - a_1 \end{pmatrix} \quad \text{where} \quad a_1 = \frac{\pi_1}{\pi_2}. \tag{38}$$

For the case in which $\pi_1 \geq \pi_2$, the transition probability matrix $\boldsymbol{P}$ can be written in the form

$$\boldsymbol{P} = \begin{pmatrix} a_1 & 1 - a_1 \\ 1 & 0 \end{pmatrix} \quad \text{where} \quad a_1 = 1 - \frac{\pi_2}{\pi_1}. \tag{39}$$

3.3. Semi-PH process with hyper-exponential distribution

Consider a random variable X distributed according to a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$, where

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2), \quad \boldsymbol{Q} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \quad \boldsymbol{q}^t = (\lambda_1, \lambda_2).$$

Equivalently, the X is distributed according to hyper exponential distribution with two parameter λ_1 and λ_2, where $0 < \lambda_1 < \lambda_2$. The LST of X is written in the form

$$E(e^{-\theta X}) = \frac{N(\theta)}{D(\theta)} = \frac{\alpha_1\lambda_1}{\theta + \lambda_1} + \frac{\alpha_2\lambda_2}{\theta + \lambda_2}$$

so that $\boldsymbol{g}^* = (\alpha_1\lambda_1, \alpha_2\lambda_2)$.

Let Z_1 and Z_2 be random variables distributed according to exponential distribution with parameter λ_1 and λ_2, respectively. Thus we have

$$E(e^{-\theta Z_1}) = \frac{\lambda_1}{\theta + \lambda_1}, \qquad E(e^{-\theta Z_2}) = \frac{\lambda_2}{\theta + \lambda_2}.$$

We compose kernel variables such that

$$\begin{cases} U_1 = & Z_1 + & Z_2 \\ U_2 = & & Z_2. \end{cases}$$

The LST and its partial fraction expansion form for U_1 are given as

$$E(e^{-\theta U_1}) = \left(\frac{\lambda_1}{\theta + \lambda_1}\right)\left(\frac{\lambda_2}{\theta + \lambda_2}\right) = \frac{\beta(1,1)}{(\theta + \lambda_1)} + \frac{\beta(1,2)}{\theta + \lambda_2}$$

Setting coefficient vector $\boldsymbol{g} = \boldsymbol{g}^* = (\alpha_1\lambda_1, \alpha_2\lambda_2)$ and calculating and arranging each coefficient $\beta(i,j)$ into matrix form, we obtain the coefficient matrix $\boldsymbol{C}$ in the form

$$\boldsymbol{C} = (\beta(i,j)) = \begin{pmatrix} \dfrac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1} & \dfrac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \\ 0 & \lambda_2 \end{pmatrix}.$$

Solving the linear equation $\boldsymbol{\pi}\boldsymbol{C} = \boldsymbol{g}$, the stationary probabilities $\boldsymbol{\pi} = (\pi_i)_{i=1}^2$ are obtained in the form

$$\pi_1 = \alpha_1(1 - \frac{\lambda_1}{\lambda_2}), \quad \pi_2 = \alpha_2 - \alpha_1(\frac{\lambda_1}{\lambda_2}).$$

Consequently,

$$E(e^{-\theta X_n}) = \alpha_1(1 - \frac{\lambda_1}{\lambda_2})E(e^{-\theta U_1}) + (\alpha_2 - \alpha_1(\frac{\lambda_1}{\lambda_2}))E(e^{-\theta U_2}) = E(e^{-\theta X}) \quad \text{for} \quad n = 1, 2, \ldots$$

If $\alpha_1 \leq \alpha_2$, then $\pi_1 \leq \pi_2$, else $\pi_1 \geq \pi_2$. Therefore, for the case in which $\pi_1 \leq \pi_2$, the transition probability matrix $\boldsymbol{P}$ can be written in the form

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 \\ a_1 & 1 - a_1 \end{pmatrix} \quad \text{where} \quad a_1 = \frac{\pi_1}{\pi_2}. \tag{40}$$

For another case in which $\pi_1 \geq \pi_2$, the transition probability matrix $\boldsymbol{P}$ can be written in the form

$$\boldsymbol{P} = \begin{pmatrix} a_1 & 1 - a_1 \\ 1 & 0 \end{pmatrix} \quad \text{where} \quad a_1 = 1 - \frac{\pi_2}{\pi_1}. \tag{41}$$

Note that we use no additional random variable for this case, i.e., $M = 0$.

Remark 3.1 As shown in this section, there exist two examples, one of which requires no additional random variable to generate the objective semi-PH process, while the other requires at least one additional random variable. The questions of how many additional random variables are required to generate an objective semi-PH process and what condition determines the minimal number of additional random variables arise. Unfortunately, the answers to these questions remain unclear. ∎

4. Heuristic Algorithm

4.1. Genetic algorithm

In section 2, we investigate how to determine the stationary probability vector $\boldsymbol{\pi}$ to generate the objective semi-PH process. In this section, we study the semi-PH process which satisfies an additional constraint in addition to the constraint giving in the form of a target distribution at the same time. We consider the problem of how to fix the transition probability $\boldsymbol{P}$ of the embedded Markov chain such that $\boldsymbol{P}$ satisfies the relation $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$ and $\boldsymbol{P}$ provides given values of the correlation coefficient through the relation (30) at the same time. Finding an analytic approach to accomplishing this task is difficult and remains an open problem. We therefore examine this problem using a heuristic approach rather than an analytic approach. In this section, we use a genetic algorithm(GA)[9].

Consider a transition probability matrix P as an individual and let ϕth individual estimate be $\hat{P}_\phi = (\hat{p}_{ij}^{(\phi)})$ for $\phi = 1, 2, \ldots, \Phi$. To obtain the best estimate, we search for the smallest evaluated value among Φ individuals and Ψ generations. Each generation has evaluation and operation steps.

Each individual has $32N(N-1)$ bits' data f where N is order of $\hat{P}$ and they correspond to a transition probability matrix $\hat{P}$.

$$\text{bitdata}f : \overbrace{\underbrace{011\ldots10}_{\hat{p}_{11}}|\underbrace{110\ldots00}_{\hat{p}_{12}}|\ldots|\underbrace{001\ldots11}_{\hat{p}_{N,N-1}}}^{32N(N-1)[\text{bits}]}$$

Each element $\hat{p}_{ij} (0 \le \hat{p}_{ij} \le 1)$ in $\hat{P}$ is assigned to 32 bits from f. These bits are transformed to a $[0, 1]$ real number. Since order of $\hat{P}$ is N, $32N^2$ bits are required. However, from stochastic condition we define $\hat{p}_{iN} = 1 - \hat{p}_{i1} - \hat{p}_{i2} - \cdots - \hat{p}_{i,N-1}$. If we get a s for $\hat{p}_{i1} + \hat{p}_{i2} + \cdots + \hat{p}_{is} > 1$ on the transformation process of $\hat{p}_{ij}$, then we define

$$\begin{aligned} \hat{p}_{is} &= 1 - \hat{p}_{i1} - \hat{p}_{i2} - \cdots - \hat{p}_{i,s-1} \quad \text{and} \\ \hat{p}_{it} &= 0 \quad \text{for } t = s+1, s+2, \ldots, N. \end{aligned}$$

In the evaluation step, we first evaluate the error between the stationary probability vector $\boldsymbol{\pi}$ and $\hat{\boldsymbol{\pi}}^{(\phi)}$ by calculating $v_1^{(\phi)}$;

$$v_1^{(\phi)} = |\hat{\pi}_1^{(\phi)} - \pi_1| + |\hat{\pi}_2^{(\phi)} - \pi_2| + \cdots + |\hat{\pi}_N^{(\phi)} - \pi_N|.$$

If $v_1^{(\phi)}$ is greater than ϵ_1, then let the evaluated value be maximum and we do not perform evaluation again. When $v_1^{(\phi)} \le \epsilon_1$, we calculate the correlation coefficients $\hat{\boldsymbol{\rho}}^{(\phi)} = (\hat{\rho}_1^{(\phi)}, \hat{\rho}_2^{(\phi)}, \ldots, \hat{\rho}_h^{(\phi)})$ and then define the evaluated value $v_2^{(\phi)}$ of ϕth individual;

$$v_2^{(\phi)} = (\hat{\rho}_1^{(\phi)} - \rho_1)^2 + (\hat{\rho}_2^{(\phi)} - \rho_2)^2 + \cdots + (\hat{\rho}_h^{(\phi)} - \rho_h)^2,$$

where $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_h)$ are assigned correlation coefficients. In the operation step, we perform crossover with the individual that has the minimal evaluated value and then apply mutations to every individual. The entire algorithm is given below.

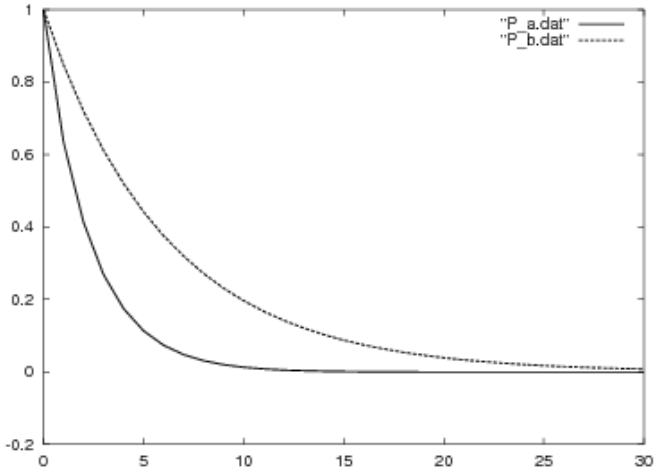

Figure 2: Two correlation functions with $\rho_2 = 0.7$ and $\rho_2 = 0.9$

Algorithm K:

A-1 Generate initial individuals $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_\Phi$ such that each $\hat{P}_\phi$ is a probability matrix of which each element is chosen randomly.

A-2 Calculate $v_1^{(\phi)}$ and $v_2^{(\phi)}$ for $\phi = 1, 2, \ldots, \Phi$ and let $v_{min} = \min_\phi\{v_2^{(\phi)}\}$.

A-3 Stop searching if either v_{min} is smaller than the given criterion ϵ_2 or the number of generations achieved at the given Ψ.

A-4 Renumber the index ϕ of $\hat{P}_\phi$ in the increasing order of $v_2^{(\phi)}$, i.e., $\phi_1 \leq \phi_2$ if $v_2^{(\phi_1)} \leq v_2^{(\phi_2)}$.

A-5 Replace $\{\hat{P}_\phi\}_{\phi=[\frac{\Phi}{2}]}^{\Phi}$ with $\hat{P}_1$, where $[\xi]$ is the maximum integer less than ξ.

A-6 Do *operations* in the sense of GA except $\hat{P}_1$.

A-6 Increment the number of generations and go to A-2.

4.2. Numerical example

Let the target distribution of a random variable X be a PH distribution with the representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$, where

$$\boldsymbol{\alpha} = (1, 0), \quad \boldsymbol{Q} = \begin{pmatrix} -0.1 & 0.05 \\ 0 & -0.3 \end{pmatrix}.$$

The LST of X is written as

$$E(e^{-\theta X}) = \frac{0.125}{\theta + 0.1} + \frac{-0.075}{\theta + 0.3}$$

so that $\lambda_1 = 0.1$, $\lambda_2 = 0.3$ and $\boldsymbol{g}^* = (0.125, -0.075)$. Adding one additional random variable Z_3, we define three exponentially distributed random variables Z_1, Z_2 and Z_3 with parameter $\lambda_1 = 0.1$, $\lambda_2 = 0.3$ and $\lambda_3 = 0.7$, respectively. Composing generic random variables $U_1 = Z_1 + Z_2 + Z_3$, $U_2 = Z_2 + Z_3$ and $U_3 = Z_3$, decomposing LST of those random variables into partial fraction form, calculating each coefficient of each partial fraction, we derive matrix $\boldsymbol{C}$ in the form

$$\boldsymbol{C} = \begin{pmatrix} 0.175 & -0.263 & 0.088 \\ 0 & 0.525 & -0.525 \\ 0 & 0 & 0.700 \end{pmatrix}.$$

Setting $\boldsymbol{g} = (\boldsymbol{g}^*, 0)$ and solving the linear equation of $\boldsymbol{\pi C} = \boldsymbol{g}$, we obtain

$$\boldsymbol{\pi} = (0.7143, 0.2143, 0.0714).$$

We keep two constraints in order to determine the transition probability matrix $\boldsymbol{P}$. One of these constraints is that $\boldsymbol{P}$ must satisfy the relation $\boldsymbol{\pi} = \boldsymbol{\pi P}$. The other constraint is that the value of the correlation function at $n = 2$ satisfies a given value, i.e., ρ_2 of the correlation function is equal to the given value. We examine two cases in which $\rho_2 = 0.7$ and $\rho_2 = 0.9$. Applying the GA algorithm **K** given in the previous subsection, transition probability matrices $\boldsymbol{P}_a$ for $\rho_2 = 0.7$ and $\boldsymbol{P}_b$ for $\rho_2 = 0.9$ are obtained such that

$$P_a = \begin{pmatrix} 0.9080 & 0.0920 & 0.0000 \\ 0.2886 & 0.4000 & 0.3113 \\ 0.0527 & 0.8831 & 0.0642 \end{pmatrix} \quad \text{and} \quad P_b = \begin{pmatrix} 0.9526 & 0.0474 & 0.0000 \\ 0.1379 & 0.7807 & 0.0815 \\ 0.0601 & 0.1836 & 0.7563 \end{pmatrix}$$

respectively. Using the results for $\boldsymbol{P}_a$ and $\boldsymbol{P}_b$, we can compute correlation function ρ_n for $n = 2, 3, \ldots$, as shown in Fig.2 in which the solid line shows the correlation function with $\boldsymbol{P}_a$ and the dotted line shows correlation function with $\boldsymbol{P}_b$.

5. Conclusion

The method to generate the semi-PH process was discussed in this paper. By using the proposed method, we can generate a sequence of random variables that have a given marginal distribution and are not independent but dependent. Future studies are listed in the followings: Analytic method to find the transition matrix that satisfy both conditions for a given marginal distribution and for a given correlation coefficients at the same time; Study for application of queueing systems with correlation characteristics by using the semi-PH process; Study for extension to a PH-subgenerator with complex conjugate eigenvalues.

Acknowledgement: The authors wish to acknowledge their grateful thanks to anonymous referees for their helpful comments on the paper.

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