Journal of the Operations Research
Society of Japan
2006, Vol. 49, No. 2, 117-129

NOTE ON STOCHASTIC INVENTORY MODELS WITH SERVICE LEVEL CONSTRAINT

Robert Huang-Jing Lin
Oriental Institute of Technology

Peter Chu
Central Police University

(Received March 31, 2005; Revised December 5, 2005)

Abstract This note examines the inventory model with service level constraint in which the lead time, the reordered point and order quantity are treated as decision variables. The previous researchers believed that the objective function is concave in the lead time such that the minimum must occur on the boundary points of each sub-domain. In this note, we will show that their belief contains questionable results. In a recent paper in *Journal of the Operations Research Society of Japan*, Ouyang and Chuang studied this problem. However, their algorithm might not find the optimal solution due to flaws in their solution procedure. We developed some lemmas to reveal the parameter effects and then presented the complete procedures for finding the optimal solution for the inventory model in which the lead time demand is unknown and for solving by the Gallego's minimax distribution free procedure. The savings are illustrated by solving the same examples from Ouyang and Chuang's paper to demonstrate a 30% improvement by our revised algorithm.

Keywords: Inventory, minimax distribution free procedure, lead time, crashing cost

1. Introduction

In the traditional inventory model, the lead time is considered as a predetermined constant or a stochastic variable as in Silver and Peterson [14] such that lead time is not controllable. Liao and Shyu [5] and Ben-Daya and Raouf [1] decomposed the lead time into several components, each having a different piecewise linear crash cost function for lead time reduction; therefore, the lead time becomes a new decision variable. Gallego [3] created a wonderful two point distribution to serve as the most unfavorable case among the distributions with the same mean and variance to estimate the expected cost of the lost sales such that the minmax distribution free approach of Scarf [13] can apply to the stochastic inventory models. Moon and Gallego [7] extended the minmax distribution free approach for stochastic inventory model with backorders and lost sales. Ouyang *et al.* [11] generalized Ben-Daya and Raouf's [1] assumption allowing shortages. Moon and Choi [6] and Lan *et al.* [4] pointed out the problem in Ouyang *et al's.* method [11]. Ouyang and Wu [10] extended the Ouyang *et al.* [11] article to apply the minimax distribution free procedure. Ouyang and Chuang [8] studied stochastic inventory models with service level constraint which are solving by the minimax distribution free procedure. Wu and Tsai [15] studied inventory models with a mixed normal distribution from different customers. Pan and Hsiao [12] developed the model with backorder discount to ensure that customers would be willing to wait for backorders. Ouyang and Wu [9] extended to inventory model with service level constraint. Chu, *et al.* [2] improved the results of Ouyang and Wu [9] first for lead time demand following a normal distribution and then extended the minmax distribution free procedure to solve the problem.

For the inventory models with crashable lead time, there are many generalized extensions to apply for more realistic inventory models. However, there is a disputable result that deserves more detailed discussion. Ouyang *et al.* [11] proved that the expected annual cost is a concave down function in lead time such that the minimum value will occur on the boundary points of each sub-domain. It is an excellent discovery that dramatically simplifies the solution procedure. To clearly indicate this property, we denoted it as follows: the minimum values for concave down functions degenerate to the boundary points on the sub-domain of the crash cost.

However, Ouyang *et al.* [11] considered the inventory model without service level constraint. The researchers who followed them believed that this property also holds with the service level constraint.

In this note, we will point out that the degeneracy to the boundary points for the concave down function requires more detailed examination. We will construct a correct and efficient algorithm to find the optimal order quantity, reorder point and lead time, develop lemmas to reveal the parameter effects and illustrate our improvement by solving the same numerical example in Ouyang and Chuang [8] to indicate that sometimes their algorithm does not find the optimal solution.

2. Notation and Assumptions

We used the same notation and assumptions as Ouyang and Chuang [8] and several new notations to simplify the expression.

Notation:

A = Ordering cost per order.
D = Expected demand per year.
h = Holding cost per unit per year.
L = Length of lead time.
Q = Order quantity.
X = The lead time demand which has a distribution function F with finite mean μL and standard derivation $\sigma\sqrt{L}(>0)$.
x^+ = Maximum value of x and 0, i.e. $x^+ = \max\{x, 0\}$.
α = Proportion of demands that are not met from stock so $1-\alpha$ is the service level.
β = Fraction of the demand during the stock-out period that will be backordered.
M_β = Expected value of β.
$C(L)$ = Lead time crashing cost.
$L(Q,k)$ = The boundary points of lead time that satisfies the service level constraint.
$EAC(Q,k,L)$ = Total expected annual cost.
$MID(x,y,z)$ = The middle term in x, y and z.
$k_i(Q) = \frac{\sigma^2 L_i - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_i}}$, for $0 < 2Q\alpha < \sigma\sqrt{L_i}$, the least value of safety factor that satisfies the service level constraint.

Assumptions:

(1) The reorder point r = expected demand during lead time + safety stock (SS), and $SS = k\sigma\sqrt{L}$, that is, $r = \mu L + k\sigma\sqrt{L}$ where k is the safety factor and satisfies $P(X > r) = q$, q representing the allowable stock-out probability during L.

(2) $B(r) = E[X-r]^+$ is the expected demand shortage at the end of cycle. Hence, $\beta B(r)$ are the backordered quantities and $(1-\beta)B(r)$ are the lost sales. Therefore, the total demand during lead time period equals $\mu L - (1-\beta)B(r)$ and the expected net inventory level just before the order arrives is $r - (\mu L - (1-\beta)B(r))$. Moreover, the expected net inventory level at the beginning of the cycle is $Q + r - (\mu L - (1-\beta)B(r))$, so the expected holding cost per cycle is

$$\frac{Q}{D}\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)B(r)\right].$$

(3) Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point r.

(4) The lead time L has n mutually independent components. The ith component has a minimum duration a_i, and normal duration b_i, and a crash cost per unit time c_i. Further, we assume that $c_1 \le c_2 \le \ldots \le c_n$.

(5) The lead time components are crashed one at a time starting with the least c_i component and so on.

(6) If we let $L_0 = \sum_{j=1}^{n} b_j$ and L_i the length of lead time with components $1, 2, \ldots, i$ crash to their minimum durations, then $L_i = \sum_{j=i+1}^{n} b_j + \sum_{j=1}^{i} a_j$. The lead time crash cost $C(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$, is given by $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.

(7) For technical reasons, we assume that both $\frac{h}{2} + h\alpha(1-M_\beta) - 4c_i\frac{D\alpha^2}{\sigma^2} > 0$ and $h - \frac{4c_iD\alpha}{\sigma^2} > 0$ are valid. This is confirmed by the numerical examples in Ouyang and Chuang [8].

3. Review of Previous Results

We studied the inventory model of Ouyang and Chuang [8] such that the order quantity, Q, length of lead time, L, and reorder point, r, are decision variables. Their objective is to minimize the expected annual cost, subject to a constraint on service level as follows

$$Min\ \ EAC(Q,k,L) = \frac{D}{Q}\left[A + C(L)\right] + \frac{hQ}{2} + h\left[r - \mu L + (1-M_\beta)E(X-r)^+\right] \tag{1}$$

subject to

$$E(x-r)^+ \le Q\alpha \tag{2}$$

for $L_i \le L \le L_{i-1}$ with $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.

Since the distribution of lead time demand is unknown, researchers could not find the exact value of $E(x-r)^+$. Hence, they used the minimax distribution free procedure of Gallego and Moon [6] to find a tight upper bound for $E(x-r)^+$ and use the safety factor, k, as the new variable to replace the reorder point, r, then the problem is reduced to

$$Min\ \ EAC(Q,k,L) = \frac{D[A + C(L)]}{Q} + \frac{hQ}{2} + h\sigma\sqrt{L}\left[k + \frac{1-M_\beta}{2}(\sqrt{1+k^2} - k)\right] \tag{3}$$

subject to

$$\sigma\sqrt{L}(\sqrt{1+k^2} - k) \le 2Q\alpha. \tag{4}$$

Ouyang and Chuang [8] used a nonnegative slack variable, S^2 to convert the inequality in constraint into equality such that they considered

$$\begin{aligned} EAC(Q,k,L,\lambda,S) \ &= \ \frac{D[A + C(L)]}{Q} + \frac{hQ}{2} + h\sigma\sqrt{L}\left[k + \frac{1}{2}(1-M_\beta)(\sqrt{1+k^2} - k)\right] \\ &\quad + \lambda\left[\sigma\sqrt{L}(\sqrt{1+k^2} - k) + S^2 - 2Q\alpha\right]. \end{aligned} \tag{5}$$

They derived that

$$\frac{\partial^2}{\partial L^2} EAC(Q,k,L,\lambda,S) = \frac{-h\sigma}{4\sqrt{L^3}}\left[k + \frac{1-M_\beta}{2(\sqrt{1+k^2}+k)}\right] - \frac{\lambda\sigma}{4\sqrt{L^3}}(\sqrt{1+k^2}-k) < 0. \tag{6}$$

From $\frac{\partial^2}{\partial L^2}EAC(Q,k,L,\lambda,S) < 0$, they implied that $EAC(Q,k,L,\lambda,S)$ is a concave down function in variable L for $L_i \le L \le L_{i-1}$ so they directly assume that the minimum value will occur at the boundary points $L = L_i$ and $L = L_{i-1}$.

They claimed that they could further prove that, for any given $L \in [L_i, L_{i-1}]$, $EAC(Q,k,L,\lambda,S)$ satisfies the Kuhn-Tucker necessary conditions for minimization problem and it obtains the slack variable $S^2 = 0$. Hence, with the variable $S = 0$, for a given $L = L_i$, they considered the partial derivatives with respect to Q, k, and λ, to derive that $\frac{\partial}{\partial Q}EAC(Q,k,L_i,\lambda,S) = 0$, $\frac{\partial}{\partial k}EAC(Q,k,L_i,\lambda,S) = 0$, and $\frac{\partial}{\partial \lambda}EAC(Q,k,L_i,\lambda,S) = 0$, so they found the solution for the partial derivative system, say Q_i, k_i and λ_i, respectively for $L = L_i$ as follows

$$Q_i = \left[\frac{2D(A+C(L_i))}{h-4\lambda_i\alpha}\right]^{\frac{1}{2}} \tag{7}$$

$$\lambda_i = h\left[\frac{\sqrt{1+k_i^2}}{\sqrt{1+k_i^2}-k_i} - \frac{1-M_\beta}{2}\right] \tag{8}$$

and

$$\sigma\sqrt{L_i}\left(\sqrt{1+k_i^2}-k_i\right) = 2Q_i\alpha. \tag{9}$$

They sophisticatedly combined Equations (7), (8) and (9) to obtain Q_i as

$$Q_i = \left[\frac{4D\alpha(A+C(L_i)) + h\sigma^2 L_i}{2h\alpha(1-2M_\beta\alpha)}\right]^{\frac{1}{2}}. \tag{10}$$

Plugging Q_i into Equations (9) and (8), they had k_i and λ_i, respectively. They compared $EAC(Q_i,k_i,L_i,\lambda_i)$ for $i = 0, 1, \ldots, n$, to locate the minimum value. Here, we wish to point out that Ouyang and Chuang [8] did not explain why k_i is nonnegative, when Q_i is plugged into Equation (9). In the next section, we will offer an alternative method to solve the minimum problem and indicate that the solution procedure of Ouyang and Chuang [8] contains debatable results.

4. Our Improvement

In the beginning, the domain for (Q,k,L) is $Q > 0$, $k \ge 0$ and $L_n \le L \le L_0$. However, the cost for crashed lead time is defined for each interval $L \in [L_i, L_{i-1}]$, $i = 1, ..., n$, respectively. Hence, we need to consider $EAC(Q,k,L)$ for $L \in [L_i, L_{i-1}]$, $i = 1, ..., n$, respectively. However, we will only consider $L \in [L_1, L_0]$ for the time being. The generalization from $L \in [L_1, L_0]$ to $L \in [L_i, L_{i-1}]$ is an easy task that will be discussed at the end of Section 4. We try to solve the minimum problem for stochastic inventory model with service level constraint by the minimax distribution free procedure of Gallego and Moon [6]

$$Min \;\; EAC(Q,k,L) = \frac{D[A+C(L)]}{Q} + \frac{hQ}{2} + h\sigma\sqrt{L}\left[k + \frac{1-M_\beta}{2}(\sqrt{1+k^2}-k)\right] \tag{11}$$

subject to $\sigma\sqrt{L}(\sqrt{1+k^2}-k) \le 2Q\alpha$, $0 < Q$, $0 \le k$ and $L_1 \le L \le L_0$.

Similar to Equation (6), we still have

$$\frac{\partial^2}{\partial L^2}EAC(Q,k,L) = \frac{-h\sigma}{4\sqrt{L^3}}\left[k + \frac{1-\beta}{2(\sqrt{1+k^2}+k)}\right] < 0. \tag{12}$$

Our solution procedure is explained as follows. First, we will fix (Q, k) for the moment, and find the possible domain of L such that the service level constraint $\sigma\sqrt{L}(\sqrt{1+k^2}-k) \le 2Q\alpha$ is satisfied. Second, depending on partition on the domain of Q, we will reduce the domain of k from $0 \le k < \infty$ to a single point. Third, for a restricted domain, we will face the problem in one variable of Q to prove that it is a convex function in Q. Fourth, we will check the minimum solution in the third step whether or not it satisfies the restricted domain of Q to find the minimum solution that satisfies the condition of the restricted sub-domain of Q. Fifth, we will combine all local minimum solutions in each case to derive the minimum solution for $L \in [L_1, L_0]$. We will generalize it to the general case for $L \in [L_i, L_{i-1}]$. Finally, we will compare the minimum solution for $L \in [L_i, L_{i-1}]$ with $i = 1, 2, ..., n$ to find the optimal solution.

We may rewrite the service level constraint as

$$\sigma\sqrt{L} \le 2Q\alpha\left(\sqrt{1+k^2}+k\right). \tag{14}$$

Motivated by Equation (14), we will partition $\{(Q, K) : 0 < Q,\ 0 \le k < \infty\}$ into the following three regions:

(1) R1 is defined as $\left\{(Q,k) : 0 < Q, 0 \le k < \infty,\ 2Q\alpha(k+\sqrt{1+k^2}) < \sigma\sqrt{L_1}\right\}$.

(2) R2 is defined as $\left\{(Q,k) : 0 < Q, 0 \le k < \infty,\ \sigma\sqrt{L_1} \le 2Q\alpha(k+\sqrt{1+k^2}) \le \sigma\sqrt{L_0}\right\}$.

(3) R3 is defined as $\left\{(Q,k) : 0 < Q, 0 \le k < \infty,\ \sigma\sqrt{L_0} < 2Q\alpha(k+\sqrt{1+k^2})\right\}$.

For those points (Q, k) in R1, for any L with $L_1 \le L \le L_0$ we know that (Q, k, L) cannot satisfy the service $\sigma\sqrt{L} \le 2Q\alpha(\sqrt{1+k^2}+k)$ so there is no possible local minimum solution in R1. On the other hand for those points (Q, k) in R3, for any L with $L_1 \le L \le L_0$ we will show that (Q, k, L) always satisfies the service $\sigma\sqrt{L} \le 2Q\alpha(\sqrt{1+k^2}+k)$, then from the concave down property for $EAC(Q, k, L)$ in variable L of Equation (12), we will obtain that the minimum value will happen on the two boundary points at $L = L_1$ or $L = L_0$, so we will derive the next lemma.

Lemma 1 *Given a point in R3, such as (Q_a, k_a), the minimum value of $\{(Q_a, K_a, L) : L_1 \le L \le L_0\}$ is equal to the minimum value of $\{(Q_a, K_a, L) : L = L_1,$ or $L = L_0\}$.*

Next, we consider a point, such as (Q_b, k_b) in R2. For those values of L with $\sigma\sqrt{L} \le 2Q_b\alpha(\sqrt{1+k_b^2}+k_b)$, (Q_b, k_b, L) can be treated as a possible candidate for the local minimum solution. For those values of L with $\sigma\sqrt{L} > 2Q_b\alpha(\sqrt{1+k_b^2}+k_b)$, (Q_b, k_b, L) cannot be treated as a possible candidate for the local minimum solution. Hence by the concave down property for $EAC(Q, k, L)$ in variable L of Equation (12), we derived the second lemma.

Lemma 2 *From (Q_b, k_b) in R2, the minimum value of $\{EAC(Q_b, k_b, L) : L_1 \le L \le L_0\}$ is equal to the minimum value of $\left\{EAC(Q_b, k_b, L) : L = \frac{4Q_b^2\alpha^2}{\sigma^2}\left(k_b+\sqrt{1+k_b^2}\right)^2,\ \text{or } L = L_1\right\}$.*

Based on Lemmas 1 and 2, we know that we can simplify the domain of lead time, L, from an interval, $[L_1, L_0]$ to reduce it to a single point. To be more specific, we indicated the single point as $L(Q, k)$. Hence, we will solve the minimum problem of

$$\{EAC(Q, k, L(Q, k)) : 0 < Q, 0 \le k < \infty\}$$

for four different cases as follows.

Case (a): For (Q, k) in R2, with $L(Q, k) = L_1$,

Case (b): For (Q, k) in R2, with $L(Q, k) = \frac{4\alpha^2}{\sigma^2}Q^2\left(\sqrt{1+k^2}+k\right)^2$,

Case (c): For (Q, k) in R3, with $L(Q, k) = L_1$,

Case (d): For (Q, k) in R3, with $L(Q, k) = L_0$.

4.1. For Case (a), (Q, k) in R2, with $L(Q, k) = L_1$

To discuss Case (a), we will begin by pointing out that from (Q, k) in R2 the domain of Q must be changed from $0 < Q$ into $0 < Q \leq \frac{\sigma\sqrt{L_0}}{2\alpha}$.

Lemma 3 *From (Q, k) in R2, the domain of Q becomes $0 < Q \leq \frac{\sigma\sqrt{L_0}}{2\alpha}$.*

Proof. Since $\sqrt{1+k^2}+k$ increases in k, if the value of Q is as big as $\sigma\sqrt{L_0} < 2Q\alpha$, then we have that $\frac{4\alpha^2}{\sigma^2}Q^2(\sqrt{1+k^2}+k)^2 \geq \frac{4\alpha^2}{\sigma^2}Q^2 > L_0$ which means (Q, k) in R3. This contradicts the assumption of (Q, k) in R2. Hence, the domain of Q is reduced from $0 < Q$ to $0 < Q \leq \frac{\sigma\sqrt{L_0}}{2\alpha}$.

Based on Lemma 3, we rewrite Case (a) as the following problem: to find the minimum for $EAC(Q, k, L_1)$ under the conditions

$$(Q, k) \text{ in R2, } 0 < Q \leq \frac{\sigma\sqrt{L_0}}{2\alpha}, \text{ and } 0 \leq k < \infty. \tag{15}$$

Here, we will begin our next simplification process to divide the domain of Q into two sub-domains such that we can reduce the domain of k from an interval, $0 \leq k < \infty$, to a single point.

We know that $0 \leq M_\beta \leq 1$, $-1 \leq \frac{k}{\sqrt{1+k^2}} - 1 < 0$ and $\frac{-1}{2} \leq \frac{1-M_\beta}{2}\left(\frac{k}{\sqrt{1+k^2}} - 1\right) < 0$, hence

$$\frac{\partial}{\partial k}EAC(Q, k, L_1) = h\sigma\sqrt{L_1}\left[1 + \frac{1}{2}(1-M_\beta)\left(\frac{k}{\sqrt{1+k^2}} - 1\right)\right] \geq \frac{h\sigma\sqrt{L_1}}{2} > 0. \tag{16}$$

By Equation (16), when Q is fixed for the moment, the minimum will occur at the minimum value of k that satisfies the conditions in Equation (15). Hence, we will further divide Case (a) into two sub-cases to assume that Case (a1) $\frac{\sigma\sqrt{L_1}}{2\alpha} \leq Q \leq \frac{\sigma\sqrt{L_0}}{2\alpha}$ and Case (a2) $0 < Q < \frac{\sigma\sqrt{L_1}}{2\alpha}$.

Case (a1) yields that $\sigma\sqrt{L_1} \leq 2Q\alpha \leq \sigma\sqrt{L_0}$, so we imply that

$$\frac{4\alpha^2}{\sigma^2}Q^2\left(\sqrt{1+k^2}+k\right)^2 \geq \frac{4\alpha^2}{\sigma^2}Q^2\left(\sqrt{1+0^2}+0\right)^2 = \frac{4\alpha^2}{\sigma^2}Q^2 \geq L_1. \tag{17}$$

Equation (17) means that when Q is in Case (a1), then $(Q, k = 0)$ is in Case (a1). Therefore, to take the minimum value of k with $k = 0$. Here, we face the following minimum problem:

$$EAC(Q, 0, L_1) = \frac{a}{Q} + bQ + c, \tag{19}$$

with $a = D(A + C(L_1))$, $b = \frac{h}{2}$ and $c = \frac{h}{2}(1-M_\beta)\sigma\sqrt{L_1}$, for $\frac{\sigma\sqrt{L_1}}{2\alpha} \leq Q \leq \frac{\sigma\sqrt{L_0}}{2\alpha}$.

We may rewrite Equation (19) as

$$EAC(Q, 0, L_1) = \left(\sqrt{\frac{a}{Q}} - \sqrt{bQ}\right)^2 + \sqrt{2ab} + c. \tag{20}$$

From Equation (20), we obtained that without considering the condition $\sigma\sqrt{L_1} \le 2Q\alpha \le \sigma\sqrt{L_0}$, the minimum solution is

$$Q = \sqrt{\frac{a}{b}} = \sqrt{\frac{2D(A + C(L_1))}{h}}. \tag{21}$$

However, we must consider the condition $\sigma\sqrt{L_1} \le 2Q\alpha \le \sigma\sqrt{L_0}$. To simplify the expression, we defined a new operator, $MID(x, y, z)$ such that $MID(x, y, z)$ is the middle term in x, y and z. For example, $MID(2, 6, 5) = 5$, $MID(5, 8, \infty) = 8$ and $MID(6, 2, 2) = 2$. Since $EAC(Q, 0, L_1)$ is a convex function in Q, we have three cases, namely, Case (a11) $\sqrt{\frac{a}{b}} \le \frac{\sigma\sqrt{L_1}}{2\alpha}$, Case (a12) $\frac{\sigma\sqrt{L_1}}{2\alpha} \le \sqrt{\frac{a}{b}} \le \frac{\sigma\sqrt{L_0}}{2\alpha}$, and Case (a13) $\frac{\sigma\sqrt{L_0}}{2\alpha} \le \sqrt{\frac{a}{b}}$.

For Case (a11), $EAC(Q, 0, L_1)$ increases on $\frac{\sigma\sqrt{L_1}}{2\alpha} \le Q \le \frac{\sigma\sqrt{L_0}}{2\alpha}$ so the minimum occurs at $\frac{\sigma\sqrt{L_1}}{2\alpha}$. For Case (a12), the minimum solution for the unrestricting problem still is the minimum solution for the restricting problem. For Case (a13) $EAC(Q, 0, L_1)$ decreases on $\frac{\sigma\sqrt{L_1}}{2\alpha} \le Q \le \frac{\sigma\sqrt{L_0}}{2\alpha}$ so the minimum occurs at $\frac{\sigma\sqrt{L_0}}{2\alpha}$. If we observe these three cases (a11), (a12) and (a13), then the minimum solution always occurs at $MID\left(\frac{\sigma\sqrt{L_1}}{2\alpha}, \frac{\sigma\sqrt{L_0}}{2\alpha}, \sqrt{\frac{a}{b}}\right)$. Hence, we summarized the results in the next lemma.

Lemma 4 *For Case (a1), the minimum solution for the order quantity is* $MID\left(\frac{\sigma\sqrt{L_1}}{2\alpha}, \frac{\sigma\sqrt{L_0}}{2\alpha}, \sqrt{\frac{a}{b}}\right)$ *with* $a = D(A + C(L_1))$ *and* $b = \frac{h}{2}$*, and the safety factor* $k = 0$ *for the expected annual cost* $EAC(Q, 0, L_1)$*.*

We considered Case (a2) for $0 < Q < \frac{\sigma\sqrt{L_1}}{2\alpha}$. By equation (16), we know that $EAC(Q, k, L_1)$ is an increasing function of k so we will make k as small as possible while satisfying the service level constraint. If we still take $k = 0$, then

$$\frac{4\alpha^2}{\sigma^2}Q^2\left(\sqrt{1+k^2}+k\right)^2 = \frac{4\alpha^2}{\sigma^2}Q^2\left(\sqrt{1+0^2}+0\right)^2 = \frac{4\alpha^2}{\sigma^2}Q^2 < L_1.$$

This means the service level constraint is not satisfied, so $k = 0$ is not acceptable. We need to find the least value of k such that $\frac{4\alpha^2}{\sigma^2}Q^2(\sqrt{1+k^2}+k)^2 \ge L_1$ which implies that if we solve $\sigma\sqrt{L_1} = 2Q\alpha(\sqrt{1+k^2}+k)$ for variable k then we will obtain the least value of k. Since $1 = (\sqrt{1+k^2}-k)(\sqrt{1+k^2}+k)$ then $\frac{\sigma\sqrt{L_1}}{2Q\alpha} = \sqrt{1+k^2}+k$ and $\frac{2Q\alpha}{\sigma\sqrt{L_1}} = \sqrt{1+k^2}-k$, so $k = \frac{\sigma^2 L_1 - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_1}}$. To clearly indicate this relation, we assume a new expression, say $k_1(Q)$ such that

$$k_1(Q) = \frac{\sigma^2 L_1 - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_1}}. \tag{22}$$

Here, for $0 < 2Q\alpha < \sigma\sqrt{L_1}$, we face the following minimum problem

$$EAC(Q, k_1(Q), L_1) = \frac{d}{Q} + eQ, \tag{23}$$

with $d = D[A + C(L_1)] + \frac{h}{4\alpha}\sigma^2 L_1$ and $e = \frac{1}{2}(1 - 2\alpha M_\beta)h$.

We may rewrite Equation (23) as

$$EAC(Q, k_1(Q), L_1) = \left(\sqrt{\frac{d}{Q}} - \sqrt{eQ}\right)^2 + \sqrt{2de}. \tag{24}$$

From Equation (24), we obtain that without considering the condition $0 < 2Q\alpha < \sigma\sqrt{L_1}$, then the minimum solution is

$$Q = \sqrt{\frac{d}{e}} = \sqrt{\frac{4\alpha D[A + C(L_1)] + h\sigma^2 L_1}{2\alpha h(1 - 2\alpha M_\beta)}}. \tag{25}$$

However, we must consider the condition $0 < 2Q\alpha < \sigma\sqrt{L_1}$. Since $EAC(Q, k_1(Q), L_1)$ is a convex function in Q, we have two cases, Case (a21) $\frac{\sigma\sqrt{L_1}}{2\alpha} \leq \sqrt{\frac{d}{e}}$, and Case (a22) $0 < \sqrt{\frac{d}{e}} \leq \frac{\sigma\sqrt{L_1}}{2\alpha}$.

For Case (a21), $EAC(Q, k(Q), L_1)$ decreases for $0 < Q < \frac{\sigma\sqrt{L_1}}{2\alpha}$ so the minimum occurs at $\frac{\sigma\sqrt{L_1}}{2\alpha}$. For Case (a22), the minimum solution for the unrestricting problem still is the minimum solution for the restricting problem. If we observe these two cases (a21), (a22), then the minimum solution always occurs at $MID\left(0, \frac{\sigma\sqrt{L_1}}{2\alpha}, \sqrt{\frac{d}{e}}\right)$. Hence, we summarize the results in the next Lemma.

Lemma 5 *For Case (a2), the minimum solution for the order quantity is* $Q = MID\left(0, \frac{\sigma\sqrt{L_1}}{2\alpha}, \sqrt{\frac{d}{e}}\right)$ *with* $d = D[A + C(L_1)] + \frac{h}{4\alpha}\sigma^2 L_1$, $e = \frac{1}{2}(1 - 2\alpha M_\beta)h$ *and the safety factor is* $k_1(Q) = \frac{\sigma^2 L_1 - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_1}}$ *for the expected annual cost* $EAC(Q, k_1(Q), L_1)$.

4.2. For Case (b), (Q, k) in R2, with $L(Q, k) = \frac{4\alpha^2}{\sigma^2}Q^2(\sqrt{1 + k^2} + k)^2$

From $L(Q, k) = \frac{4\alpha^2}{\sigma^2}Q^2(\sqrt{1 + k^2} + k)^2$, we imply that $(Q, k, L(Q, k))$ satisfies the following relation $\sigma\sqrt{L(Q, k)} = 2Q\alpha(\sqrt{1 + k^2} + k)$ such that the service level constraint is satisfied. According to Lemma 3, we will solve the following problem to minimize $EAC(Q, k, L(Q, k))$ where (Q, k) in R2, $0 < 2Q\alpha \leq \sigma\sqrt{L_0}$, $0 \leq k < \infty$ and $L(Q, k) = \frac{4\alpha^2}{\sigma^2}Q^2(\sqrt{1 + k^2} + k)^2$. Here, we begin our next simplification process to divide the domain of Q into two sub-domains such that we can reduce the domain of k from an interval, $0 \leq k < \infty$, to a single point. We know that

$$\begin{aligned} EAC(Q, k, L(Q, k)) &= \frac{D[A + c_1 L_0]}{Q} \\ &+ \left[\frac{h}{2} + 2hk\alpha\left(\sqrt{1 + k^2} + k\right) + h\alpha(1 - M_\beta) - 4c_1\frac{D\alpha^2}{\sigma^2}\left(\sqrt{1 + k^2} + k\right)^2\right]Q. \end{aligned} \tag{26}$$

We find that

$$\frac{\partial}{\partial k}EAC(Q, k, L(Q, k)) = \frac{2Q\alpha\left(\sqrt{1 + k^2} + k\right)^2}{\sqrt{1 + k^2}}\left[h - \frac{4c_1 D\alpha}{\sigma^2}\right]. \tag{27}$$

Based on Equation (27), we estimate the value of $h\sigma^2 - 4c_i D\alpha$. From the practical point of view, we will claim that $h - \frac{4c_i D\alpha}{\sigma^2} > 0$ and $\frac{h}{2} + h\alpha(1 - M_\beta) - 4c_i\frac{D\alpha^2}{\sigma^2} > 0$.

Observation 6 *From the numerical example of Ouyang and Chuang [8], we may claim that* $h - \frac{4c_i D\alpha}{\sigma^2} > 0$ *and* $\frac{h}{2} + h\alpha(1 - M_\beta) - 4c_i\frac{D\alpha^2}{\sigma^2} > 0$.

Explanation for Observation 6. From Ouyang and Chuang [8]. We may give the values for each parameter as follows: $h = 20$ $c_1 = 0.4$ $c_2 = 1.2$ $c_3 = 5$ $D = 600$ $\alpha = 0.015$ $\sigma = 7$ and $M_\beta = 0.5$ such that we may claim that the Observation 6 holds.

From our Observation 6, we may conclude that in Equation (27), $\frac{\partial}{\partial k}EAC(Q,k,L(Q,k)) > 0$ such that $EAC(Q,k,L(Q,k))$ is an increasing function of k so we will make k as small as possible to find the minimum value. We will divide the domain of Q from $0 < Q \le \frac{\sigma\sqrt{L_0}}{2\alpha}$ into two cases: Case (b1) $\frac{\sigma\sqrt{L_1}}{2\alpha} \le Q \le \frac{\sigma\sqrt{L_0}}{2\alpha}$ and Case (b2) $0 < Q < \frac{\sigma\sqrt{L_1}}{2\alpha}$.

For Case (b1), if we take $k = 0$, then $L(Q,0) = \frac{4Q^2\alpha^2}{\sigma^2}$ which is inside the interval $[L_1, L_0]$ so $L(Q,0)$ can be considered as a feasible solution for lead time and then we further simplify the minimum problem to find the minimum for $EAC(Q,0,L(Q,0))$ under the condition

$$\frac{\sigma\sqrt{L_1}}{2\alpha} \le Q \le \frac{\sigma\sqrt{L_0}}{2\alpha}. \tag{28}$$

Here, we face the following minimum problem:

$$EAC\left(Q, k=0, L=\frac{4Q^2\alpha^2}{\sigma^2}\right) = \frac{f}{Q} + gQ \tag{29}$$

with $f = D[A + c_1 L_0]$ and $g = \frac{h}{2} + h\alpha(1-M_\beta) - 4c_1\frac{D\alpha^2}{\sigma^2}$. In Observation 6, we have $g > 0$. If we compare Equations (19) and (29), work through the similar procedure as Equations (20) and (21) and divide into three cases similar to Cases (a11), (a12) and (a13), then we can derive the similar result as Lemma 4 in the following:

Lemma 7 *For Case (b1), the minimum solution for the order quantity is* $MID\left(\frac{\sigma\sqrt{L_1}}{2\alpha}, \frac{\sigma\sqrt{L_0}}{2\alpha}, \sqrt{\frac{f}{g}}\right)$ *with* $f = D[A + c_1 L_0]$ *and* $g = \frac{h}{2} + h\alpha(1-M_\beta) - 4c_1\frac{D\alpha^2}{\sigma^2}$*, and the safety factor* $k = 0$ *for the expected annual cost* $EAC(Q,0,L(Q,0))$.

Remark. If we compute $L(Q,0)$, then $L(Q,0) = MID\left(L_1, L_0, \frac{4f^2\alpha^2}{g^2\sigma^2}\right)$ such that the local minimum solution for lead time sometimes will not equal to L_1 or L_0. This indicates that in Ouyang and Chuang's [8] claim, that the lead time must reduce to boundary points L_1 or L_0 is sometimes not true.

Next, for Case (b2), with $0 < Q < \frac{\sigma\sqrt{L_1}}{2\alpha}$, by Equation (27) and Observation (6), we accepted that $EAC(Q,k,L(Q,k))$ is an increasing function of k so we will make k as small as possible to find the minimum value. We cannot directly quote the results because the discussion for Case (a2) in Equation (22) requires a new explanation to show why we will take the least k in R2 that satisfies the service level constraint and then $k = k_1(Q)$ to imply that

$$L(Q, k_1(Q)) = L_1. \tag{30}$$

If we still take $k = 0$, then $L(Q,0) = \frac{4\alpha^2Q^2}{\sigma^2} < L_1$ such that $L(Q,0)$ is not in $[L_1, L_0]$ so $L(Q,0)$ cannot be accepted as a possible solution for lead time. Hence, for Case (b2), we cannot take $k = 0$ and instead we will take the least k such that $L(Q,k) = \frac{4\alpha^2}{\sigma^2}Q^2(\sqrt{1+k^2}+k)^2$ in $[L_1, L_0]$. This yields that $L(Q,k) = \frac{4\alpha^2}{\sigma^2}Q^2(\sqrt{1+k^2}+k)^2 = L_1$. The discussion before Equation (22) implies $k = k_1(Q)$ and Equation (30). Hence, we face the following minimum problem:

$$EAC(Q, k_1(Q), L_1) = \frac{d}{Q} + eQ, \tag{31}$$

where $d = \frac{4\alpha D[A+C(L_1)]+h\sigma^2 L_1}{4\alpha}$ and $e = \frac{h}{2}(1-2\alpha M_\beta)$ under the condition

$$0 < Q < \frac{\sigma\sqrt{L_1}}{2\alpha}. \tag{32}$$

Hence, comparing Equations (23) and (32) and following the same calculation as Lemma 5, we find the next Lemma.

Lemma 8 *For Case (b2), the minimum solution for the order quantity is* $Q = MID\left(0, \frac{\sigma\sqrt{L_1}}{2\alpha}, \sqrt{\frac{d}{e}}\right)$ *with* $d = D[A + C(L_1)] + \frac{h_1}{4\alpha}\sigma^2 L_1$, $e = \frac{1}{2}(1 - 2\alpha M_\beta)h$ *and the safety factor is* $k_1(Q) = \frac{\sigma^2 L_1 - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_1}}$ *for the expected annual cost* $EAC(Q, k_1(Q), L_1)$.

4.3. For Case (c), (Q, k) in R3, with $L(Q, k) = L_1$

From (Q, k) in R3, we know that in the beginning the possible range for Q is $(0, \infty)$ and consequently divide the domain of Q into the following two cases: (c1) $\frac{\sigma\sqrt{L_1}}{2\alpha} \le Q < \infty$ and (c2) $0 < Q < \frac{\sigma\sqrt{L_1}}{2\alpha}$.

For Case (c1), by the method similar to Case (a1), we derive the next lemma.

Lemma 9 *For Case (c1), the minimum solution for the order quantity is* $MID\left(\infty, \frac{\sigma\sqrt{L_1}}{2\alpha}, \sqrt{\frac{a}{b}}\right)$ *with* $a = D[A + C(L_1)]$ *and* $b = \frac{h}{2}$, *for the expected annual cost* $EAC(Q, 0, L_1)$.

Next, for Case (c2), we directly quote Lemma 5 to imply the next results.

Lemma 10 *For Case (c2), the minimum solution for the order quantity is* $Q = MID\left(0, \frac{\sigma\sqrt{L_1}}{2\alpha}, \sqrt{\frac{d}{e}}\right)$ *with* $d = D[A + C(L_1)] + \frac{h\sigma^2 L_1}{4\alpha}$, $e = \frac{1}{2}(1 - 2\alpha M_\beta)h$ *and the safety factor is* $k_1(Q) = \frac{\sigma^2 L_1 - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_1}}$ *for the expected annual cost* $EAC(Q, k_1(Q), L_1)$.

4.4. 4.4 Case (d), (Q, k) in R3, with $L(Q, k) = L_0$

We divided the domain of Q into the following two cases: (d1) $\frac{\sigma\sqrt{L_0}}{2\alpha} \le Q < \infty$ and (d2) $0 < Q < \frac{\sigma\sqrt{L_0}}{2\alpha}$. For Case (d1), we know the next Lemma.

Lemma 11 *For Case (d1), the minimum solution for the order quantity is* $MID\left(\infty, \frac{\sigma\sqrt{L_0}}{2\alpha}, \sqrt{\frac{m}{b}}\right)$ *with* $m = D[A + C(L_0)]$ *and* $b = \frac{h}{2}$, *for the expected annual cost* $EAC(Q, 0, L_0)$.

Next, for Case (d2), by a small modification of Equation (16) to change L_1 to L_0 we still have that $EAC(Q, k, L_0)$ increases in k. Moreover, we will make a small modification of Equation (22) so we assume a new notation, say $k_0(Q)$, such that

$$k_0(Q) = \frac{\sigma^2 L_0 - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_0}}, \tag{33}$$

for $0 < Q < \frac{\sigma\sqrt{L_0}}{2\alpha}$, and further simplify the minimum problem to find the minimum for $EAC(Q, k_0(Q), L_0)$ under the condition for $0 < Q < \frac{\sigma\sqrt{L_0}}{2\alpha}$. Similarly to Case (a2), we derive the next Lemma.

Lemma 12 *For Case (d2), the minimum solution for the order quantity is* $MID\left(0, \frac{\sigma\sqrt{L_0}}{2\alpha}, \sqrt{\frac{r}{e}}\right)$ *with* $r = D[A + C(L_0)] + \frac{h\sigma^2}{4\alpha}L_0$, $e = \frac{h}{2}(1 - 2\alpha M_\beta)$ *and the safety factor* $k_0(Q) = \frac{\sigma^2 L_0 - 4Q^2\alpha^2}{4Q\alpha\sigma\sqrt{L_0}}$ *for the expected annual cost* $EAC(Q, k_0(Q), L_0)$.

For easy comparison among different cases, we combined our previous results in the following table 1, with the same expression, except for simplifying the space, then we further assume that $S_1 = \frac{\sigma\sqrt{L_1}}{2\alpha}$, $S_0 = \frac{\sigma\sqrt{L_0}}{2\alpha}$, $U = \sqrt{\frac{a}{b}}$, $V = \sqrt{\frac{d}{e}}$, $W = \sqrt{\frac{f}{g}}$, $X = \sqrt{\frac{m}{b}}$ and $Y = \sqrt{\frac{r}{e}}$.

Table 1: The minimums for different cases when L is restricted with $L_1 \le L \le L_0$.

Case a	(a1) $S_1 \le Q \le S_0$	(a11) $U \le S_1$	S_1	Min $= MID(S_1, S_0, U)$
		(a12) $S_1 \le U \le S_0$	U	
		(a13) $S_0 \le U$	S_0	
	(a2) $0 < Q < S_1$	(a21) $S \le V$	S_1	Min $= MID(0, S_1, V)$
		(a22) $0 < V \le S_0$	V	
Case b	(b1) $S_1 \le Q \le S_0$	(b11) $W \le S_1$	S_1	Min $= MID(S_1, S_0, W)$
		(b12) $S_1 \le W \le S_0$	W	
		(b13) $S_0 \le W$	S_0	
	(b2) $0 < Q < S_1$	(b21) $S_1 \le V$	S_1	Min $= MID(0, S_1, V)$
		(b22) $V \le S_1$	V	
Case c	(c1) $S_1 \le Q \le \infty$	(c11) $U \le S_1$	S_1	Min $= MID(\infty, S_1, U)$
		(c12) $S_1 \le U$	U	
	(c2) $0 < Q < S_1$	(c21) $V \le S_1$	V	Min $= MID(0, S_1, V)$
		(c22) $S \le V$	S_1	
Case d	(d1) $S_0 \le Q < \infty$	(d11) $X \le S_0$	S_0	Min $= MID(\infty, S_0, X)$
		(d12) $S_0 \le X$	X	
	(d2) $0 < Q < S_0$	(d21) $Y \le S_0$	Y	Min $= MID(0, S_0, Y)$
		(d22) $S_0 \le Y$	S_0	

Now, we combined the previous results for $L \in [L_1, L_0]$ since Lemmas 5, 8 and 10 imply the same result, there are six local minimum solutions for the minimum problem. We compared the values for these six values to determine the minimum solution for $L \in [L_1, L_0]$.

Here, we discussed how to generalize the results for the interval $L \in [L_1, L_0]$ to the general case $L \in [L_i, L_{i-1}]$. We change L_1 to L_i, L_0 to L_{i-1}, c_1 to c_i, and $k_1(Q)$ for $0 < 2Q\alpha < \sigma\sqrt{L_1}$ to $k_i(Q)$ for $0 < 2Q\alpha < \sigma\sqrt{L_i}$. Moreover, we modified $c_1 L_0$ to $c_i L_{i-1} + \sum_{j=1}^{i-1} c_j(b_j - a_j)$. All the previous results can be derived under the new expression.

After we found the minimum solution for $L \in [L_i, L_{i-1}]$ with $i = 1, 2, ..., n$, we compared these local minimum solutions to find the optimal solution.

5. Numerical Example

To illustrate our improvement, we considered the same numerical example as Ouyang and Chuang [8] with the following data: $D = 600$ units/years, $A = \$200$ per order, $h = \$20$ /unit/year, $\mu = 11$ units/week, $\sigma = 7$ units/week, the service level $1 - \alpha = 0.985$, i.e., the proportion of demand that are not met from stock is at most $\alpha = 0.015$. The lead time has three components with $c_1 = \$0.4$ /day, $a_1 = 6$ days, $b_1 = 20$ days, $c_2 = \$1.2$ /day, $a_2 = 6$ days, $b_2 = 20$ days, $c_3 = \$5.0$ /day, $a_3 = 9$ days, $b_{31} = 16$ days, the backorder rate β during the stockout period has a uniform distribution, i.e., the probability density function of β is $g(\beta) = 1$ for $0 \le \beta \le 1$ and $g(\beta) = 0$, otherwise. Hence, the mean of β is $M_\beta = 0.5$.

We computed the local minimum value from Lemmas 4, 5, 7, 9, 11 and 12 and list them in the following Table 2. We derived that the minimum value occurs at $Q^* = 111.068$, $k^* = 0$ and $L^* = L_1 = 6$ with $EAC(Q^*, k^*.L^*) = 2307.08$.

Ouyang and Chuang [8] found that $Q^* = 142$, $k^* = 1.49$ and $L^* = L_2 = 4$ with $EAC(Q^*, k^*.L^*) = 2798.23$. We may say that by our improved method we save \$691.15 which means a 29.96% saving.

Table 2: Local minimum value for each case

	$L \in [L_1, L_0]$	$L \in [L_2, L_1]$	$L \in [L_3, L_2]$
Lemma 4	6017.04	5022.61	4484.21
Lemma 5	2953.03	2798.51	2831.17
Lemma 7	6017.04	5022.61	4484.21
Lemma 9	2307.08	6018.95	5058.23
Lemma 11	2422.19	2396.06	5022.61
Lemma 12	3142.65	2953.23	2798.51

6. Conclusion

In the above discussions we pointed out the questionable results in the paper of Ouyang and Chuang [8] such that the minimum may not occur at the boundary points. We offered the corrected algorithm to find the optimal solution. Our refined algorithms are easy to use and mathematically sound and provide the optimal replenishment solution for decision makers. From the numerical example, we achieved an excellent saving of 30%.

Acknowledgments

The authors are grateful to the two anonymous referees for constructive comments that led to important improvement in this note. All their suggestions were incorporated directly into the text. In addition, the authors wish to express appreciation to Miss Dana Ter for her assistance of English improvement.

References

[1] M. Ben-Daya and A. Raouf: Inventory models involving lead time as decision variable. *Journal of the Operational Research Society*, **45** (1994), 579-582.

[2] P. Chu P., K.L. Yang and P.S. Chen, Improved inventory models with service level and lead time. *Computers and Operations Research*, **32** (2005), 285-296.

[3] G. Gallego: A minmax distribution-free procedure for the (Q, R) inventory model. *Operations Research Letters*, **11** (1992), 55-60.

[4] S.P. Lan, P. Chu, K.J. Chung, and K.J. Wan: A simple method to locate the optimal solution of inventory model with variable lead time. *Computer and Operations Research*, **26** (1999), 599-605.

[5] C.J. Liao and C.H. Shyu: An analytical determination of lead time with normal demand. *International Journal of Operations Production Management*, **11** (1991), 72-78.

[6] I. Moon and S. Choi: A note on lead time and distribution assumptions in continuous reviews inventory models. *Computer and Operations Research*, **25** (1998), 1007-1012.

[7] I. Moon and G. Gallego: Distribution free procedures for some inventory models. *Journal of the Operational Research Society*, **45** (1994), 651-658.

[8] L.Y. Ouyang and B.R. Chuang: A minimax distribution free procedure for stochastic inventory models with a random backorder rate. *Journal of the Operations Research Society of Japan*, **42** (1999), 342-351.

[9] L.Y. Ouyang and K.S. Wu: Mixture inventory model involving variable lead time with a service level constraint. *Computers and Operations Research*, **24** (1997), 875-882.

[10] L.Y. Ouyang and K.S. Wu: A mixture distribution free procedure for mixed inventory model with variable lead time. *International Journal of Production Economics*, **56** (1998),

511-516.

[11] L.Y. Ouyang, N.C. Yeh and K.S. Wu: Mixture inventory models with backorders and lost sales for variable lead time. *Journal of the Operational Research Society*, **47** (1996), 829-832.

[12] C.H. Pan and Y.C. Hsiao: Inventory models with back-order discounts and variable lead time. *International Journal of Systems Science*, **32** (2001), 925-929.

[13] H. Scarf: *A Min-max Solution of an Inventory Problem, Studies in The Mathematical Theory of Inventory and Production*, Chapter 12, (Stanford Univ. Press., Stanford, CA, 1958).

[14] E.A. Silver and R. Peterson: *Decision Systems for Inventory Management and Production Planning* (John Wiley, New York, 1985).

[15] J.K. Wu and H.Y. Tsai: Mixture inventory model with back orders and lost sales for variable lead time demand with the mixture of normal distribution. *International Journal of Systems Science*, **32** (2001), 259-268.

Robert Lin
Dept of General Education Center,
Oriental Institute of Technology
Post Box 16-092, Banciao,
Taipei County, 22099, Taiwan (R.O.C.)
E-mail: hjlin@mail.oit.edu.tw