

AN OPTIMAL INVESTIGATION IN TWO STAGE SEARCH WITH RECOGNITION ERRORS

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Abstract We consider an optimal search plan for a movable target. A searcher conducts a two-stage search consisting of a broad search and an investigating search. In the broad search phase, the searcher seeks for many uncertain signals featuring target. They are called contacts. There are three possible entities of the origin of a contact. They are a true target, false object or noise. When a contact is gained, the searcher executes the investigating search in the following step. He gives a judgment on the contact after it. At the moment, the searcher could make the errors of the first and the second kind. We model the search procedure taking account of these three possible entities and the two kinds of human errors and derive conditions of the optimal investigating time so as to maximize the detection probability of a target. By numerical examples, we show some optimal investigating plans and explain the properties of these optimal plans.

Keywords: Search, dynamic programming, two stage search

1. Introduction

We will consider an optimal search plan. A searcher seeks for a movable target within a certain area and within some limited time. We suppose that the area is large enough compared to the detection range of search sensors. In such a huge area, the searcher usually conducts a two-stage search consisting of a broad search and an investigating search with suitable sensors. The two-stage search procedure is shown in Figure 1. In the first broad search phase, the searcher uses long-range sensors to get rough information about the target. It is called as ‘contacts’. When he gets a contact, it is classified as the i -th class by the feature of the signal. He must investigate it precisely to confirm what the source of it is. This second stage search is called as investigating search, or investigation.

As we have supposed that the target is movable, it can run away or hide itself so the searcher must investigate the gained contact as soon as possible. There are three possible sources of a contact. They are a true target, false objects or system noise. When he executes the investigation of the contact for relatively long time period and cannot get any information from it, he wastes the operational time. It means that the target has moved away or the contact originates from noise. The searcher cannot tell them apart. On the other hand, if he quits the investigation too shortly, he may lose the chance to detect a true target. So it is often the case that the searcher decides reasonable investigating time period τ_i before he starts an investigation. Then the problem is how long the searcher should execute an investigation. Within the planned time τ_i , the searcher may finish the investigation or not. If he cannot finish the investigation, he must come back to the broad search. If he can, he gives judgment on a contact. But he can make the errors of the first and the second kind at the same time. When the searcher makes the error of the second

kind, he chases a vacant entity and loses his limited operational time.

In this paper, we will make optimal investigating plans under circumstances stated above. They are considering three possible sources and two kinds of errors in judgment. When we make an optimal plan, we maximize the detection probability of the true target until the end of the operational time.

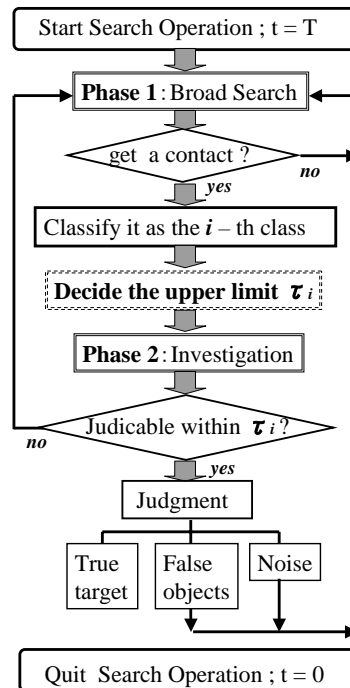


Figure 1: Procedure of the two-stage search operation

We generalize former studies focused on this investigating time. Several authors have studied the optimal investigating problem with false contacts. As for search problems with the false contacts by real objects resembling the true target, Stone and Stanshine [7], Stone et al. [8] and Dobbie [1] dealt with the optimal broad search including the investigating search uninterrupted until finishing the investigation. These studies are compiled in the textbook [9]. Sakurai [6] studied the optimal investigating plan both with time restriction and without time restriction. He assumed that contacts are caused by a true target or false objects. In the case of no time restriction, he made an optimal plan to minimize the expected time to detect a true target. When there is the time restriction, he made another plan to maximize the expected number of true targets and he studied the relation between the limited search time and the optimal investigating plan. Kisi [5] also studied both cases of the time restriction. But he studied them under the condition that the source of contacts was true target or noise. In the case of limitless search time, he assumed that there is a single target in a search area and the searcher gets many contacts of noise. He derived an optimal investigating plan to minimize the expected time by the detection of the target. In the case of limited time, he assumed that several targets are in the search area and he found an optimal search plan to maximize the expected number of the targets detected during the time period. Iida et al. [2] studied the time duration between the moment of the contact had happened and the beginning of the investigation. They made an optimal investigating plan so as to minimize the expected time until the searcher detects the target

without time restriction. Iida [3] expanded the model and considered the positional error and the multi-classes of the contact. Iida et al. [4] also studied the two-stage search in which noise contacts possibly happen. They showed the necessary and sufficient conditions for the optimal investigating plan. They made the optimal investigating plans so as to maximize the detection probability under the time restriction. These studies treat one of the false-target type contact or the noise type but none of both. However it often happens that both of them would take place in real operations. We will consider the models that contain those target types simultaneously.

In the next section, we describe the investigating search problem more precisely and present some assumptions of the model. We also explain variables used in our model. In Section 3, we make the investigating model. We adopt the dynamic programming procedure to formulate the search process. In Section 4, the properties of the optimal investigating search are elucidated. We show some numerical examples in Section 5.

2. Assumptions and Notation

We define the problem mentioned above and make some assumptions for our model.

Investigating Search Problem : When a searcher is taking place a broad search for a movable target, he gets a contact classified as the i -th class at time t left. How long should he investigate it? We will decide the optimal investigating time $\tau_i(t)$ so as to maximize the detection probability of the target until the end of the search operation.

[Assumptions]

1. Total operational time is T . The time is consumed continuously and is counted in the reverse order. Time t means that t is left until the end of the search operation.
2. A searcher searches a target within a limited area A . The target is sinking in water. So the target is assumed to distribute uniformly in the area. The searcher starts from the broad search in the random search manner; he searches the area for a target randomly with speed v and by a sensor with sweep width W . The search pattern is random in the sense that the path can be thought of as having its different portions placed independently and randomly of one another in A . By these assumptions, the searcher gets a contact with a Poisson rate $\lambda_0 = vW/A$ [4]. Parameter λ_0 is composed by Poisson arrival of signals from a true target and false objects. The searcher also gets system noise randomly. That process is also regarded as Poisson arrival process and we suppose the rate is λ_n . By the reproductivity, a contact originated from those three entities happens with Poisson rate $\lambda = \lambda_0 + \lambda_n$.
3. When the searcher gets a contact, it is classified into one of m classes according to the characteristic of the signal or certainty of being true. The probability that the contact belongs to the i -th class is p_{C_i} , $\sum_{i=1}^m p_{C_i} = 1$, $p_{C_i} \geq 0$ for all $i = 1, \dots, m$. The searcher knows the distribution of p_{C_i} from the past, accumulated data.
4. In the i -th contact class, the ratio that source of a gotten contact originated from a true target, false objects or noise is also known to the searcher by the previous data. From now on, we call the contact caused by a true target true contact. We also call false contact and noise contact in the same way. The ratios that the contact belongs to each source, each class are p_{T_i} , p_{F_i} and p_{N_i} respectively and they satisfy $0 \leq p_{T_i}, p_{F_i}, p_{N_i} \leq 1$ and $p_{T_i} + p_{F_i} + p_{N_i} = 1$ for each class i .
5. We suppose that the searcher can determine the upper limit of an investigating time $\tau_i(t)$ ($; t \in [0, T], 0 \leq \tau_i(t) \leq t$) as soon as he gets a contact at time t and belongs to the

- class i . At the same time, he can also start the investigating search if he decides that he should do it. There is no time lag among these actions.
6. The time consumed for the investigation is a random variable. If the contact belongs to the i -th class, it depends on the probability density functions $h_{T_i}(\tau)$ for the true contact and $h_{F_i}(\tau)$ for the false contact. We assume that both density functions are continuous. The cumulative density functions of them are denoted by $H_{T_i}(\tau)$ and $H_{F_i}(\tau)$.
 7. If the searcher cannot judge a contact within $\tau_i(t)$, he must return to the broad search immediately and search for a new contact. We assume that the searcher cannot get any other contact during the investigating phase.
 8. When the searcher can give judgment on a contact, the sequential search process branches into some cases. The details are depicted in Figure 2.
 - (a) When the searcher gets a true contact and judges it as ‘true’, the investigation will finish successfully and he moves on the next step action such as chasing and/or ready for attack. In the case that he misjudges the contact as ‘false’, he makes the error of the first kind and he returns to the broad search. The probability that he makes the mistake is denoted by a_T . We assumed that this value is constant through the operational period and independent of the contact class i and t .
 - (b) When the searcher gets a false contact and he judges it as ‘false’, he returns to the broad search. But if he makes a mistake and judges it as ‘true’, he steps into the next phase. He starts tracking of the target or preparing for attack. The probability that he makes the mistake is a_F . This value has the same properties of a_T . When the searcher is engaged himself in the next phase action, he cannot search for a true target and wastes the operational time. We call the period penalty time x . The penalty time x is a random variable depending on some continuous probability density function $g(x)$ ($; x \in [0, \max(t - \tau_i(t), t - t_i^A)]$). The variable t_i^A denotes actual investigating time, $0 \leq t_i^A \leq \tau_i(t)$. Just after the time x , the searcher realizes that he is chasing an imaginary object and he comes back to the broad search.
 - (c) If the searcher gets a noise contact, he may not be able to get any evidence of the target through $\tau_i(t)$, and after that period, he returns to the broad search.

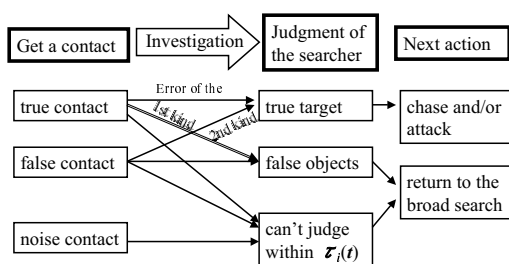


Figure 2: The sequential actions after a judgment

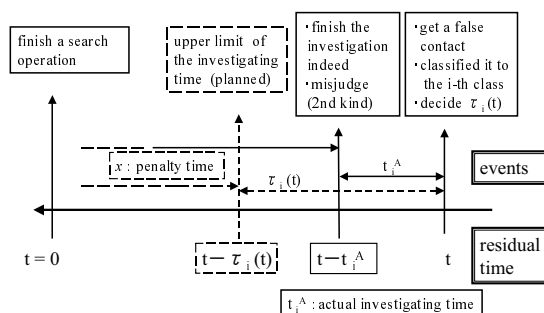


Figure 3: The procedure in the case of the error of the 2nd kind

3. Formulation and Algorithm

In this section, we formulate the two-stage search problem. We define two probabilities here. $G_i(t, \tau_i(t))$ denotes the conditional detection probability of the true target given that a class i of contact happens at t and the searcher investigates according to the decisions $\tau_i(t)$. $P(t)$ is the detection probability of the true target when the search is conducted according to the

optimal investigating plan $\{\tau_i^*(t)\}$ after time t . $\tau_i^*(t)$ is the optimal investigating time and should be decided by the searcher at t , that maximizes the conditional detection probability $G_i(t, \tau_i(t))$, i.e.,

$$G_i(t, \tau_i^*(t)) = \max_{0 \leq \tau_i(t) \leq t} G_i(t, \tau_i(t)). \quad (3.1)$$

During fractional time period $[t, t + \Delta t]$, the searcher may get a contact with probability $\lambda \Delta t$. Therefore we can estimate $P(t + \Delta t)$ as follows.

$$P(t + \Delta t) = \{1 - \lambda \Delta t\} P(t) + \lambda \Delta t \sum_{i=1}^m p_{C_i} G_i(t, \tau_i^*(t)). \quad (3.2)$$

In the right-hand side of Eq.(3.2), the first term is a product of the probabilities between the probability that no contact occurs during the fractional time and the detection probability after then. The second term indicates the probability that the contact does happen during $[t, t + \Delta t]$ and the searcher detects a true target by optimal search plans after then.

$G_i(t, \tau_i(t))$ consists of three parts; originates from a true contact, a false contact and a noise contact as follows.

$$\begin{aligned} G_i(t, \tau_i(t)) &= p_{T_i} \left[\int_0^{\tau_i(t)} h_{T_i}(u) \{ (1 - a_T) + a_T P(t - u) \} du + \{ 1 - H_{T_i}(\tau_i(t)) \} P(t - \tau_i(t)) \right] \\ &+ p_{F_i} \left[\int_0^{\tau_i(t)} h_{F_i}(u) \{ (1 - a_F) P(t - u) + a_F \int_0^{t-u} g(x) P(t - u - x) dx \} du \right. \\ &\quad \left. + \{ 1 - H_{F_i}(\tau_i(t)) \} P(t - \tau_i(t)) \right] + p_{N_i} P(t - \tau_i(t)) \quad (3.3) \end{aligned}$$

$$\begin{aligned} &= p_{T_i} \int_0^{\tau_i(t)} h_{T_i}(u) \{ (1 - a_T) + a_T P(t - u) \} du + p_{F_i} (1 - a_F) \int_0^{\tau_i(t)} h_{F_i}(u) P(t - u) du \\ &+ p_{F_i} a_F \int_0^{\tau_i(t)} h_{F_i}(u) \left\{ \int_0^{t-u} g(x) P(t - u - x) dx \right\} du \\ &+ \{ 1 - p_{T_i} H_{T_i}(\tau_i(t)) - p_{F_i} H_{F_i}(\tau_i(t)) \} P(t - \tau_i(t)). \quad (3.4) \end{aligned}$$

The first term of Eq.(3.3), starting with p_{T_i} , can be still divided into two parts. The former integral part means that the investigating search ends up to $\tau_i(t)$. At time u , the searcher finishes the investigation and makes a right judgment with probability $1 - a_T$. If he makes a mistake with probability a_T , he returns to the broad search but he has operational time $t - u$ left in hand. The latter part indicates the detection probability that the investigation does not finish by $\tau_i(t)$ and the searcher starts the broad search again. The second term is also divided into some pieces and each piece can be interpreted in the same manner as the first term. If the searcher makes the error of the second kind with probability a_F at time u , he moves into the next penalty phase action. He must be kept in the penalty phase for x . The penalty time could last for $t - u$ long at most. The third term indicates the case of a noise contact. The searcher investigates the contact in vain for $\tau_i(t)$ to return to the broad search finally.

We can derive the following differential equation from Eq.(3.2).

$$\frac{dP(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = \lambda \left\{ \sum_{i=1}^m p_{C_i} G_i(t, \tau_i^*(t)) - P(t) \right\}. \quad (3.5)$$

The initial conditions of Eq.(3.5) are as follows.

$$P(0) = G_i(0, 0) = 0 \quad (i = 1, \dots, m). \tag{3.6}$$

We will obtain the detection probability $P(t)$ at every operational time $t \in [0, T]$. To do this, beginning with conditions (3.6), we first calculate $G_i(t, \tau_i(t))$ for some t by changing $\tau_i(t)$ within $[0, t]$ and get the maximal value $G_i(t, \tau_i^*(t))$ for each i . Then we substitute those $G_i(t, \tau_i^*(t))$ s into Eq.(3.5) and calculate the increment $dP(t)$, or $dP(t)/dt \times dt$. Finally we can get the extended value $P(t + dt)$ by adding the increment to $P(t)$.

When $t \leq 0$, we define $P(t) = 0$. While there is no time to search, the searcher cannot increase the detection probability so the definition is natural. And we assume that there exists a $P(t)$ for any t within $[0, T]$. Then $G_i(t, \tau_i(t))$ are continuous and $dP(t)/dt$, convex combination of them, is also continuous. As a result, the detection probability $P(t)$ is continuous on $[0, T]$ and differentiable on $[0, T)$. $G_i(t, \tau_i(t))$ are defined on $[0, t]$ and bounded from above, there exist at least one maximal value $G_i(t, \tau_i^*(t))$ and the optimal investigating time $\tau_i^*(t)$ in every class i .

To get the maximal values of $G_i(t, \tau_i(t))$, we have to solve Eq.(3.4) optimally with respect to $\tau_i(t)$ but as shown in Eq.(3.4), $G_i(t, \tau_i(t))$ is so complicated that it is hard to solve and to get the optimal values of $G_i(t, \tau_i^*(t))$. From now on, we will adopt a discrete algorithm to get a numerical solution. By following the algorithm, we can get the maximized detection probability $P(T)$ and the optimal investigating plans $\{\tau_i^*(t)\}$ ($i = 1, \dots, m$) for every $t \in \{0, \Delta, 2\Delta, \dots, r\Delta(= T)\}$.

In the discrete algorithm, we divide the whole operational time $[0, T]$ into r fractional parts equally at first. We redefine functions $P(t)$, $G_i(t, \tau_i(t))$ at instant $t = l\Delta$ of the discretized search time, where $\Delta = T/r$ and modify other notation as follows.

$$\tau_{i,l} = \tau_i(l\Delta), \quad P_l = P(l\Delta), \quad G_{i,l}(\tau_{i,l}) = G_i(l\Delta, \tau_i(l\Delta)), \quad l = 0, 1, \dots, r.$$

Equation (3.5) is rewritten in the following difference equation.

$$P_{l+1} - P_l = \lambda \left\{ \sum_{i=1}^m p_{Ci} G_{i,l}^* - P_l \right\} \Delta, \quad \text{where} \quad G_{i,l}^* = \max_{\tau_{i,l} \in \{0, \Delta, \dots, l\Delta\}} G_{i,l}(\tau_{i,l}). \tag{3.7}$$

$G_{i,l}(\tau_{i,l})$ of Eq.(3.4) is also replaced by the following difference equation.

$$\begin{aligned} G_{i,l}(\tau_{i,l}) = & p_{Ti} \left\{ \sum_{k=0}^{\tau_{i,l}} h_{Ti}(k\Delta) \{ (1 - a_T) + a_T P_{l-k} \} \right\} \Delta + p_{Fi} \left\{ \sum_{k=0}^{\tau_{i,l}} h_{Fi}(k\Delta) (1 - a_F) P_{l-k} \right\} \Delta \\ & + p_{Fi} a_F \left\{ \sum_{k=0}^{\tau_{i,l}} h_{Fi}(k\Delta) \sum_{j=0}^{l-k} g(j\Delta) P_{l-k-j} \right\} \Delta^2 + \{ 1 - p_{Ti} H_{Ti}(\tau_{i,l}) - p_{Fi} H_{Fi}(\tau_{i,l}) \} P_{l-\tau_{i,l}}. \end{aligned} \tag{3.8}$$

From Eq.(3.7), we get the following relation.

$$P_{l+1} = P_l + \lambda \left\{ \sum_{i=1}^m p_{Ci} G_{i,l}^* - P_l \right\} \Delta \quad (l = 0, \dots, r). \tag{3.9}$$

To obtain $G_{i,l}^*$, we evaluate $G_{i,l}(\tau_{i,l})$ with changing $\tau_{i,l}$ for $0, \Delta, \dots, l\Delta$, using Eq.(3.8). By Eq.(3.9), the iterative calculation proceeds in the following manner from initial conditions $P_0 = G_{i,0}^* = 0$.

$$P_0 \rightarrow G_{i,0}^* \rightarrow P_1 \rightarrow G_{i,1}^* \rightarrow \dots \rightarrow P_l \rightarrow G_{i,l}^* \rightarrow P_{l+1} \rightarrow \dots \rightarrow G_{i,r}^* \rightarrow P_r.$$

Finally, we summarize the discrete algorithm described above.

- 1** Initialize $P_0 = 0$, $G_{i,0}^* = 0$, $i = 1, \dots, m$.
- 2** Set iteration index $l = 0$. Start with class index $i = 1$.
- 3** Set $\tau_{i,l} = 0$, $G_{i,l}^* = G_{i,l}(0) = P_l$, $\tau_{i,l}^* = 0$.
- 4** Calculate $G_{i,l}(\tau_{i,l})$ by Eq.(3.8). If $G_{i,l}(\tau_{i,l}) > G_{i,l}^*$ then $G_{i,l}^* = G_{i,l}(\tau_{i,l})$, $\tau_{i,l}^* = \tau_{i,l}$.
- 5** If $\tau_{i,l} < l\Delta$, then $\tau_{i,l} = \tau_{i,l} + \Delta$ and return **4**.
- 6** If $i < m$ then $i = i + 1$ and return **3**.
Else if $i = m$, substitute $G_{i,l}^*$ and P_l into Eq.(3.9), calculate P_{l+1} .
- 7** If $l < r$, then let $l = l + 1$, $i = 1$ and return **3**.
Else if $l = r$, output $\{\tau_{i,l}^*\}$ and quit.

Note that the optimal investigating plans $\{\tau_{i,l}^*\}$, calculated by the above procedure, are not optimal ones in exact sense. In each class i , there exists some difference between the exact solution $\tau_i^*(t)$ in continuous case and $\tau_{i,l}^*$ as much as $|\tau_i^*(t) - \tau_{i,l}^*| \leq \Delta$. But if we calculate along with the above algorithm and divide the time period $[0, T]$ as much as possible, it means that $\Delta \rightarrow 0$, the differences tend to zero and the numerical solutions will converge to the solutions in continuous case.

4. Conditions for Optimal Investigation

We analyze $G_i(t, \tau_i(t))$ here to get some knowledge of the optimal stopping time of the investigating search. We partially differentiate Eq.(3.4) with respect to $\tau_i(t)$.

$$\begin{aligned} \frac{\partial G_i(t, \tau_i(t))}{\partial \tau_i(t)} &= (1 - a_T)p_{T_i}h_{T_i}(\tau_i(t))\{1 - P(t - \tau_i(t))\} \\ &\quad - a_F p_{F_i} h_{F_i}(\tau_i(t))\left\{P(t - \tau_i(t)) - \int_0^{t - \tau_i(t)} g(x)P(t - \tau_i(t) - x)dx\right\} \\ &\quad - P'(t - \tau_i(t))\{1 - p_{T_i}H_{T_i}(\tau_i(t)) - p_{F_i}H_{F_i}(\tau_i(t))\}. \end{aligned} \quad (4.1)$$

When $\tau_i(t) = 0$, Eq.(4.1) is rewritten as follows.

$$\begin{aligned} \left[\frac{\partial G_i(t, \tau_i(t))}{\partial \tau_i(t)}\right]_{\tau_i(t)=0} &= (1 - a_T)p_{T_i}h_{T_i}(0)\{1 - P(t)\} \\ &\quad - a_F p_{F_i} h_{F_i}(0)\left\{P(t) - \int_0^t g(x)P(t - x)dx\right\} - P'(t). \end{aligned} \quad (4.2)$$

Theorem 4.1 The necessary condition that the searcher should start the investigating search is the following inequality.

$$(1 - a_T)p_{T_i}h_{T_i}(0) > \frac{P'(t) + a_F p_{F_i} h_{F_i}(0)\{P(t) - \int_0^t g(x)P(t - x)dx\}}{1 - P(t)}. \quad (4.3)$$

Proof If the sign of Eq.(4.2) is positive, $G_i(t, \tau_i(t))$ has a positive derivative at $\tau_i(t) = 0$. From Eq.(3.4), $G_i(t, 0) = P(t)$. Substitute them into Eq.(3.5), it is clear that $dP(t)/dt$ is an increasing function of $\tau_i(t)$. \square

Theorem 4.2 When the searcher takes place the investigation for some time, the optimal investigating time $\tau_i^0(t)$ must satisfy the following relation.

$$\begin{aligned} \frac{(1 - a_T)p_{T_i}h_{T_i}(\tau_i^0(t))}{1 - p_{T_i}H_{T_i}(\tau_i^0(t)) - p_{F_i}H_{F_i}(\tau_i^0(t))} &= \frac{P'(t - \tau_i^0(t))}{1 - P(t - \tau_i^0(t))} \\ &\quad + \frac{a_F p_{F_i} h_{F_i}(\tau_i^0(t))}{1 - p_{T_i}H_{T_i}(\tau_i^0(t)) - p_{F_i}H_{F_i}(\tau_i^0(t))} \times \frac{P(t - \tau_i^0(t)) - \int_0^{t - \tau_i^0(t)} g(x)P(t - \tau_i^0(t) - x)dx}{1 - P(t - \tau_i^0(t))}. \end{aligned} \quad (4.4)$$

Proof Eq.(4.4) is given by setting Eq.(4.1) equals zero. □

Note that the uniqueness of $\tau_i^0(t)$ is unproved, so there may be some solutions satisfying Eq.(4.4). In such a case, we will adopt a minimal solution as $\tau_i^0(t)$ for the economy of search.

The left-hand side of Eq.(4.4) indicates the marginal detection probability that a true contact is judged properly as a true target under the conditions that the investigation has not finished until $\tau_i^0(t)$. The searcher finishes the investigation of the true contact just at $\tau_i^0(t)$. On the right-hand side, the first term is the increasing rate of the detection probability at time $t - \tau_i^0(t)$ under the condition that the search cannot detect the true target after $t - \tau_i^0(t)$ and he does not get any contact at that time. The second term indicates the detection probability of a ‘quasi-true’ target. The searcher mistakes the false contact as if it were true with probability a_F . In the numerator, $P(t - \tau_i^0(t))$ is subtracted by the detection probability in the penalty state for x time period. At every moment, $\tau_i^0(t)$ must be determined with balancing the short-term marginal detection probability (left-hand side) and the future detection probability, regarding the loss of the detection probability during the penalty time, until the end of the search operation. As we have mentioned in the previous section, we will adopt the discrete algorithm and get numerical solutions.

5. Numerical Examples

In this section, we show some numerical examples. We will set the values of each parameter as follows. TU stands for a time unit.

- Total operational time period: $T = 160$ [TUs]
- Time increment: $\Delta = 1$ [TU]
- Contact ratio : $\lambda = 0.05$ [times/ TU] (= 8 [times/ total operational time])
- Probabilities that the searcher makes the error of the 1st and 2nd kind :
 $a_T = a_F = 0.4$
- Number of the contact classes : $m = 4$

The penalty time changes depending on the exponential distribution with mean $1/\mu$ or it is set a constant.

- The case of the exponential distribution : $\mu = 1/60$
- The case of constant penalty time : $t_L = 60$ TUs

The mean of both cases are same value. We will compare the optimal investigating plans for both cases. When the searcher executes the investigation, it takes the exponential investigating time $h_{T_i}(\tau) = \alpha_{T_i} \exp(-\alpha_{T_i}\tau)$ and $h_{F_i}(\tau) = \alpha_{F_i} \exp(-\alpha_{F_i}\tau)$.

In each of the four contact classes, probabilities $p_{T_i}, p_{F_i}, p_{N_i}$, the investigating parameters $\alpha_{T_i}, \alpha_{F_i}$, and the probabilities that the contact belongs to the class i , p_{C_i} are set as shown in Table 1. We assume that p_{N_i} is independent of the class number i because we want to make up the same situation for noise environment. As the class number i increases, the certainty that the true contact is actually true becomes higher.

Table 1: Set of parameter values

Class	$(p_{T_i}, p_{F_i}, p_{N_i})$	$(\alpha_{T_i}, \alpha_{F_i})$	p_{C_i}
1	(0.1, 0.4, 0.5)		
2	(0.2, 0.3, 0.5)	(0.05, 0.05)	1/4
3	(0.3, 0.2, 0.5)		
4	(0.4, 0.1, 0.5)		

5.1. The relation between contact class and optimal investigating plan

We will investigate the relation between the certainty of the contact p_{T_i} and the optimal investigating plan. Here we consider only the case of the exponential penalty time, *i.e.* $g(x) = \mu e^{-\mu x}$. By the discrete algorithm explained in Section 3, the curves of $G_1(t, \tau_1(t))$ for Class 1 are shown in Figure 4. These curves are illustrated for $t = 20, 40, 60, 80, 100, 120, 140, 160$. Each curve has a negative gradient at $\tau_1(t) = 0$. But the sign of its gradient changes positive near the terminal points $\tau_1(t) = t$. In the case of $t = 20$, $G_1(t, t)$ is larger than $G_1(t, 0)$. In this situation, the optimal investigating time $\tau_1^*(t)$ is not zero and the searcher should take place the investigation for all residual time t . On the other hand, for larger t , the values at $\tau_1(t) = 0$ are much larger than the values at the terminal points $\tau_1(t) = t$. In those situations, the searcher should not execute investigating search, *i.e.* $\tau_1^*(t) = 0$.

The curves of $G_2(t, \tau_2(t))$ s are increasing around $\tau_2(t) = 0$ as shown in Figure 5. Optimal investigation time $\tau_2^*(t)$ becomes terminal point in case of $t = 20, 40$, whereas the inner points in another cases. The curves of $G_i(t, \tau_i(t))$ ($i = 3, 4$) have similar properties of Class 2.

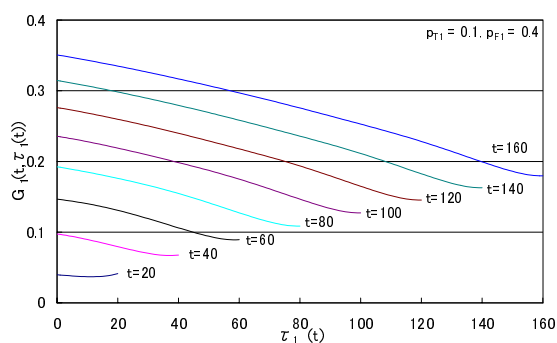


Figure 4: $G_1(t, \tau_1(t))$

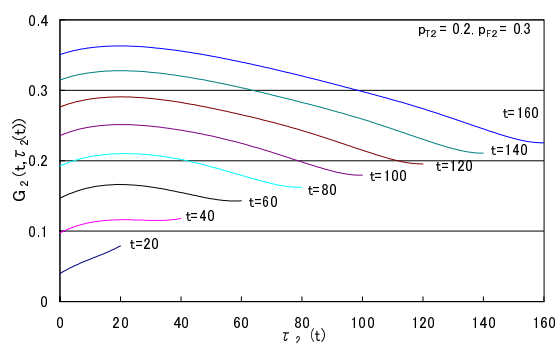


Figure 5: $G_2(t, \tau_2(t))$

The optimal investigating plans $\{\tau_i^*(t)\}$ are shown in Figure 6. From the figure, we can conclude that the searcher should spend all the time on the investigation for small residual time t . It means that when t is small, there is little chance to get a new contact and the searcher should spend all t on the investigation. On the other hand, for larger time t , the optimal investigating times are almost constant because of the maximal points of $G_2(t, \tau_2(t))$, also in the Class 3 and the Class 4, are almost constant when t is large. This phase transition of $\tau_i^*(t)$ occurs drastically but those properties have already seen in the former studies [4, 5]. As the certainty of the true contact becomes bigger, the searcher must use much time to the investigation, whereas the contact belongs to the Class 1, that class contains much uncertainty, the searcher should not investigate until the end stage of the operation. Figure 7 shows the curves of $G_i(t, \tau_i(t))$ for each Class i at $t = 100$. The peak of each curve moves from left to right as the Class number increases. When the searcher executes a search operation according to the optimal investigating plans shown in Figure 6, the detection probability of the true target $P(t)$ is maximized time by time as shown in Figure 8.

5.2. The effect of the distribution of penalty time

We will consider here the effect of the penalty time. The penalty time tends to be influenced by the operation of the investigation. Here we consider the case of constant penalty time in addition to the exponential case. When the penalty time is constant t_L , we modify the

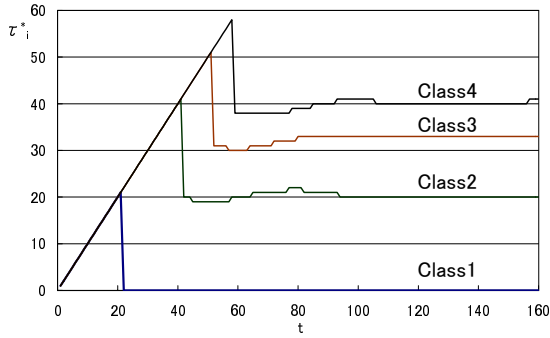


Figure 6: $\{\tau_i^*(t)\}$

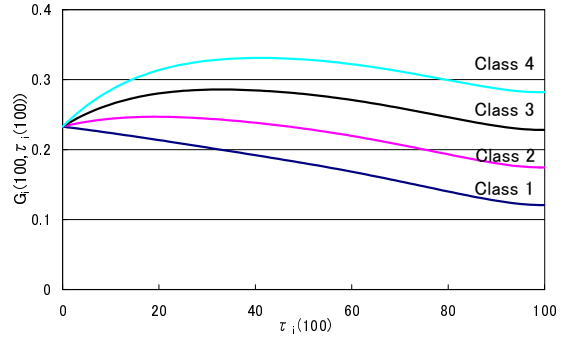


Figure 7: $G_i(100, \tau_i(100))$

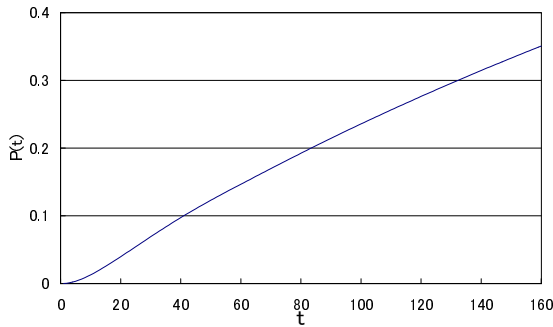


Figure 8: $P(t)$

third term of Eq.(3.4) by substituting $g(x) = \delta(x - t_L)$, where $\delta(\cdot)$ is Dirac's delta function.

$$p_{F_i} a_F \int_0^{\tau_i(t)} h_{F_i}(u) \left\{ \int_0^{t-u} g(x) P(t-u-x) dx \right\} du = \begin{cases} p_{F_i} a_F \int_0^{\tau_i(t)} h_{F_i}(u) P(t-u-t_L) du & (t > t_L) \\ 0 & (t \leq t_L) \end{cases} \quad (5.1)$$

Resultant $G_2(t, \tau_2(t))$ s are shown in Figure 9. The curves are almost same as Figure 5. These similarities are also seen for other classes. When we compare $G_2(t, \tau_2(t))$ s between the two cases of penalty time for the same time t , the values in the constant penalty case are slightly smaller than the exponential case. The main reason of this difference can be explained as follows. In the constant penalty case, the searcher must be in the penalty state for exact t_L and he cannot increase the detection probability within the period. On the other hand, in the exponential case, the searcher always has a chance to come back the broad search phase when he is in the penalty state and the detection probability can be stored even within t_L .

But the difference of $G_i(t, \tau_i(t))$ is so little that it does not affect $\{\tau_i^*(t)\}$ and $P(t)$. The detection probability $P(t)$ in the constant penalty case is exhibited in Figure 10. Compare it with Figure 8, the difference is not seen clearly. For the discrete algorithm, the case of the exponential penalty time requires more calculations than the case of constant time. So it is better to use the constant case as a substitution of the exponential case if precise computation is not required. From the next section, we will consider only the constant penalty case.

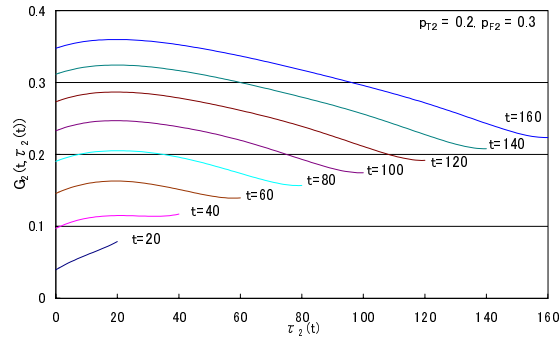


Figure 9: $G_2(t, \tau_2(t))$ (Constant penalty case)

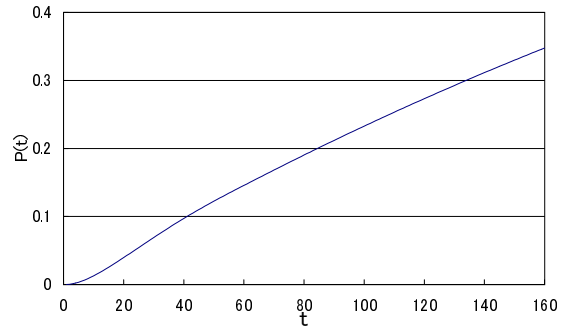


Figure 10: $P(t)$ (Constant penalty case)

5.3. The effect of source of a contact

In Section 5.1. and Section 5.2., we assumed that a contact could be caused by three types of sources. Here we will omit one of false or noise contact and make it clear that the relationship between the sources of contact and the optimal investigating time in the case of constant penalty time. We will suppose the following two environments in which a contact is caused by only two sources. One is False-Type environment and the other is Noise-Type. In the False-Type environment, a contact is caused by only a true target or false objects. In the Noise-Type one, a contact is caused by a true target or noise. We set up parameters for the two type environments as in Table 2. Figure 11 shows $G_4(t, \tau_4(t))$ s of the two environments. The left figure is for False-Type and the right is for Noise-Type.

The curves of False-Type tend to flatten as t becomes larger. On the other hand, they tend to bend downward greatly in the Noise-Type case. These phenomena can be explained by Eq.(3.4). In the False case, Eq.(3.4) is written as the same, while in the Noise case, it is rewritten as follows.

$$G_i(t, \tau_i(t)) = p_{Ti} \int_0^{\tau_i(t)} h_{Ti}(u) \{ (1 - a_T) + a_T P(t - u) \} du + \{ 1 - p_{Ti} H_{Ti}(\tau_i(t)) \} P(t - \tau_i(t)). \quad (5.3)$$

Table 2: Set of parameter values in False-Type and Noise-Type environments

Type	False-Type	Noise-Type	
probabilities	(p_{Ti}, p_{Fi})	(p_{Ti}, p_{Ni})	p_{Ci}
Class 1	(0.1, 0.9)	(0.1, 0.9)	
Class 2	(0.2, 0.8)	(0.2, 0.8)	1/4
Class 3	(0.3, 0.7)	(0.3, 0.7)	
Class 4	(0.4, 0.6)	(0.4, 0.6)	

Compare it with Eq.(3.4), the second and the third terms of it vanish because of the value of $p_{Fi} = 0$. In Eq.(3.4), those terms are positive definite and are increasing functions of $\tau_i(t)$. By lack of those terms, the curves of $G_i(t, \tau_i(t))$ of the Noise-Type case bend greatly when $\tau_i(t)$ becomes large.

In the False case, $G_i(t, \tau_i(t))$ is written as same as Eq.(3.4), but the probability of each class p_{Fi} s are bigger than those in Table 1, the second and the third term of Eq.(3.4) cause pushing up $G_i(t, \tau_i(t))$ s and the curves of them become flat compare to Figure 5.

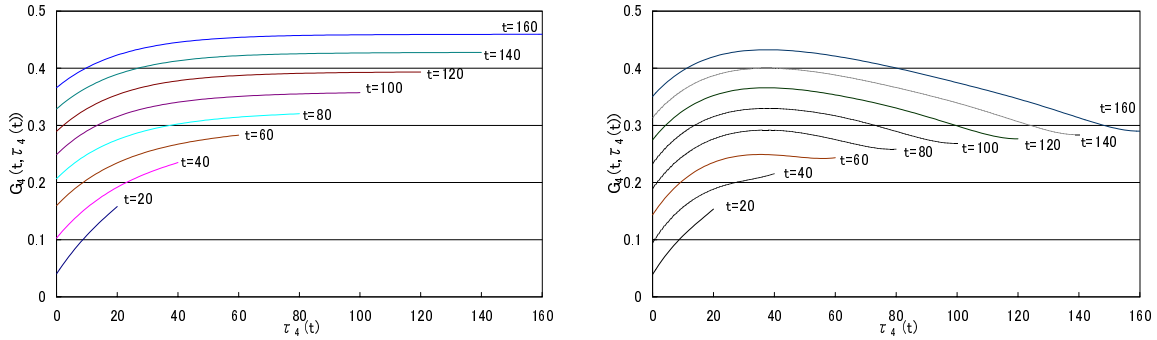


Figure 11: $G_4(t, \tau_4(t))$ (left : False-Type right : Noise-Type)

The optimal investigating plans, $\{\tau_i^*(t)\}$, for the each contact type and each contact class are depicted in Figure 12. As the curves of $G_i(t, \tau_i(t))$ increase even if t is big enough in the False-Type case, they take the maximal values at the terminal points. In this case, there is no probability that the contact is caused by noise so the searcher tends to keep investigation until the source of the contact becomes clear. The optimal investigating time $\tau_i^*(t)$ are proportional to t in the classes where p_T are large. It is concluded that if the searcher has high performance sensors, they make contact signals rarely from noise sources and the optimal investigating period tends to be long.

Contrarily, in the Noise case, $G_i(t, \tau_i(t))$ s become maximal at an inner point for large residual time. The optimal investigating plans $\{\tau_i^*(t)\}$ have the same properties as depicted in Figure 6. We can conclude that if noise is mixed in contact signals, it is better to finish the investigation early without inspecting it too much.

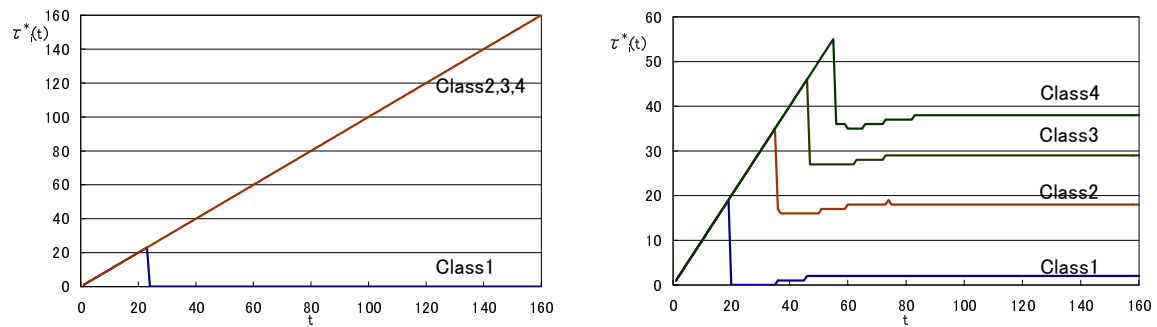


Figure 12: Optimal investigating plans $\{\tau_i^*(t)\}$ (left: False-Type right: Noise-Type)

5.4. The effect of contact ratio λ

We will study here the effect of the contact ratio λ on $\tau_i^*(t)$. We keep the same parameters as in Section 5.1. but change λ from 0.05 to 0.1, 0.025 or 0.0125. It means that the expectation of contact is changed from 8 times to 16,4 times or twice during the whole operational time. Under those conditions we will assume that the penalty time is constant and the contact is caused by three different sources.

The results are shown in Figure 13. The left figure is the optimal investigating plans for Class 1 and the right is for Class 4. When λ is small, it is difficult to get a contact so the investigating time for a gained contact tends to be long. When the contact ratio λ becomes large, it is better to finish the investigation early and to return to the broad search to get a new contact. Especially for the Class 1 and $\lambda = 0.1, 0.05$, the searcher should not

investigate a contact at all until the final stage of the operation.

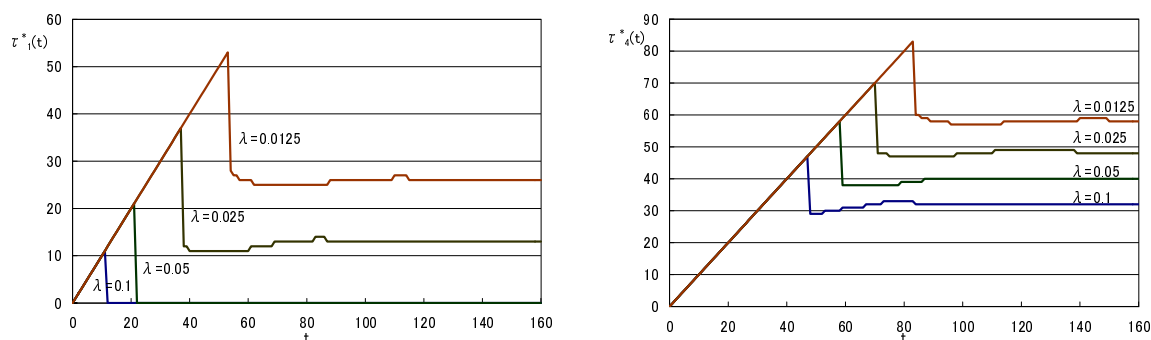


Figure 13: λ dependency of $\tau_1^*(t)$ (left) and $\tau_4^*(t)$ (right)

6. Conclusions

In this paper, we consider a two-stage sequential search operation which consists of a broad search and an investigation. We propose a method for the optimal investigating plans so as to maximize the detection probability until the end of the operation. The method takes into account the errors of two kinds that may happen when the judgment is given on a contact. Three possible entities are also considered in the method. By numerical examples, we show some properties of the optimal investigating plans and make sure that they are reasonable comparing with real operation. When the searcher misjudges a false target as a true one, he must be in the penalty state for a while. We assumed that the penalty period obeys two different distributions. By the numerical experiment, it becomes clear that the difference of them affects very little on the optimal investigating plans. So it is worth to use constant penalty case in practice.

As future studies, we would like to apply this model to the real world. The real operational field is thought to be composed of many heterogeneous regions. Some parameters such as λ or penalty time are different in each region. We must modify this model and apply it to the multi-area circumstances.

There are some preceding studies that treat the two-stage search problem with another criteria such as the minimization of the expected time until a searcher detects a true target[1, 3]. They considered only two possible contacts but don't consider the errors of two kinds. It is also a future study that we make a new model with this criterion incorporating the concept of three possible entities and the errors of two kinds.

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