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CLOSED FORM OF PH-DISTRIBUTION

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Abstract In this paper a new approach of Jordan canonical formulation for analysis of a PH-subgenerator is proposed. Based on the new method, this paper shows that if all eigenvalues of a PH-subgenerator are real then the closed form of the PH-distribution is a linear combination of Erlang distributions, while if the subgenerator has complex eigenvalues then the closed form of the PH-distribution is no longer such a simple form but more complex form that includes trigonometric functions in addition to a linear combination of Erlang distributions. This paper also shows that the dominant parameter for degree of a PH-distribution is not the size but the degree of minimal polynomial of the PH-subgenerator.

Keywords: Markov process, phase-type distribution, Jordan canonical form, minimal representation

1. Introduction

Phase-type distribution (PH-distribution) was proposed by Neuts [5] as the distribution of absorption time in a finite-state Markov process. Since then, PH-distribution has been used widely in a variety of fields in stochastic modelling such as queueing theory, risk analysis and reliability theory [3],[6]. Many authors have contributed to the study of PH-distributions and related issues including the minimal PH-representation problem, the triangular PH-distribution, PH-simplicity and PH-majorizaton, polytope and matrix-exponential distribution [1],[7],[8],[9]. Survey of current status of the studies about PH-distribution is given in O'Cinneide [10].

In this paper, we propose a new approach that is different from those of previous studies, and derive the closed form of PH-distribution using the new approach. Representation of probability distribution function and probability density function for a PH-distribution include the matrix exponential term that expressed in the infinite series of the matrices, so that they are not closed but open forms. In order to derive the closed form of them, systematic analysis for the matrix structure of the PH-subgenerator is required. To achieve systematic study for PH-distribution, we use Jordan canonical form which is a well established formulation that provides complete analysis of matrix structures, particularly for matrices with multiplicated eigenvalues.

Based on the closed form of PH-distribution, we show that if all eigenvalues of a PH-subgenerator are real, then the closed form of the PH-distribution is a linear combination of Erlang distributions, while if a PH-subgenerator has complex eigenvalues, then the closed form of the PH-distribution is no longer a linear combination of Erlang distributions but more complex form that includes trigonometric functions.

This paper is organised as follows. In Section 2, we consider a PH-subgenerator $\boldsymbol{Q}$ all of which eigenvalues are real. In Subsection 2.1, we introduce some basic concepts of PH-subgenerator and related issues. In Subsection 2.2, for the simplest case, we consider a PH-subgenerator $\boldsymbol{Q}$ with one real multiplicated eigenvalue. In Subsection 2.3, generalizing the results obtained in Subsection 2.2, we consider a PH-subgenerator $\boldsymbol{Q}$ with multiple real eigenvalues which may be multiplicated and derive the closed form of the PH-distribution. Two examples are given in Subsection 2.4. In Section 3, we consider a PH-subgenerator which has conjugate complex eigenvalues in addition to real eigenvalues. In Subsection 3.1, we introduce some basic concepts of conjugate complex eigenvalues of a PH-subgenerator and provide preliminary propositions. In Subsection 3.2, we consider a PH-subgenerator $\boldsymbol{Q}$ that has one multiplicated real eigenvalue and a pair of multiplicated conjugate complex eigenvalues for the simplest case. One example is given in this Subsection. In Subsection 3.3, generalizing the results obtained in Subsection 3.2, we consider a PH-subgenerator $\boldsymbol{Q}$ which has multiplicated multiple real eigenvalues and multiplicated multiple pairs of conjugate complex eigenvalues. The main result of this paper is given in Section 4. We conclude the paper in Section 5. Through this paper, we refer to Han and Iri[2] and Meyer[4] for matrix theory and Jordan canonical form.

2. PH-subgenerator with Real Eigenvalues

2.1. Preliminary

PH-subgenerator: Let $\boldsymbol{Q}$ be a $(n \times n)$ matrix with a negative diagonal, non-negative off-diagonal elements, non-positive row sums, and at least one negative row sum. The matrix $\boldsymbol{Q}$ is called a subgenerator in the Markov processes literature. Based on $\boldsymbol{Q}$, we define a continuous time Markov process with $(n+1)$ states $(0, 1, 2, \ldots, n)$ and with an infinitesimal generator $\boldsymbol{Q}^*$ of the following form:

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{q} & \boldsymbol{Q} \end{pmatrix} \tag{2.1}$$

where the 0 state is an absorption state, the states $(1, 2, \ldots, n)$ are transient, $\mathbf{0}$ is a row vector with all elements being 0, $\boldsymbol{q} = -\boldsymbol{Q}\mathbf{1}$ and $\mathbf{1}$ is a column vector with all elements being 1. We write $\boldsymbol{\alpha}^* = (\alpha_0, \boldsymbol{\alpha})$ to denote the initial probability vector of $\boldsymbol{Q}^*$. The PH-distribution represented by $(\boldsymbol{\alpha}, \boldsymbol{Q})$ is defined as the distribution of the absorption time to the absorption state 0 with the initial probability $\boldsymbol{\alpha}^*$ in the Markov process. We call $\boldsymbol{Q}$ a PH-subgenerator of the Markov process and $(\boldsymbol{\alpha}, \boldsymbol{Q})$ a PH-representation of the PH-distribution. The probability distribution function $F(t)$ and the density function $f(t)$ for a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ are given as

$$F(t) = 1 - \boldsymbol{\alpha} \exp(\boldsymbol{Q}t)\mathbf{1} \tag{2.2}$$

$$f(t) = \boldsymbol{\alpha} \exp(\boldsymbol{Q}t)\boldsymbol{q} \tag{2.3}$$

for $t \in [0, \infty)$. If $\alpha_0 \neq 0$, then the PH-distribution has a mass at time 0. In the case that $\alpha_0 = 0$, we can simply add $f(0) = \alpha_0$ to an absolutely continuous part of $f(t)$, $t > 0$ to obtain $f(t)$ for $t \in [0, \infty)$. In this paper, we therefore assume that $\alpha_0 = 0$ or, equivalently, $\boldsymbol{\alpha}\mathbf{1} = 1$. As shown in (2.2) and (2.3), $F(t)$ and $f(t)$ include the matrix exponential $\exp(\boldsymbol{Q}t)$ expressed in the infinite series of matrices, thus they are not closed forms, but rather open forms.

Characteristic polynomial and minimal polynomial: The characteristic polynomial for a matrix $\boldsymbol{Q}$ of a PH-subgenerator is defined as $c(t) = \det(t\boldsymbol{I} - \boldsymbol{Q})$, where $\boldsymbol{I}$ is the identity matrix and $c(t)$ is a monic polynomial, i.e., its leading coefficient is 1. Monic polynomials $p(t)$ that satisfy $p(\boldsymbol{A}) = \boldsymbol{0}$ are said to be annihilating polynomials for a square matrix $\boldsymbol{A}$, in general. It is well known that there is a unique annihilating polynomial of minimal degree for matrix $\boldsymbol{A}$, and this polynomial is called the minimal polynomial for matrix $\boldsymbol{A}$. We write $m(t)$ to denote the minimal polynomial for the PH-subgenerator $\boldsymbol{Q}$, i.e., $m(t)$ is a unique monic polynomial of minimal degree such that $m(\boldsymbol{Q}) = \boldsymbol{0}$. The Cayley-Hamilton theorem guarantees that dig$[m(t)] \leq n$. The minimal polynomial $m(t)$ divides the characteristic polynomial $c(t)$. It is well known that $\boldsymbol{Q}$ is diagonalisable if and only if $m(t)$ is a product of distinct linear factors, otherwise $\boldsymbol{Q}$ is similar to a Jordan canonical form.

Eigenvalues: The matrix $\boldsymbol{Q}$ has n eigenvalues, including multiplicity, and these form the solution of the characteristic equation $c(t) = 0$. The eigenvalues may be either real or complex. The index for an eigenvalue λ of $\boldsymbol{Q}$ is defined as the smallest positive integer k such that rank $\left[(\boldsymbol{Q} - \lambda \boldsymbol{I})^k\right] =$ rank $\left[(\boldsymbol{Q} - \lambda \boldsymbol{I})^{k+1}\right]$ and is written as index$(\lambda) = k$. The geometric multiplicity for an eigenvalue λ is equal to the maximum number of linearly independent eigenvectors associated with λ and is equivalent to the number of Jordan cells for λ in the Jordan canonical form of $\boldsymbol{Q}$.

From the Perron-Frobenius theory, we have the following lemma.

Lemma 2.1 The PH-subgenerator $\boldsymbol{Q}$ has at least one negative eigenvalue and the real part of the other eigenvalues are strictly negative.

Multiplicated eigenvalues: If every eigenvalue of a given PH-subgenerator $\boldsymbol{Q}$ is distinct from every other eigenvalue, or the minimal polynomial of $\boldsymbol{Q}$ is a product of distinct factors, then $\boldsymbol{Q}$ is simply diagonalisable and the problem becomes easy to analyse. PH-subgenerators, however, may have multiplicated eigenvalues and some may have a different eigenvalue index. For example, a PH-subgenerator $\boldsymbol{Q}_{(k)}$ of the k-Erlang distribution has one real eigenvalue λ, the algebraic multiplicity of which is k and the index of which is $k-1$. On the other hand, a PH-subgenerator $\boldsymbol{Q}_{(k_1+k_2)}$ of a mixture of k_1-Erlang and k_2-Erlang distributions, where $k_1 + k_2 = k$, has the same eigenvalue λ with the same algebraic multiplicity k as $\boldsymbol{Q}_{(k)}$. However, the index of the eigenvalue is different from $\boldsymbol{Q}_{(k)}$ in general. Thus, even though $\boldsymbol{Q}_{(k)}$ and $\boldsymbol{Q}_{(k_1+k_2)}$ have the same spectrum (a set of distinct eigenvalues) and the same algebraic multiplicity, the structure of these matrices are different. In order to determine the matrix structures of PH-subgenerators and their difference, we need the Jordan canonical forms of $\boldsymbol{Q}_{(k)}$ and $\boldsymbol{Q}_{(k_1+k_2)}$.

2.2. Multiplicated single real eigenvalue

In this subsection, we consider the most simple case, in which the PH-subgenerator $\boldsymbol{Q}$ $(n \times n)$ has only one multiplicated real eigenvalue λ with index m, so that the characteristic polynomial and the minimal polynomial are given in the form

$$c(t) = (t - \lambda)^n \quad \text{and} \quad m(t) = (t - \lambda)^m \tag{2.4}$$

for $m \leq n$. From Proposition 2.1, $\lambda < 0$. In this case, the algebraic multiplicity of λ is n and the largest Jordan cell size is m. The geometric multiplicity of λ is defined as $\dim N(\boldsymbol{Q} - \lambda \boldsymbol{I})$ where $N(\boldsymbol{Q} - \lambda \boldsymbol{I})$ is the nullspace of $(\boldsymbol{Q} - \lambda \boldsymbol{I})$, i.e., $N(\boldsymbol{Q} - \lambda \boldsymbol{I}) = \{\boldsymbol{x} | (\boldsymbol{Q} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}\}$.

The standard theory for the Jordan canonical form shows that $\boldsymbol{Q}$ is similar to the Jordan canonical form

$$\boldsymbol{J}(\lambda) = \bigoplus_{\ell=1}^{\ell(\lambda,1)} \boldsymbol{J}_\ell(\lambda, 1) \oplus \bigoplus_{\ell=1}^{\ell(\lambda,2)} \boldsymbol{J}_\ell(\lambda, 2) \oplus \cdots \oplus \bigoplus_{\ell=1}^{\ell(\lambda,m)} \boldsymbol{J}_\ell(\lambda, m) \tag{2.5}$$

where $\ell(\lambda, k)$ is the number of Jordan cells of size k for $k = 1, 2, \ldots, m$. In other words, there exists a regular matrix $\boldsymbol{U}$ such that

$$\boldsymbol{U}^{-1}\boldsymbol{Q}\boldsymbol{U} = \boldsymbol{J}(\lambda) \quad \text{or equivalently} \quad \boldsymbol{Q} = \boldsymbol{U}\boldsymbol{J}(\lambda)\boldsymbol{U}^{-1}. \tag{2.6}$$

In expression (2.5), if $\ell(\lambda, k) = 0$, then we simply skip $\boldsymbol{J}_\ell(\lambda, k)$, i.e., we assume $\boldsymbol{J}_\ell(\lambda, k) = \emptyset$ for $\ell(\lambda, k) = 0$. We also use the following notations in expression (2.5). We denote a Jordan cell of size k associated with the eigenvalue λ and a nilpotent matrix with index k (the smallest positive integer such that $(\boldsymbol{N}_k)^k = \boldsymbol{0}$) to give

$$\boldsymbol{J}_\ell(\lambda, k) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \quad \text{and} \quad \boldsymbol{N}_k = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

for $k \geq 2$. Both of these are $(k \times k)$ matrices. We define $\boldsymbol{N}_1 = (0)$ and $\boldsymbol{J}_\ell(\lambda, 1) = \lambda$. Note that,

$$\boldsymbol{J}_\ell(\lambda, k) = \lambda \boldsymbol{I}_k + \boldsymbol{N}_k \tag{2.7}$$

where $\boldsymbol{I}_k$ is an identity matrix of size k. The subscript ℓ in $\boldsymbol{J}_\ell(\lambda, k)$ is used to distinguish Jordan cells of the same size and same eigenvalue for $\ell = 1, 2, \ldots$ We write a block-diagonal matrix $\boldsymbol{A}$ with block diagonals of square matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ in the following forms

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & & & \\ & \boldsymbol{A}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{A}_n \end{bmatrix} = \boldsymbol{A}_1 \oplus \boldsymbol{A}_2 \oplus \cdots \oplus \boldsymbol{A}_n \quad \text{or} \quad \boldsymbol{A} = \bigoplus_{\ell=1}^{n} \boldsymbol{A}_\ell.$$

We apply the Jordan chain technique to accomplish this similarity transformation (2.6) (see Appendix A in details). Let $\boldsymbol{U}_\ell(\lambda, k)$ be the right-side Jordan chain block associated with Jordan cell $\boldsymbol{J}_\ell(\lambda, k)$, then the regular matrix $\boldsymbol{U}$ is given in the form

$$\boldsymbol{U} = \left((\boldsymbol{U}_\ell(\lambda, k))_{\ell=1}^{\ell(\lambda,k)} \right)_{k=1}^{m} \tag{2.8}$$

Note that $\boldsymbol{U}_\ell(\lambda, k)$ is an $(n \times k)$ matrix that consists of k right-side Jordan chain column vectors. In the form (2.8), we write $\boldsymbol{U} = (\boldsymbol{U}_{\boldsymbol{i}})_{i=1}^{\ell}$ to denote the matrix $\boldsymbol{U} = (\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_\ell)$ for the matrices $\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_\ell$ having the same row size.

Matrix exponential form: The matrix exponential form of $\boldsymbol{Q}t$ is written in the infinite series of the following matrices:

$$\exp(\boldsymbol{Q}t) = \boldsymbol{I} + t\boldsymbol{Q} + \frac{t^2}{2!}\boldsymbol{Q}^2 + \cdots + \frac{t^n}{n!}\boldsymbol{Q}^n + \cdots \tag{2.9}$$

From similarity transformation (2.6), the n-th power of matrix $\boldsymbol{Q}$ is given in the form

$$\boldsymbol{Q}^n = \boldsymbol{U}\boldsymbol{J}(\lambda)^n\boldsymbol{U}^{-1}. \tag{2.10}$$

Substituting (2.10) into (2.9), we can express matrix exponential $\exp(\boldsymbol{Q}t)$ in the following form:

$$\begin{aligned}\exp(\boldsymbol{Q}t) &= \boldsymbol{U}\exp(\boldsymbol{J}(\lambda)t)\boldsymbol{U}^{-1}\\ &= \boldsymbol{U}\left(\bigoplus_{k=1}^{m}\bigoplus_{\ell=1}^{\ell(\lambda,k)}\exp\left(\boldsymbol{J}_\ell(\lambda,k)t\right)\right)\boldsymbol{U}^{-1}. \qquad (2.11)\end{aligned}$$

Using the relation (2.7) and the property of nilpotent matrix $\boldsymbol{N}_k$, we derive that

$$\begin{aligned}\exp\left(\boldsymbol{J}_\ell(\lambda,k)\right) &= \exp(\lambda t\boldsymbol{I}_k)\exp(\boldsymbol{N}_k t)\\ &= e^{\lambda t}\exp(\boldsymbol{N}_k)\\ &= e^{\lambda t}\left(\boldsymbol{I}_k + t\boldsymbol{N}_k + \frac{t^2}{2!}\boldsymbol{N}_k^2 + \cdots + \frac{t^{k-1}}{(k-1)!}\boldsymbol{N}_k^{k-1}\right). \qquad (2.12)\end{aligned}$$

Together with (2.11) and (2.12), we can derive the closed form expression of matrix exponential $\exp(\boldsymbol{Q}t)$ to provide

$$\exp(\boldsymbol{Q}t) = e^{\lambda t}\boldsymbol{U}\left(\bigoplus_{k=1}^{m}\bigoplus_{\ell=1}^{\ell(\lambda,k)}\exp\left(\boldsymbol{N}_k t\right)\right)\boldsymbol{U}^{-1}. \qquad (2.13)$$

Density function: Let matrix $\boldsymbol{V}$ be an inverse matrix of $\boldsymbol{U}$, i.e., $\boldsymbol{V} = \boldsymbol{U}^{-1}$. We define a row vector and a column vector as

$$\boldsymbol{g}_\ell(\lambda,k) = \boldsymbol{\alpha}\boldsymbol{U}_\ell(\lambda,k) \quad \text{and} \quad \boldsymbol{h}_\ell(\lambda,k) = \boldsymbol{V}_\ell(\lambda,k)\boldsymbol{q}$$

for $k = 1, 2, \dots, m$, $\ell = 1, 2, \dots, \ell(\lambda, m)$, respectively, where $\boldsymbol{V}_\ell(\lambda,k)$ is a $(k \times n)$ submatarix in $\boldsymbol{U}^{-1}$ that appears at the corresponding position at which $\boldsymbol{U}_\ell(\lambda,k)$ appears in ${}^t\boldsymbol{U}$. Note that $\boldsymbol{V}_\ell(\lambda,k)$ consists of the generalised left-side eigenvectors associated with the Jordan cell $\boldsymbol{J}_\ell(\lambda,k)$. Hereinafter, we use the notation ${}^t\boldsymbol{A}$ to denote the transposition of $\boldsymbol{A}$, which is either a matrix or a vector.

To represent each element of $\boldsymbol{g}_\ell(\lambda,k)$ and $\boldsymbol{h}_\ell(\lambda,k)$, we denote

$$\boldsymbol{g}_\ell(\lambda,k) = (\gamma_{\ell,j}(\lambda,k))_{j=1}^{k} \quad \text{and} \quad \boldsymbol{h}_\ell(\lambda,k) = [\delta_{\ell,j}(\lambda,k)]_{j=1}^{k}$$

where we write a row vector $\boldsymbol{u} = (u_1, u_2, \dots, u_k)$ in the form $\boldsymbol{u} = (u_i)_{i=1}^k$, and a column vector $\boldsymbol{v} =^t (v_1, v_2, \dots, v_k)$ in the form $\boldsymbol{v} = [v_i]_{i=1}^k$.

Note that

$$\boldsymbol{g} = \boldsymbol{\alpha}\boldsymbol{U} \quad \text{and} \quad \boldsymbol{h} = \boldsymbol{V}\boldsymbol{q} \quad \left(= \boldsymbol{U}^{-1}\boldsymbol{q}\right)$$

where a row vector $\boldsymbol{g}$ and a column vector $\boldsymbol{h}$ are defined as

$$\boldsymbol{g} = \left((\boldsymbol{g}_\ell(\lambda,k))_{\ell=1}^{\ell(\lambda,k)}\right)_{k=1}^{m} \quad \text{and} \quad \boldsymbol{h} = \left[[\boldsymbol{h}_\ell(\lambda,k)]_{\ell=1}^{\ell(\lambda,k)}\right]_{k=1}^{m}.$$

Using this notation, we have the following proposition.

Proposition 2.1 The probability density function $f(t)$ of a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ is given in the closed form

$$f(t) = \sum_{n=0}^{m-1}\varphi_n(\lambda)\frac{t^n}{n!}e^{\lambda t} \qquad (2.14)$$

where

$$\varphi_n(\lambda) = \sum_{k=n+1}^{m} \sum_{\ell=1}^{\ell(\lambda,k)} \sum_{s=1}^{k-n} \gamma_{\ell,s}(\lambda,k) \cdot \delta_{\ell,s+n}(\lambda,k) \quad \text{for} \quad n = 0, 1, \ldots, m-1. \tag{2.15}$$

Proof. Since N_k is a nilpotent matrix, we can write $\exp(\boldsymbol{N}_k t)$ in the form

$$\begin{aligned}
\exp(\boldsymbol{N}_k t) &= \boldsymbol{I}_k + t\boldsymbol{N}_k + \frac{t^2}{2!}\boldsymbol{N}_k^2 + \cdots + \frac{t^{k-1}}{(k-1)!}\boldsymbol{N}_k^{k-1} \\
&= \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \frac{t^2}{2!} & & \vdots \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & & \ddots & t \\ & & & & & 1 \end{bmatrix}.
\end{aligned} \tag{2.16}$$

Substituting the matrix exponential form (2.13) into (2.3) and using (2.16), we obtain

$$\begin{aligned}
f(t) &= \boldsymbol{a}\exp(\boldsymbol{Q}t)\boldsymbol{q} \\
&= e^{\lambda t}\boldsymbol{a}\boldsymbol{U}\left(\bigoplus_{k=1}^{m}\bigoplus_{\ell=1}^{\ell(\lambda,k)} \exp(\boldsymbol{N}_k t)\right)\boldsymbol{U}^{-1}\boldsymbol{q} \\
&= e^{\lambda t}\boldsymbol{g}\left(\bigoplus_{k=1}^{m}\bigoplus_{\ell=1}^{\ell(\lambda,k)} \exp(\boldsymbol{N}_k t)\right)\boldsymbol{h} \\
&= e^{\lambda t}\sum_{k=1}^{m}\sum_{\ell=1}^{\ell(\lambda,k)} \boldsymbol{g}_\ell(\lambda,k)\exp(\boldsymbol{N}_k t)\,\boldsymbol{h}_\ell(\lambda,k) \\
&= \sum_{n=0}^{m-1}\frac{t^n}{n!}e^{\lambda t}\sum_{k=n+1}^{m}\sum_{\ell=1}^{\ell(\lambda,k)}\sum_{s=1}^{k-n}\gamma_{\ell,s}(\lambda,k)\cdot\delta_{\ell,s+n}(\lambda,k) \qquad \blacksquare
\end{aligned}$$

Remark 2.1 If we write $\mathcal{E}_{n,\lambda}$ for Erlang(n, λ) density

$$\mathcal{E}_{n,\lambda}(t) = \frac{(-\lambda)^n t^n}{(n-1)!}e^{\lambda t}, \quad t \geq 0,$$

then we can rewrite the density function (2.14) in the form

$$f(t) = \sum_{n=1}^{m} \zeta_n(\lambda)\mathcal{E}_{n,\lambda}(t) \quad \text{where} \quad \zeta_n(\lambda) = \frac{\varphi_{n-1}(\lambda)}{(-\lambda)^n}. \tag{2.17}$$

The expression (2.17) implies that the density function $f(t)$ is a linear combination of Erlang distributions. Here, we use the term “linear combination” instead of “mixture ” because the coefficient $\zeta_n(\lambda)$ may be either positive or negative. $\blacksquare$

Remark 2.2 Laplace-Stieltjes transform (LST) of a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$, with $\boldsymbol{Q}$ an $(n \times n)$ matrix, is given as

$$\phi(\theta) = \boldsymbol{\alpha} \left(\theta \boldsymbol{I} - \boldsymbol{Q}\right)^{-1} \boldsymbol{q} \quad \text{for} \quad \mathrm{Re}(\theta) > 0$$

which is a rational function of θ. The degree of a PH-distribution is defined to be the degree of the denominator of $\phi(\theta)$. Proposition 2.1 asserts that the degree $D(\boldsymbol{\alpha}, \boldsymbol{Q})$ of the PH-distribution is not equal to the order n of the PH-distribution, but rather is smaller or equal to degree m of the minimal polynomial, i.e., $D(\boldsymbol{\alpha}, \boldsymbol{Q}) \leq m$. In other words, the dominant parameter of a PH-distribution is not the order of the PH-subgenerator, but rather the degree of the minimal polynomial. ∎

2.3. Multiplicated multiple real eigenvalues

Generalising the results obtained in the previous section, we consider a PH-subgenerator $\boldsymbol{Q}$ $(n \times n)$ with r $(0 < r \leq n)$ real and different eigenvalues λ_1, λ_2, ..., λ_r with index m_1, m_2, ... , m_r and with algebraic multiplicity n_1, n_2, ... , n_r, respectively. Thus, the characteristic polynomial and the minimal polynomial of $\boldsymbol{Q}$ are

$$c(t) = \prod_{i=1}^{r} (t - \lambda_i)^{n_i} \quad \text{and} \quad m(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}, \quad (m_i \leq n_i). \tag{2.18}$$

The standard theory for the Jordan canonical form shows again that $\boldsymbol{Q}$ is similar to the Jordan canonical form

$$\boldsymbol{J}(\lambda_1, \lambda_2, \ldots, \lambda_r) = \bigoplus_{i=1}^{r} \boldsymbol{J}(\lambda_i) \tag{2.19}$$

where

$$\boldsymbol{J}(\lambda_i) = \bigoplus_{\ell=1}^{\ell(\lambda_i,1)} \boldsymbol{J}_\ell(\lambda_i, 1) \oplus \bigoplus_{\ell=1}^{\ell(\lambda_i,2)} \boldsymbol{J}_\ell(\lambda_i, 2) \oplus \cdots \oplus \bigoplus_{\ell=1}^{\ell(\lambda_i,m_i)} \boldsymbol{J}_\ell(\lambda_i, m_i) \tag{2.20}$$

and $\ell(\lambda_i, k)$ is the number of Jordan cells of size k associated with eigenvalue λ_i for $k = 1, 2, \ldots, m_i$. If $\ell(\lambda_i, k) = 0$, then $\boldsymbol{J}_\ell(\lambda_i, k) = \emptyset$. There exists a regular matrix $\boldsymbol{U}$ such that

$$\boldsymbol{U}^{-1}\boldsymbol{Q}\boldsymbol{U} = \boldsymbol{J}(\lambda_1, \lambda_2, \ldots, \lambda_r) \quad \text{or equivalently} \quad \boldsymbol{Q} = \boldsymbol{U}\boldsymbol{J}(\lambda_1, \lambda_2, \ldots, \lambda_r)\boldsymbol{U}^{-1}. \tag{2.21}$$

Applying a procedure similar to that described in the previous subsection to each eigenvalue λ_i, we can build Jordan chains associated with the Jordan segment $\boldsymbol{J}(\lambda_i)$ for $i = 1, 2, \ldots, r$.

Let $\boldsymbol{U}_\ell(\lambda_i, k)$ be the right-side Jordan chains associated with Jordan cell $\boldsymbol{J}_\ell(\lambda_i, k)$, then the regular matrix $\boldsymbol{U}$ is given in the form

$$\boldsymbol{U} = \left(\left((\boldsymbol{U}_\ell(\lambda_i, k))_{\ell=1}^{\ell(\lambda_i,k)} \right)_{k=1}^{m_i} \right)_{i=1}^{r}$$

The matrix exponential form of $\boldsymbol{Q}t$ is given such that

$$\begin{aligned} \exp(\boldsymbol{Q}t) &= \boldsymbol{U} \exp\left(\boldsymbol{J}(\lambda_1, \lambda_2, \ldots, \lambda_r)t\right) \boldsymbol{U}^{-1} \\ &= \boldsymbol{U} \left(\bigoplus_{i=1}^{r} \bigoplus_{k=1}^{m_i} \bigoplus_{\ell=1}^{\ell(\lambda_i,k)} \exp\left(\boldsymbol{J}_\ell(\lambda_i, k)t\right) \right) \boldsymbol{U}^{-1} \\ &= \boldsymbol{U} \left(\bigoplus_{i=1}^{r} \bigoplus_{k=1}^{m_i} \bigoplus_{\ell=1}^{\ell(\lambda_i,k)} e^{\lambda_i t} \cdot \exp\left(\boldsymbol{N}_k t\right) \right) \boldsymbol{U}^{-1}. \end{aligned} \tag{2.22}$$

We define matrix $\boldsymbol{V}$ as the inverse matrix $\boldsymbol{U}^{-1}$, i.e., $\boldsymbol{UV} = \boldsymbol{I}$, such that

$$\boldsymbol{V} = \left[\left[[\boldsymbol{V}_\ell(\lambda_i, k)]_{\ell=1}^{\ell(\lambda_i,k)} \right]_{k=1}^{m_i} \right]_{i=1}^{r}$$

where $\boldsymbol{V}_\ell(\lambda_i, k)$ is a $(k \times n)$ submatarix in $\boldsymbol{U}^{-1}$ that appears at the corresponding position at which $\boldsymbol{U}_\ell(\lambda_i, k)$ appears in ${}^t\boldsymbol{U}$. Note that $\boldsymbol{V}_\ell(\lambda_i, k)$ consists of generalised left-side eigenvectors associated with the Jordan cell $\boldsymbol{J}_\ell(\lambda_i, k)$.

The row vector $\boldsymbol{g}$ and the column vector $\boldsymbol{h}$ are defined as

$$\boldsymbol{g} = \boldsymbol{\alpha U} \quad \text{and} \quad \boldsymbol{h} = \boldsymbol{Vq} \quad \left(= \boldsymbol{U}^{-1}\boldsymbol{q}\right).$$

To represent each element of $\boldsymbol{g}$ and $\boldsymbol{h}$, we denote

$$\boldsymbol{g}_\ell(\lambda_i, k) = (\gamma_{\ell,j}(\lambda_i, k))_{j=1}^{k} \quad \text{and} \quad \boldsymbol{h}_\ell(\lambda_i, k) = [\delta_{\ell,j}(\lambda_i, k)]_{j=1}^{k}.$$

Then, we have

$$\boldsymbol{g} = \left(\left((\boldsymbol{g}_\ell(\lambda_i, k))_{\ell=1}^{\ell(\lambda_i,k)} \right)_{k=1}^{m_i} \right)_{i=1}^{r} \quad \text{and} \quad \boldsymbol{h} = \left[\left[[\boldsymbol{h}_\ell(\lambda_i, k)]_{\ell=1}^{\ell(\lambda_i,k)} \right]_{k=1}^{m_i} \right]_{i=1}^{r}.$$

Using this notation, we have the following proposition:

Proposition 2.2 The probability density function $f(t)$ of a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ is given in the following closed form:

$$f(t) = \sum_{i=1}^{r} \sum_{n=0}^{m_i-1} \varphi_n(\lambda_i) \frac{t^n}{n!} e^{\lambda_i t} \tag{2.23}$$

where

$$\varphi_n(\lambda_i) = \sum_{k=n+1}^{m_i} \sum_{\ell=1}^{\ell(\lambda_i,k)} \sum_{s=1}^{k-n} \gamma_{\ell,s}(\lambda_i, k) \cdot \delta_{\ell,s+n}(\lambda_i, k) \quad \text{for} \quad n = 0, 1, \ldots, m_i - 1. \quad \blacksquare \tag{2.24}$$

The proof is similar to that given in Proposition 2.1 and so will not be presented here.

Remark 2.3 We can rewrite the density function (2.23) as a linear combination of Erlang distribution $\mathcal{E}_{n,\lambda}$ in the form

$$f(t) = \sum_{i=1}^{r} \sum_{n=1}^{m_i} \zeta_n(\lambda_i) \mathcal{E}_{n,\lambda_i}(t) \quad \text{where} \quad \zeta_n(\lambda_i) = \frac{\varphi_{n-1}(\lambda_i)}{(-\lambda_i)^n}. \quad \blacksquare \tag{2.25}$$

Remark 2.4 Proposition 2.2 asserts that the degree $D(\boldsymbol{\alpha}, \boldsymbol{Q})$ of the PH-distribution is smaller than or equal to the summation of index for each eigenvalue, i.e., $D(\boldsymbol{\alpha}, \boldsymbol{Q}) \leq m_1 + m_2 + \cdots + m_r$.$\blacksquare$

2.4. Examples

Example 2.1 for a PH-subgenerator with a multiplicated single eigenvalue. Consider a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$, where for $\nu > 0$

$$\boldsymbol{Q} = \boldsymbol{Q}_1 \oplus \boldsymbol{Q}_2 \oplus \boldsymbol{Q}_3, \tag{2.26}$$

$$\boldsymbol{Q}_1 = \begin{bmatrix} -3\nu & \nu \\ 0 & -3\nu \end{bmatrix}, \quad \boldsymbol{Q}_2 = \begin{bmatrix} -3\nu & 2\nu \\ 0 & -3\nu \end{bmatrix}, \quad \boldsymbol{Q}_3 = \begin{bmatrix} -3\nu & 2\nu & \nu \\ 0 & -3\nu & \nu \\ 0 & 0 & -3\nu \end{bmatrix},$$

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_7)$, $\alpha_i > 0$ for $i = 1, 2, \ldots, 7$ and $\boldsymbol{q} = -\boldsymbol{Q}\boldsymbol{1} = ^t (2\nu, 3\nu, \nu, 3\nu, 0, 2\nu, 3\nu)$.

In this example, the characteristic polynomial and the minimal polynomial for $\boldsymbol{Q}$ are

$$c(t) = (t + 3\nu)^7 \quad \text{and} \quad m(t) = (t + 3\nu)^3 \tag{2.27}$$

so that algebraic multiplicity of the eigenvalue -3ν is 7 and index$(-3\nu) = 3$, i.e., $m = 3$. Thus the largest Jordan cell size is 3. Applying Jordan chain technique, we can derive Jordan chains $\boldsymbol{U}_1(-3\nu, 2)$, $\boldsymbol{U}_2(-3\nu, 2)$, $\boldsymbol{U}_1(-3\nu, 3)$ in the following procedure (see Appendix for the definition of notations M_k and N_k). Solving linear equation $(\boldsymbol{Q}+3\nu\boldsymbol{I})^k\boldsymbol{x} = \boldsymbol{0}$ for $k = 3, 2, 1$, we see that $M_3 = \{\boldsymbol{e}_7, \boldsymbol{e}_6, \ldots, \boldsymbol{e}_1\}$, $M_2 = \{\boldsymbol{e}_6, \boldsymbol{e}_5, \ldots, \boldsymbol{e}_1\}$ and $M_1 = \{\boldsymbol{e}_5, \boldsymbol{e}_3, \boldsymbol{e}_1\}$, where $\boldsymbol{e}_i$ is a unit i-th column vector for $i = 1, 2, \ldots, 7$. Consequently complementary spaces N_3, N_2 and N_1 are fixed such that $N_3 = \{\boldsymbol{e}_7\}$, $N_2 = \{\boldsymbol{e}_6, \boldsymbol{e}_4, \boldsymbol{e}_2\}$ and $N_1 = \{\boldsymbol{e}_5, \boldsymbol{e}_3, \boldsymbol{e}_1\}$, respectively. Since $\ell(-3\nu, 3) = \dim N_3 = 1$, $\ell(-3\nu, 2) = \dim N_2 - \dim N_3 = 2$, $\ell(-3\nu, 1) = \dim N_1 - \dim N_2 = 0$, there exist one Jordan cell of size 3, two Jordan cells of size 2 and no Jordan cell of size 1. To accomplish Jordan chains $\boldsymbol{U}_1(-3\nu, 3)$ $\boldsymbol{U}_1(-3\nu, 2)$ and $\boldsymbol{U}_2(-3\nu, 2)$, we derive that $(\boldsymbol{Q} + 3\nu\boldsymbol{I})\boldsymbol{e}_7 = ^t(0, 0, 0, 0, \nu, \nu, 0)$, $(\boldsymbol{Q} + 3\nu\boldsymbol{I})^2\boldsymbol{e}_7 = ^t (0, 0, 0, 0, 2\nu^2, 0, 0)$, $(\boldsymbol{Q} + 3\nu\boldsymbol{I})\boldsymbol{e}_4 = {}^t(0, 0, 2\nu, 0, 0, 0, 0)$ and $(\boldsymbol{Q} + 3\nu\boldsymbol{I})\boldsymbol{e}_2 = {}^t(\nu, 0, 0, 0, 0, 0, 0)$. Using these Jordan chains, the regular matrix $\boldsymbol{U}$ is accomplished in the form

$$\begin{aligned}
\boldsymbol{U} &= (\boldsymbol{U}_1(-3\nu, 2), \boldsymbol{U}_2(-3\nu, 2), \boldsymbol{U}_1(-3\nu, 3)) \\
&= \left[\begin{bmatrix} \nu & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2\nu & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\nu^2 & \nu & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} \nu & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 2\nu & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 2\nu^2 & \nu & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{2.28}$$

so that the inverse matrix of $\boldsymbol{U}^{-1}(= \boldsymbol{V})$ is given in the form

$$\boldsymbol{U}^{-1} = \begin{bmatrix} \frac{1}{\nu} & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{2\nu} & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{2\nu^2} & -\frac{1}{2\nu^2} & 0 \\ 0 & \frac{1}{\nu} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.29}$$

We can obtain the Jordan canonical form

$$\begin{aligned}
\boldsymbol{J}(-3\lambda) &= \boldsymbol{U}^{-1}\boldsymbol{Q}\boldsymbol{U} \\
&= \begin{bmatrix} -3\nu & 1 \\ 0 & -3\nu \end{bmatrix} \oplus \begin{bmatrix} -3\nu & 1 \\ 0 & -3\nu \end{bmatrix} \oplus \begin{bmatrix} -3\nu & 1 & 0 \\ 0 & -3\nu & 1 \\ 0 & 0 & -3\nu \end{bmatrix}.
\end{aligned} \tag{2.30}$$

Since

$$\exp(\boldsymbol{N}_2 t) = \boldsymbol{I} + t\boldsymbol{N}_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \exp(\boldsymbol{N}_3 t) = \boldsymbol{I} + t\boldsymbol{N}_3 + \frac{t^2}{2!}\boldsymbol{N}_3^2 = \begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

we can derive the relation

$$\begin{aligned}\exp(\boldsymbol{Q}t) &= e^{-3\nu t}\boldsymbol{U}\left(\exp(\boldsymbol{N}_2)\oplus\exp(\boldsymbol{N}_2)\oplus\exp(\boldsymbol{N}_3)\oplus\right)\boldsymbol{U}^{-1}\\ &= e^{-3\nu t}\boldsymbol{U}\left(\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & t & \frac{t^2}{2!}\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}\right)\boldsymbol{U}^{-1}\end{aligned}$$

Simple calculation yields

$$\boldsymbol{g} = \boldsymbol{\alpha}\boldsymbol{U} = \big(\alpha_1\nu, \alpha_2, 2\alpha_3\nu, \alpha_4, 2\alpha_5\nu^2, (\alpha_5+\alpha_6)\nu, \alpha_7\big)$$

and

$$\boldsymbol{h} = \boldsymbol{U}^{-1}\boldsymbol{q} =^t \left(2,\ 3\nu,\ \frac{1}{2},\ 3\nu,\ -\frac{1}{\nu},\ 2,\ 3\nu\right).$$

Consequently, closed form of the PH-distribution is derived such that

$$\begin{aligned}f(t) &= \boldsymbol{\alpha}\exp(\boldsymbol{Q}t)\boldsymbol{q}\\ &= e^{-3\nu}\boldsymbol{g}\left(\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & t & \frac{t^2}{2!}\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}\right)\boldsymbol{h}\\ &= \varphi_0(-3\nu)e^{-3\nu t} + \varphi_1(-3\nu)te^{-3\nu t} + \varphi_2(-3\nu)\frac{t^2}{2!}e^{-3\nu t}\end{aligned}$$

where

$$\begin{aligned}\varphi_0(-3\nu) &= (2\alpha_1+3\alpha_2+\alpha_3+3\alpha_4+2\alpha_6+3\alpha_7)\,\nu\\ \varphi_1(-3\nu) &= (3\alpha_1+6\alpha_3+7\alpha_5+3\alpha_6)\,\nu^2\\ \varphi_2(-3\nu) &= 6\alpha_5\nu^3.\end{aligned}$$

One can see immediately that the degree $D(\boldsymbol{\alpha},\boldsymbol{Q})$ of the PH-distribution given in this Example is equal to not the order 7 of the PH-subgenerator but smaller than or equal to the degree 3 of the minimal polynomial, i.e., $D(\boldsymbol{\alpha},\boldsymbol{Q}) \le 3$. Thus one can confirm Remark 2.4. ∎

Example 2.2 for PH-subgenerator with multiplicated multiple eigenvalues. Consider a PH-distribution with representation $(\boldsymbol{\alpha},\boldsymbol{Q})$, where for $\nu > 0$

$$\boldsymbol{Q} = \begin{bmatrix}-\nu & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\nu & -\nu & 0 & 0 & 0 & 0\\ 0 & 0 & -2\nu & \nu & 0 & 0 & 0\\ 0 & 0 & \nu & -2\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -3\nu & \nu & 0\\ 0 & 0 & 0 & 0 & 0 & -3\nu & 2\nu\\ 0 & 0 & 0 & 0 & 0 & 0 & -3\nu\end{bmatrix} \tag{2.31}$$

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_7)$, $\alpha_i > 0$ for $i = 1, 2, \ldots, 7$, and $\boldsymbol{q} = -\boldsymbol{Q}\mathbf{1} =^t (\nu, 0, \nu, \nu, 2\nu, \nu, 3\nu)$. Note that this example contains a feedback loop, i.e., $\boldsymbol{Q}$ is not a triangular matrix. Eigenvalues of $\boldsymbol{Q}$ are $-\nu$ and -3ν. The characteristic polynomial and the minimal polynomial for $\boldsymbol{Q}$ are

$$c(t) = (t+\nu)^3(t+3\nu)^4 \quad \text{and} \quad m(t) = (t+\nu)^2(t+3\nu)^3 \tag{2.32}$$

so that the algebraic multiplicity of eigenvalues $-\nu$ and -3ν are 3 and 4, index$(-\nu) = 2$ and index$(-3\nu) = 3$, i.e., $m_1 = 2$, $m_2 = 3$, respectively. The largest Jordan cell sizes for $-\nu$ and -3ν are 2 and 3, respectively.

Proceeding for each eigenvalue according to a procedure similar to that of the previous example, we see that there exist two Jordan cells of sizes 1 and 2 for the eigenvalue $-\nu$, and two Jordan cells of sizes 1 and 3 for the eigenvalue -3ν, respectively. Solving linear equations, we can build Jordan chains in the form

$$(\boldsymbol{U}_1(-\nu,1), \boldsymbol{U}_1(-\nu,2)) = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \nu & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right]$$

and

$$(\boldsymbol{U}_1(-3\nu,1), \boldsymbol{U}_1(-3\nu,3)) = \left[\begin{bmatrix} 0 \\ \frac{1}{2} \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\nu^2 & 0 & 0 \\ 0 & 2\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

so that

$$\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\nu^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{U}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\nu} & \dfrac{1}{4\nu} & -\dfrac{1}{4\nu} & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{2} & \dfrac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{2} & \dfrac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{2\nu^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{2\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.33}$$

Consequently, the Jordan canonical form of $\boldsymbol{Q}$ is given in the form

$$\begin{aligned} \boldsymbol{J}(-\nu,-3\nu) &= \boldsymbol{U}^{-1}\boldsymbol{Q}\boldsymbol{U} \\ &= \boldsymbol{J}_1(-\nu,1) \oplus \boldsymbol{J}_1(-\nu,2) \oplus \boldsymbol{J}_1(-3\nu,1) \oplus \boldsymbol{J}_1(-3\nu,3) \\ &= \begin{bmatrix} -\nu \end{bmatrix} \oplus \begin{bmatrix} -\nu & 1 \\ 0 & -\nu \end{bmatrix} \oplus \begin{bmatrix} -3\nu \end{bmatrix} \oplus \begin{bmatrix} -3\nu & 1 & 0 \\ 0 & -3\nu & 1 \\ 0 & 0 & -3\nu \end{bmatrix}. \end{aligned} \tag{2.34}$$

We can obtain the relation

$$\begin{aligned} \exp(\boldsymbol{Q}t) &= \boldsymbol{U}\left(e^{-\nu t}(\exp(\boldsymbol{N}_1 t) \oplus \exp(\boldsymbol{N}_2 t)) \oplus e^{-3\nu t}(\exp(\boldsymbol{N}_1 t) \oplus \exp(\boldsymbol{N}_3 t)\right)\boldsymbol{U}^{-1} \\ &= \boldsymbol{U}\left(e^{-\nu t}\left([1] \oplus \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) \oplus e^{-3\nu t}\left([1] \oplus \begin{bmatrix} 1 & t & \dfrac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\right)\right)\boldsymbol{U}^{-1}. \end{aligned}$$

Simple calculation yields that

$$\boldsymbol{g} = \boldsymbol{\alpha U} = \left(\alpha_1,\ \alpha_2\nu,\ \alpha_3+\alpha_4,\ \frac{\alpha_2}{2}-\alpha_3+\alpha_4,\ 2\alpha_5\nu^2,\ 2\alpha_6\nu,\ \alpha_7,\ \right)$$

and

$$\boldsymbol{h} = \boldsymbol{U}^{-1}\boldsymbol{q} = {}^t\left(\nu,\ 0,\ \nu,\ 0,\ \frac{1}{\nu},\ \frac{1}{2},\ 3\nu\right).$$

Therefore we can obtain closed form of the probability density function in the form

$$\begin{aligned} f(t) &= \boldsymbol{\alpha}\exp(\boldsymbol{Q}t)\boldsymbol{q} \\ &= \boldsymbol{g}\left(e^{-\nu t}\left([1]\oplus\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\right)\oplus e^{-3\nu t}\left([1]\oplus\begin{bmatrix}1 & t & \frac{t^2}{2!}\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}\right)\right)\boldsymbol{h} \\ &= \varphi_0(-\nu)e^{-\nu t}+\varphi_1(-\nu)te^{-\nu t}+\varphi_0(-3\nu)e^{-3\nu t}+\varphi_1(-3\nu)te^{-3\nu t}+\varphi_2(-3\nu)\frac{t^2}{2!}e^{-3\nu t} \end{aligned}$$

where

$$\begin{aligned} \varphi_0(-\nu) &= (\alpha_1+\alpha_3+\alpha_4)\nu & \varphi_1(-\nu) &= \alpha_2\nu^2 \\ \varphi_0(-3\nu) &= (2\alpha_5+\alpha_6+3\alpha_7)\nu & \varphi_1(-3\nu) &= (\alpha_5+6\alpha_6)\nu^2 \\ \varphi_2(-3\nu) &= 6\alpha_5\nu^3. \quad \blacksquare \end{aligned}$$

One can see immediately that the degree of the PH-distribution given in this example is not the order 7 of the PH-subgenerator but smaller than or equal to the summation of index for each eigenvalue $-\nu$ and -3ν, i.e., $D(\boldsymbol{\alpha},\boldsymbol{Q}) \le 2+3$. Thus one can confirm Remark 2.4, ∎

3. PH-subgenerator with Real and Complex Eigenvalues

3.1. Preliminary

The eigenvalues of $\boldsymbol{Q}$ may be either real or complex. We consider the cases that $\boldsymbol{Q}$ has real and complex eigenvalues and derive the general form of PH-distribution in closed form in this section. Since the PH-subgenerator $\boldsymbol{Q}$ consists of only real number elements, the complex eigenvalues occur in conjugate pairs, i.e., if σ is a complex eigenvalue of $\boldsymbol{Q}$, then $\bar{\sigma}$ is also a complex eigenvalue of $\boldsymbol{Q}$, where $\bar{\sigma}$ denotes the conjugate complex number of σ. Thus, without loss of generality, we assume that $\boldsymbol{Q}$ has r real eigenvalues $\{\lambda_i\}_{i=1}^r$ and s conjugate pairs of complex eigenvalues $\{\sigma_i,\bar{\sigma}_i\}_{i=r+1}^{r+s}$, so that the characteristic polynomial $c(t)$ can be written in the form

$$c(t) = \prod_{i=1}^{r}(t-\lambda_i)^{n_i}\prod_{i=r+1}^{r+s}((t-\sigma_i)(t-\bar{\sigma}_i))^{n_i} \tag{3.1}$$

where n_i is the algebraic multiplicity of the i-th eigenvalue for $i = 1,2,\ldots,r+s$.

In the same way, we can write the minimal polynomial $m(t)$ as

$$m(t) = \prod_{i=1}^{r}(t-\lambda_i)^{m_i}\prod_{i=r+1}^{r+s}((t-\sigma_i)(t-\bar{\sigma}_i))^{m_i} \tag{3.2}$$

where m_i is the index of the i-th eigenvalue and $m_i \le n_i$ for $i = 1,2,\ldots,r+s$.

To derive Jordan canonical form of $\boldsymbol{Q}$, the following auxiliary lemmas are required:

Lemma 3.1 If $\boldsymbol{x}$ is a generalised right-side eigenvector in a generalised right-side eigenspace $M_k(\sigma)$ for σ_i, i.e., $\boldsymbol{x} \in M_k(\sigma) = \{\boldsymbol{x}|(\boldsymbol{Q} - \sigma_i \boldsymbol{I})^k \boldsymbol{x} = \boldsymbol{0}\}$, then $\bar{\boldsymbol{x}}$ is a generalised right-side eigenvector in a generalised right-side eigenspace $M_k(\bar{\sigma})$ for $\bar{\sigma}_i$, i.e., $\bar{\boldsymbol{x}} \in M_k(\bar{\sigma}) = \{\boldsymbol{x}|(\boldsymbol{Q} - \bar{\sigma}_i \boldsymbol{I})^k \boldsymbol{x} = \boldsymbol{0}\}$.
Proof. If $(\boldsymbol{Q} - \sigma_i \boldsymbol{I})^k \boldsymbol{x} = \boldsymbol{0}$, then $\overline{(\boldsymbol{Q} - \sigma_i \boldsymbol{I})^k \boldsymbol{x}} = (\boldsymbol{Q} - \bar{\sigma}_i \boldsymbol{I})^k \bar{\boldsymbol{x}} = \boldsymbol{0}$. ∎

Lemma 3.2 Let $\ell(\sigma_i, k)$ and $\ell(\bar{\sigma}_i, k)$ be the number of size-k Jordan cells for a pair of conjugate complex eigenvalues σ_i and $\bar{\sigma}_i$, respectively, then for $i = r+1, r+2, \dots, r+s$ and $k = 1, 2, \dots, m_i$,

$$\ell(\sigma_i, k) = \ell(\bar{\sigma}_i, k).$$

Proof. Both $\ell(\sigma_i, k)$ and $\ell(\bar{\sigma}_i, k)$ are determined through the relation (A.2) given in Appendix. From lemma 3.1, the relation $\dim M_k(\sigma_i) = \dim M_k(\bar{\sigma}_i)$ is valid for $i = r+1, r+2, \dots, r+s$ and $k = 1, 2, \dots, m_i$. ∎

3.2. A multiplicated real eigenvalue and a pair of conjugate complex eigenvalues

In this subsection, we consider a simple case in which the PH-subgenerator $\boldsymbol{Q}$ $(n \times n)$ has only one multiplicated real eigenvalue λ and a pair of conjugate complex eigenvalues $(\sigma, \bar{\sigma})$ with multiplication. We assume that the characteristic polynomial and the minimal polynomial of $\boldsymbol{Q}$ are given in the form

$$c(t) = (t - \lambda)^{n_1} \left((t-\sigma)(t-\bar{\sigma})\right)^{n_2} \quad \text{and} \quad m(t) = (t-\lambda)^{m_1} \left((t-\sigma)(t-\bar{\sigma})\right)^{m_2} \tag{3.3}$$

for $m_1 \leq n_1$ and $m_2 \leq n_2$. From Lemma 2.1, $\lambda < 0$ and $\mathrm{Re}(\sigma) < 0$.

Using Lemma 3.1 and Lemma 3.2, discussion similar to that in previous sections indicates that $\boldsymbol{Q}$ is similar to the Jordan canonical form

$$\boldsymbol{J}(\lambda, \sigma, \bar{\sigma}) = \boldsymbol{J}(\lambda) \oplus \boldsymbol{J}(\sigma) \oplus \boldsymbol{J}(\bar{\sigma}) \tag{3.4}$$

where

$$\boldsymbol{J}(\lambda) = \bigoplus_{k=1}^{m_1} \bigoplus_{\ell=1}^{\ell(\lambda,k)} \boldsymbol{J}_\ell(\lambda, k), \quad \boldsymbol{J}(\sigma) = \bigoplus_{k=1}^{m_2} \bigoplus_{\ell=1}^{\ell(\sigma,k)} \boldsymbol{J}_\ell(\sigma, k) \quad \text{and} \quad \boldsymbol{J}(\bar{\sigma}) = \overline{\boldsymbol{J}(\sigma)}. \tag{3.5}$$

There exists a regular matrix $\boldsymbol{U}$ such that

$$\boldsymbol{U}^{-1} \boldsymbol{Q} \boldsymbol{U} = \boldsymbol{J}(\lambda, \sigma, \bar{\sigma}) \quad \text{or equivalently} \quad \boldsymbol{Q} = \boldsymbol{U} \boldsymbol{J}(\lambda, \sigma, \bar{\sigma}) \boldsymbol{U}^{-1}. \tag{3.6}$$

Let $\boldsymbol{U}_\ell(\lambda, k)$ and $\boldsymbol{U}_\ell(\sigma, k)$ be the right-side Jordan chains associated with Jordan cells $\boldsymbol{J}_\ell(\lambda, k)$ and $\boldsymbol{J}_\ell(\sigma, k)$, respectively, The regular matrix $\boldsymbol{U}$ is then given in the form

$$\boldsymbol{U} = (\boldsymbol{U}(\lambda), \boldsymbol{U}(\sigma), \boldsymbol{U}(\bar{\sigma}))$$

where

$$\boldsymbol{U}(\lambda) = \left((\boldsymbol{U}_\ell(\lambda, k))_{\ell=1}^{\ell(\lambda,k)} \right)_{k=1}^{m_1}, \quad \boldsymbol{U}(\lambda) = \left((\boldsymbol{U}_\ell(\sigma, k))_{\ell=1}^{\ell(\sigma,k)} \right)_{k=1}^{m_2} \quad \text{and} \quad \boldsymbol{U}(\bar{\sigma}) = \overline{\boldsymbol{U}(\sigma)}.$$

We define matrix $\boldsymbol{V}$ as the inverse matrix $\boldsymbol{U}$, i.e., $\boldsymbol{U}\boldsymbol{V} = \boldsymbol{I}$, such that

$$\boldsymbol{V} = \left[\boldsymbol{V}(\lambda), \boldsymbol{V}(\sigma), \boldsymbol{V}(\bar{\sigma})\right],$$

$$\boldsymbol{V}(\lambda) = \left[[\boldsymbol{V}_\ell(\lambda, k)]_{\ell=1}^{\ell(\lambda,k)} \right]_{k=1}^{m_1}, \quad \boldsymbol{V}(\sigma) = \left[[\boldsymbol{V}_\ell(\sigma, k)]_{\ell=1}^{\ell(\sigma,k)} \right]_{k=1}^{m_2}, \quad \boldsymbol{V}(\bar{\sigma}) = \overline{\boldsymbol{V}(\sigma)}$$

where $\boldsymbol{V}_\ell(\lambda,k)$ and $\boldsymbol{V}_\ell(\sigma,k)$ are $(k\times n)$ submatarices in $\boldsymbol{U}^{-1}$ that appear at the corresponding positions at which $\boldsymbol{U}_\ell(\lambda,k)$ and $\boldsymbol{U}_\ell(\sigma,k)$ appear in ${}^t\boldsymbol{U}$, respectively. The row vector $\boldsymbol{g}$ and the column vector $\boldsymbol{h}$ are defined as

$$\boldsymbol{g}=\boldsymbol{\alpha U} \quad \text{and} \quad \boldsymbol{h}=\boldsymbol{Vq} \quad \left(=\boldsymbol{U}^{-1}\boldsymbol{q}\right).$$

To represent each element of $\boldsymbol{g}$ and $\boldsymbol{h}$, we write

$$\boldsymbol{g}_\ell(\lambda,k)=\boldsymbol{\alpha U}_\ell(\lambda,k)=(\gamma_{\ell,j}(\lambda,k))_{j=1}^k, \quad \boldsymbol{h}_\ell(\lambda,k)=\boldsymbol{V}_\ell(\lambda,k)\boldsymbol{q}=[\delta_{\ell,j}(\lambda,k)]_{j=1}^k$$

for λ

$$\boldsymbol{g}_\ell(\sigma,k)=\boldsymbol{\alpha U}_\ell(\sigma,k)=(\gamma_{\ell,j}(\sigma,k))_{j=1}^k, \quad \boldsymbol{h}_\ell(\sigma,k)=\boldsymbol{V}_\ell(\sigma,k)\boldsymbol{q}=[\delta_{\ell,j}(\sigma,k)]_{j=1}^k$$

and

$$\boldsymbol{g}_\ell(\bar\sigma,k)=\boldsymbol{\alpha U}_\ell(\bar\sigma,k)=(\gamma_{\ell,j}(\bar\sigma,k))_{j=1}^k, \quad \boldsymbol{h}_\ell(\bar\sigma,k)=\boldsymbol{V}_\ell(\bar\sigma,k)\boldsymbol{q}=[\delta_{\ell,j}(\bar\sigma,k)]_{j=1}^k$$

for σ and $\bar\sigma$, respectively.

Using this notation, we can derive the probability density function in the form

$$\begin{aligned} f(t) &= \boldsymbol{\alpha}\exp(\boldsymbol{Q}t)\boldsymbol{q} \\ &= e^{\lambda t}\sum_{k=1}^{m_1}\sum_{\ell=1}^{\ell(\lambda,k)}\boldsymbol{g}_\ell(\lambda,k)\exp(\boldsymbol{N}_k t)\boldsymbol{h}_\ell(\lambda,k) \\ &+ e^{\sigma t}\sum_{k=1}^{m_2}\sum_{\ell=1}^{\ell(\sigma,k)}\boldsymbol{g}_\ell(\sigma,k)\exp(\boldsymbol{N}_k t)\boldsymbol{h}_\ell(\sigma,k) \\ &+ e^{\bar\sigma t}\sum_{k=1}^{m_2}\sum_{\ell=1}^{\ell(\bar\sigma,k)}\boldsymbol{g}_\ell(\bar\sigma,k)\exp(\boldsymbol{N}_k t)\boldsymbol{h}_\ell(\bar\sigma,k). \end{aligned} \tag{3.7}$$

After some calculation, we can transform (3.7) into the form

$$f(t) = \sum_{n=0}^{m_1-1}\frac{t^n}{n!}e^{\lambda t}\varphi_n(\lambda)+\sum_{n=0}^{m_2-1}\frac{t^n}{n!}\left\{e^{\sigma t}\eta_n(\sigma)+e^{\bar\sigma t}\eta_n(\bar\sigma)\right\} \tag{3.8}$$

where

$$\varphi_n(\lambda) = \sum_{k=n+1}^{m_1}\sum_{\ell=1}^{\ell(\lambda,k)}\sum_{s=1}^{k-n}\gamma_{\ell,s}(\lambda,k)\delta_{\ell,s+n}(\lambda,k) \tag{3.9}$$

$$\eta_n(\sigma) = \sum_{k=n+1}^{m_2}\sum_{\ell=1}^{\ell(\sigma,k)}\sum_{s=1}^{k-n}\gamma_{\ell,s}(\sigma,k)\delta_{\ell,s+n}(\sigma,k) \quad \text{and} \quad \eta_n(\bar\sigma)=\overline{\eta_n(\sigma)}. \tag{3.10}$$

Let

$$\mu(\sigma)=\mathrm{Re}(\sigma), \quad \nu(\sigma)=\mathrm{Im}(\sigma)$$

and for $n=0,1,\ldots,m_2-1$

$$\tau_n(\sigma)=|\eta_n(\sigma)|, \quad \theta_n(\sigma)=\arg\eta_n(\sigma), \quad \psi_n(\sigma,t)=2\tau_n(\sigma)\cos\{\nu(\sigma)t+\theta_n(\sigma)\}$$

The probability density function is then given in the following proposition.

Proposition 3.1 The probability density function $f(t)$ of a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ is given in the closed form

$$f(t) = \sum_{n=0}^{m_1-1} \varphi_n(\lambda)\frac{t^n}{n!}e^{\lambda t} + \sum_{n=0}^{m_2-1} \psi_n(\sigma, t)\frac{t^n}{n!}e^{\mu(\sigma)t}. \tag{3.11}$$

Proof. Substitution of the following relation to (3.8) yields the statement.

$$e^{\sigma t}\eta_n(\sigma) + e^{\bar{\sigma}t}\eta_n(\bar{\sigma}) = 2\mathrm{Re}\{e^{\sigma t}\eta_n(\sigma)\} = 2\tau_n(\sigma)e^{\mu(\sigma)t}\cos\{\nu(\sigma)t + \theta_n(\sigma)\}. \quad \blacksquare$$

Remark 3.1 As shown in Proposition 3.1, if $\boldsymbol{Q}$ has complex eigenvalues, then the closed form of $f(t)$ is no longer a linear combination of Erlang distributions, but instead is of more complex form that includes trigonometric functions.

Example 3.1 for a PH-subgenerator with a real eigenvalue and a pair of conjugate complex eigenvalues with multiplication. Consider a PH-distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ where

$$\boldsymbol{Q} = \begin{bmatrix} -3 & 2 & 0.5 & 0 & 0 & 0 \\ 0.5 & -4 & 1.5 & 0 & 0 & 0 \\ 3.5 & 0.5 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0.5 & 3.5 \\ 0 & 0 & 0 & 1.5 & -4 & 0.5 \\ 0 & 0 & 0 & 0.5 & 2 & -3 \end{bmatrix}, \quad \boldsymbol{\alpha} = (0.2, 0, 0, 0.5, 0.3, 0). \tag{3.12}$$

Eigenvalues λ, σ and $\bar{\sigma}$ of $\boldsymbol{Q}$ are $\lambda = -1.122$, $\sigma = -5.438 + 1.307i$ and $\bar{\sigma} = -5.438 - 1.307i$, respectively, and $\boldsymbol{q} =^t (0.5, 2, 1, 1, 2, 0.5)$. In this example, the characteristic polynomial and the minimal polynomial for $\boldsymbol{Q}$ are

$$c(t) = (t+\lambda)^2(t+\sigma)^2(t+\bar{\sigma})^2 \quad \text{and} \quad m(t) = (t+\lambda)(t+\sigma)(t+\bar{\sigma}) \tag{3.13}$$

so that all of these eigenvalues have the same algebraic multiplicity of 2 and the same index of 1, i.e., $n_1 = n_2 = 2$ and $m_1 = m_2 = 1$. Clearly, we have

$$\mu(\sigma) = -5.438 \quad \text{and} \quad \nu(\sigma) = 1.307.$$

Numerical computation shows that $\boldsymbol{U}$ is given in the form

$$\boldsymbol{U} = \begin{bmatrix} -0.637 & 0 & 0 & -0.042+0.341i & 0 & -0.042-0.341i \\ -0.440 & 0 & 0 & -0.358-0.444i & 0 & -0.358+0.444i \\ -0.632 & 0 & 0 & 0.745 & 0 & 0.745 \\ 0 & 0.632 & 0.745 & 0 & 0.745 & 0 \\ 0 & 0.440 & -0.358-0.444i & 0 & -0.358+0.444i & 0 \\ 0 & 0.637 & -0.042+0.341i & 0 & -0.042-0.341i & 0 \end{bmatrix}$$

so that the Jordan canonical form of $\boldsymbol{Q}$ is in the form

$$\begin{aligned} \boldsymbol{J}(\lambda, \sigma, \bar{\sigma}) &= \boldsymbol{U}^{-1}\boldsymbol{Q}\boldsymbol{U} \\ &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \oplus \begin{bmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{bmatrix}. \end{aligned}$$

By numerical computation, we see that

$$\begin{aligned} \boldsymbol{g} &= (-0.127, 0.448, 0.265-0.133i, -0.0084+0.0683i, 0.265+0.133i, -0.0084-0.0683i) \\ \boldsymbol{h} &= {}^t(-1.978, 1.978, -0.168+1.134i, -0.168+1.134i, -0.168-1.134i, -0.168-1.134i). \end{aligned}$$

Based on these numerical results, we derive that

$$\varphi_0(\lambda) = 1.139 \quad \text{and} \quad \eta_0(\sigma) = 0.0304 + 0.302i.$$

Consequently, we have

$$\tau_0(\sigma) = 0.305 \quad \text{and} \quad \theta_0(\sigma) = 1.470.$$

The closed form of the PH-distribution is then given in the form

$$\begin{aligned} f(t) &= \varphi_0(\lambda)e^{-\lambda} + 2\tau_0(\sigma)e^{\mu(\sigma)t}\cos\{\nu(\sigma)t + \theta_0(\sigma)\} \\ &= 1.139\ e^{-1.122t} + 0.610\ e^{-5.438t}\cos\{1.307t + 1.470\}. \qquad \blacksquare \end{aligned} \tag{3.14}$$

Remark 3.2 The PH-representation (3.12) given in example 3.1 is consistent but redundant. Representation $(\boldsymbol{\alpha},\boldsymbol{Q})$, where

$$\boldsymbol{Q} = \begin{bmatrix} -3 & 2 & 0.5 \\ 0.5 & -4 & 1.5 \\ 3.5 & 0.5 & -5 \end{bmatrix}, \quad \boldsymbol{\alpha} = (0.2, 0.3, 0.5)$$

is equivalent to the PH-representation (3.12).

4. General Form

Generalising the results described in previous sections, we can obtain the closed form of the PH-distribution for the general PH-representation of $(\boldsymbol{\alpha}, \boldsymbol{Q})$. We consider subgenerator $\boldsymbol{Q}$, the characteristic polynomial $c(t)$ and minimal polynomial $m(t)$ of which are given in the forms of (3.1) and (3.2), respectively.

Let $\boldsymbol{U}_\ell(\lambda_i, k)$, $i = 1, 2, \ldots, r$, and $\boldsymbol{U}_\ell(\sigma_i, k)$, $i = r+1, r+2, \ldots, r+s$, be the right-side Jordan chains associated with Jordan cell $\boldsymbol{J}_\ell(\lambda_i, k)$ and $\boldsymbol{J}_\ell(\sigma_i, k)$, respectively, and let $\boldsymbol{V}_\ell(\lambda_i, k)$, $i = 1, 2, \ldots, r$, and $\boldsymbol{V}_\ell(\sigma_i, k)$, $i = r+1, r+2, \ldots, r+s$ be $(k \times n)$ submatarices in $\boldsymbol{U}^{-1}$ that appear the corresponding positions at which $\boldsymbol{U}_\ell(\lambda_i, k)$ and $\boldsymbol{U}_\ell(\sigma_i, k)$ appear in ${}^t\boldsymbol{U}$. We write

$$\boldsymbol{g}_\ell(\lambda_i, k) = \boldsymbol{\alpha}\boldsymbol{U}_\ell(\lambda_i, k) = (\gamma_{\ell,j}(\lambda_i, k))_{j=1}^k, \quad \boldsymbol{h}_\ell(\lambda_i, k) = \boldsymbol{V}_\ell(\lambda_i, k)\boldsymbol{q} = [\delta_{\ell,j}(\lambda_i, k)]_{j=1}^k$$

for $i = 1.2. \ldots, r$ and

$$\boldsymbol{g}_\ell(\sigma_i, k) = \boldsymbol{\alpha}\boldsymbol{U}_\ell(\sigma_i, k) = (\gamma_{\ell,j}(\sigma_i, k))_{j=1}^k, \quad \boldsymbol{h}_\ell(\sigma_i, k) = \boldsymbol{V}_\ell(\sigma_i, k)\boldsymbol{q} = [\delta_{\ell,j}(\sigma_i, k)]_{j=1}^k$$

$$\boldsymbol{g}_\ell(\bar{\sigma}_i, k) = \boldsymbol{\alpha}\boldsymbol{U}_\ell(\bar{\sigma}_i, k) = (\gamma_{\ell,j}(\bar{\sigma}_i, k))_{j=1}^k, \quad \boldsymbol{h}_\ell(\bar{\sigma}_i, k) = \boldsymbol{V}_\ell(\bar{\sigma}_i, k)\boldsymbol{q} = [\delta_{\ell,j}(\bar{\sigma}_i, k)]_{j=1}^k$$

for $i = r+1, r+2. \ldots, r+s$. For $i = 1, 2, \ldots, r$, we define that

$$\varphi_n(\lambda_i) = \sum_{k=n+1}^{m_i} \sum_{\ell=1}^{\ell(\lambda_i,k)} \sum_{s=1}^{k-n} \gamma_{\ell,s}(\lambda_i, k)\delta_{\ell,s+n}(\lambda_i, k), \quad n = 0, 1, \ldots, m_i - 1$$

and for $i = r+1, r+2, \ldots, r+s$,

$$\begin{aligned} \eta_n(\sigma_i) &= \sum_{k=n+1}^{m_i} \sum_{\ell=1}^{\ell(\sigma_i,k)} \sum_{s=1}^{k-n} \gamma_{\ell,s}(\sigma_i, k)\delta_{\ell,s+n}(\sigma_i, k), \\ \eta_n(\bar{\sigma}_i) &= \overline{\eta_n(\sigma_i)}, \quad n = 0, 1, \ldots, m_i - 1. \end{aligned}$$

Furthermore, we define that

$$\mu(\sigma_i) = \mathrm{Re}(\sigma_i), \quad \nu(\sigma_i) = \mathrm{Im}(\sigma_i), \quad \tau_n(\sigma_i) = |\eta_n(\sigma_i)|, \quad \theta_n(\sigma_i) = \arg \eta_n(\sigma_i)$$

for $i = r+1, r+2, \ldots, r+s$ and for $n = 0, 1, \ldots, m_i - 1$.

The main result of the present paper is summarised in the following Proposition.

Proposition 4.1 The probability density function $f(t)$ of a PH-distribution with general representation $(\boldsymbol{\alpha}, \boldsymbol{Q})$ is given in the closed form

$$f(t) \;=\; \sum_{i=1}^{r}\sum_{n=0}^{m_i-1} \varphi_n(\lambda_i)\frac{t^n}{n!}e^{\lambda_i t} + \sum_{i=r+1}^{r+s}\sum_{n=0}^{m_i-1} \psi_n(\sigma_i, t)\frac{t^n}{n!}e^{\mu(\sigma_i)t} \qquad \blacksquare \tag{4.1}$$

The proof is straightforward and so is not presented here.

5. Conclusion

In this paper, we proposed a new method of Jordan canonical form to analyze a PH-subgenerator. Based on the new approach, we derive the closed form of PH-distribution and show that the closed form of PH-distribution is expressed in terms of a linear combination of Erlang distributions if all eigenvalues of the PH-subgenerator are real, while it includes ripple terms consisting of trigonometric functions in addition to a linear combination of Erlang distribution if the PH-subgenerator has complex eigenvalues. We conclude that the new method is effective and may provide clear insights into the study of PH-distribution.

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References

[1] S. Asumussen and M. Bladt: Renewal theory and queueing algorithms for matrix-exponential distributions. *Proc. MAM1*, (1996), 313-341.

[2] T. Han and M. Iri: *Jordan Canonical Form* (University of Tokyo Press, Tokyo, 1982) (in Japanese).

[3] G. Latouche and P. Taylor: *Matrix-Analytic Methods: Theory and Applications* (World Scientific, New Jersey, 2002).

[4] C.D. Meyer: *Matrix Analysis and Applied Linear Algebra* (SIAM, 2000).

[5] M.F. Neuts: *Matrix-geometric solutions in stochastic models* (Dover Publications, N.Y., 1981).

[6] M.F. Neuts: *Structured Stochastic Matrices of M/G/1 Type and Their Applications* (Marcel Dekker, 1989).

[7] C.A. O'Cinneide: Characterisation of phase-type distributions. *Stochastic Models*, **6-1** (1990), 1-57.

[8] C.A. O'Cinneide: Phase-type distributions and invariant polytope. *Advances in Applied Probability*, **23** (1991), 515-535.

[9] C.A. O'Cinneide: Triangular order of triangular phase-type distributions. *Stochastic Models*, **9-4** (1993), 507-529.

[10] C.A. O'Cinneide: Phase-type distributions: open problems and a few properties. *Stochastic Models*, **15-4** (1999), 731-757.

[11] E. Seneta: *Non-Negative Matrices: an Introduction to Theory and Applications* (John Wiley & Sons, N.Y., 1973).

Appendix

A. Jordan chain

We apply the Jordan chain technique in order to accomplish this similarity transformation (2.6) in the following manner. Let Ω_λ be the generalised right-side eigenspace for λ, i.e., $\Omega_\lambda = \{\boldsymbol{x}|(\boldsymbol{Q}-\lambda\boldsymbol{I})^m\boldsymbol{x} = \boldsymbol{0}\}$ and M_k be the subspace of Ω_λ such that

$$M_k = \{\boldsymbol{x}|(\boldsymbol{Q}-\lambda\boldsymbol{I})^k\boldsymbol{x} = \boldsymbol{0}\} \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{A.1}$$

It is clear that $\Omega_\lambda = M_m \supseteq M_{m-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = \{\boldsymbol{0}\}$ and $M_{m+1} = M_{m+2} = \cdots = \Omega_\lambda$.

All elements in these subspaces are called generalised right-side eigenvectors. The subspace M_1 is the eigenspace for eigenvalue λ. The geometric multiplicity and the algebraic multiplicity for the eigenvalue λ are equal to $\dim M_1$ and $\dim \Omega_\lambda$, respectively.

Let N_k be the complementary space of M_{k-1} in M_k for $k \geq 2$, i.e., $N_k \cup M_{k-1} = M_k$ and $N_k \cap M_{k-1} = \emptyset$ for $k = 2, 3, \ldots, m$, then for linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p$ in N_k, $(\boldsymbol{Q}-\lambda\boldsymbol{I})^\ell\boldsymbol{x}_1$, $(\boldsymbol{Q}-\lambda\boldsymbol{I})^\ell\boldsymbol{x}_2$, and $(\boldsymbol{Q}-\lambda\boldsymbol{I})^\ell\boldsymbol{x}_p$ are also linearly independent vectors, and these are generalised right-side eigenvectors in M_{k-l}. For $\ell = 1, 2, \ldots, m$, $\dim N_\ell = \dim M_\ell - \dim M_{\ell-1}$.

If we write $d_\ell = \dim N_\ell$ for $\ell = 1, 2, \ldots, m$ and $d_{m+1} = 0$, then the number of Jordan cells with size k is given as $\ell(\lambda, k) = d_k - d_{k+1}$ for $k = 1, 2, \ldots, m$, or equivalently

$$\ell(\sigma_i, k) = -\dim M_{k-1}(\sigma) + 2\dim M_k(\sigma) - \dim M_{k+1}(\sigma). \tag{A.2}$$

Thus, we can calculate $\ell(\lambda, k)$ of expression (2.5) for $k = 1, 2, \ldots, m$. Note that

$$\dim M_1 = \sum_{k=1}^{m} \ell(\lambda, k) \quad \text{and} \quad \dim \Omega_\lambda = \sum_{k=1}^{m} k \times \ell(\lambda, k).$$

In order to derive the regular matrix $\boldsymbol{U}$ in (2.6), we build Jordan chains. Since $\ell(\lambda, m) = \dim N_m$, we can fix $\ell(\lambda, m)$ bases in N_m such that

$$\boldsymbol{B}_m = \left(\boldsymbol{b}_1^{(m)}, \boldsymbol{b}_2^{(m)}, \ldots, \boldsymbol{b}_{\ell(\lambda,m)}^{(m)}\right).$$

Starting with each base $\boldsymbol{b}_\ell^{(m)} \in \boldsymbol{B}_m$, we can build a Jordan chain to provide

$$\boldsymbol{U}_\ell(\lambda, m) = \left((\boldsymbol{Q}-\lambda\boldsymbol{I})^{m-1}\boldsymbol{b}_\ell^{(m)},\ (\boldsymbol{Q}-\lambda\boldsymbol{I})^{m-2}\boldsymbol{b}_\ell^{(m)},\ \ldots, (\boldsymbol{Q}-\lambda\boldsymbol{I})\boldsymbol{b}_\ell^{(m)},\ \boldsymbol{b}_\ell^{(m)}\right) \tag{A.3}$$

for $\ell = 1, 2, \ldots, \ell(\lambda.m)$. For notation simplicity, for $j = 1, 2, \ldots, m-1$, we write

$$\boldsymbol{u}_{\ell,m-j}^{(m)} = (\boldsymbol{Q}-\lambda\boldsymbol{I})^j\boldsymbol{b}_\ell^{(m)} \quad \text{and} \quad \boldsymbol{u}_{\ell,m}^{(m)} = \boldsymbol{b}_\ell^{(m)}.$$

The Jordan chain (A.3) is then written in the form

$$\boldsymbol{U}_\ell(\lambda, m) = \left(\boldsymbol{u}_{\ell,1}^{(m)}, \boldsymbol{u}_{\ell,2}^{(m)}, \ldots, \boldsymbol{u}_{\ell,m}^{(m)}\right) \quad \text{for} \quad \ell = 1, 2, \ldots, \ell(\lambda, m). \tag{A.4}$$

The Jordan chain (A.4) is associated with the Jordan cell $\boldsymbol{J}_\ell(\lambda, m)$. Since $\{(\boldsymbol{Q}-\lambda\boldsymbol{I})\boldsymbol{b}_\ell^{(m)}\}_{\ell=1}^{\ell(\lambda.m)}$ are linearly independent vectors with index $m-1$ in M_{m-1}, there exists N_{m-1} that includes these vectors.

Therefore, we can choose additional linearly independent vectors $\{\boldsymbol{b}_\ell^{(m-1)}\}_{\ell=1}^{\ell(\lambda,m-1)}$ such that, together with $\{(\boldsymbol{Q}-\lambda\boldsymbol{I})\boldsymbol{b}_\ell^{(m)}\}_{\ell=1}^{\ell(\lambda.m)}$ $\left(=\{\boldsymbol{u}_{\ell,m-1}^{(m)}\}_{\ell=1}^{\ell(\lambda,m)}\right)$, $\{\boldsymbol{b}_\ell^{(m-1)}\}_{\ell=1}^{\ell(\lambda,m-1)}$ can constitute bases of N_{m-1}. We thus write

$$\boldsymbol{B}_{m-1}=\left(\boldsymbol{b}_1^{(m-1)},\boldsymbol{b}_2^{(m-1)},\ldots,\boldsymbol{b}_{\ell(\lambda,m-1)}^{(m-1)}\right).$$

Based on the bases of $\boldsymbol{B}_{m-1}$, we can again build a Jordan chain to provide

$$\boldsymbol{U}_\ell(\lambda,m-1)=\left((\boldsymbol{Q}-\lambda\boldsymbol{I})^{m-2}\boldsymbol{b}_\ell^{(m-1)},\ (\boldsymbol{Q}-\lambda\boldsymbol{I})^{m-3}\boldsymbol{b}_\ell^{(m-1)},\ \ldots,(\boldsymbol{Q}-\lambda\boldsymbol{I})\boldsymbol{b}_\ell^{(m-1)},\ \boldsymbol{b}_\ell^{(m-1)}\right)$$

for $\ell=1,2,\ldots,\ell(\lambda.m-1)$. For $j=1,2,\ldots,\ell(\lambda,m-1)$ $\boldsymbol{u}_{\ell,m-j}^{(m-1)}=(\boldsymbol{Q}-\lambda\boldsymbol{I})^j\boldsymbol{b}_\ell^{(m-1)}$ and $\boldsymbol{u}_{\ell,m-1}^{(m-1)}=\boldsymbol{b}_\ell^{(m-1)}$ we have

$$\boldsymbol{U}_\ell(\lambda,m-1)=\left(\boldsymbol{u}_{\ell,1}^{(m-1)},\boldsymbol{u}_{\ell,2}^{(m-1)},\ldots,\boldsymbol{u}_{\ell,m-1}^{(m-1)}\right)$$

where $\ell=1,2,\ldots,\ell(\lambda,m-1)$. Since $\{(\boldsymbol{Q}-\lambda\boldsymbol{I})^2\boldsymbol{b}_\ell^{(m)}\}_{\ell=1}^{\ell(\lambda,m)}$ and $\{(\boldsymbol{Q}-\lambda\boldsymbol{I})\boldsymbol{b}_\ell^{(m-1)}\}_{\ell=1}^{\ell(\lambda,m-1)}$ are linearly independent vectors in M_{m-2}, there exists N_{m-2} that includes these vectors.

Therefore, we can choose additional linearly independent vectors $\{\boldsymbol{b}_\ell^{(m-2)}\}_{\ell=1}^{\ell(\lambda,m-2)}$ such that, together with $\{(\boldsymbol{Q}-\lambda\boldsymbol{I})^2\boldsymbol{b}_\ell^{(m)}\}_{\ell=1}^{\ell(\lambda.m)}$ $\left(=\{\boldsymbol{u}_{\ell,m-2}^{(m)}\}_{\ell=1}^{\ell(\lambda,m)}\right)$ and $\{(\boldsymbol{Q}-\lambda\boldsymbol{I})\boldsymbol{b}_\ell^{(m-1)}\}_{\ell=1}^{\ell(\lambda,m-1)}$ $\left(=\{\boldsymbol{u}_{\ell,m-1}^{(m-1)}\}_{\ell=1}^{\ell(\lambda,m-1)}\right)$, $\{\boldsymbol{b}_\ell^{(m-2)}\}_{\ell=1}^{\ell(\lambda,m-2)}$ can form bases of N_{m-2}.

Repeating this procedure, we can obtain all Jordan chains and construct the regular matrix $\boldsymbol{U}$ as

$$\boldsymbol{U}=\left((\boldsymbol{U}_\ell(\lambda,k))_{\ell=1}^{\ell(\lambda,k)}\right)_{k=1}^{m}\quad\text{where}\quad\boldsymbol{U}_\ell(\lambda,k)=\left(\boldsymbol{u}_{\ell,1}^{(k)},\boldsymbol{u}_{\ell,2}^{(k)},\ldots,\boldsymbol{u}_{\ell,k}^{(k)}\right).$$

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