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A NEW OPTIMAL SEARCH ALGORITHM FOR THE TRANSPORTATION FLEET MAINTENANCE SCHEDULING PROBLEM

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Abstract In this study, we propose a new solution approach for the Transportation Fleet Maintenance Scheduling Problem (TFMSP). Before presenting our solution approach, we first review Goyal and Gunasekaran's [International Journal of Systems Science, **23** (1992) 655-659] mathematical model and their search procedure for determining the economic maintenance frequency of a transport fleet. To solve the TFMSP, we conduct a full analysis on the mathematical model. By utilizing our theoretical results, we propose an efficient search algorithm that finds the optimal solution for the TFMSP within a very short run time. Based on our experiments using random data, we conclude that the proposed search algorithm out-performs Goyal and Gunasekaran's search procedure.

Keywords: Scheduling, maintenance, mathematical model, algorithm

1. Introduction

In this study, we devote our efforts toward investigating a mathematical model for determining the economic maintenance frequency of a transportation fleet. We name this problem the "Transportation Fleet Maintenance Scheduling Problem", which is abbreviated as the TFMSP. The mathematical model for the TFMSP was previously proposed by Goyal and Gunasekaran [3]. We extend their work in two aspects: first, we conduct a full theoretical analysis on the theoretical properties of the mathematical model, and second, we propose an efficient search algorithm that solves for the optimal solution in Goyal and Gunasekaran's [3] model.

As mentioned in [3], the problem of determining the economic maintenance of a machine has been dealt with extensively in management science, operations research, and industrial engineering (see [2], [4], [6], [7], and [8]). But researchers pay limited attention to the problem of determining the operating and maintenance schedules for a transportation fleet.

For the rest of this section, we review Goyal and Gunasekaran's [3] mathematical model for the Transportation Fleet Maintenance Scheduling Problem (TFMSP).

Before presenting the mathematical model, we first introduce the assumptions made and the notation used later. There are m groups of vehicles, and the number of vehicles in group i is denoted as n_i. In the TFMSP, the decision maker plans the maintenance schedules of the vehicle groups in some basic period, denoted by T, (*e.g.*, in days, weeks, or bi-weeks, *etc.*). The maintenance work on a group of vehicles is carried out at a fixed, equal-time interval that is called the *maintenance cycle* for that group of vehicles. The vehicles in the i^{th} group are sent for maintenance once in k_i basic periods where k_i is a positive integer. (Therefore, the maintenance cycle for the vehicles in the i^{th} group is k_iT.) We note that the model for the TFMSP is for planned maintenance, and the model does not consider unplanned fleet vehicle failure in the scheduling of fleets.

We consider two categories of costs in the TFMSP, namely, the operating cost and the maintenance cost. The operating cost of a vehicle depends on the length of the maintenance cycle, and it is assumed to increase linearly with respect to time since the last maintenance on the vehicle. Specifically, the operating cost per unit of time at time t after the last maintenance for a vehicle in group i is given by $f_i(t) = a_i + b_i t$ where a_i is the fixed cost and b_i indicates the increase in the operating cost per unit of time. For each vehicle in group i, we also assume that it takes X_i units of time for its maintenance work. And the utilization factor of a vehicle in the i^{th} group on the road is Y_i , where X_i and Y_i are known constants. (One may refer to [10] for further discussions on the utilization factor of a vehicle.) Therefore, the actual time during which a vehicle can operate is equal to $Y_i(k_iT - X_i)$, and the total operating cost for a vehicle in group i is given by

$$\begin{aligned}\int_0^{Y_i(k_iT-X_i)} f_i(t)dt &= \int_0^{Y_i(k_iT-X_i)} (a_i + b_i t)dt \\ &= Y_i(a_i - b_iX_iY_i)k_iT + 0.5b_iY_I^2k_i^2T^2 - X_iY_i(a_i - 0.5b_iX_iY_i) \quad (1.1)\end{aligned}$$

The fixed cost of starting the maintenance for a vehicle in group i is given by s_i. On the other hand, as maintenance work is carried out at intervals of T, a fixed cost, denoted by S, will be incurred for all vehicle groups scheduled for maintenance in each basic period.

The objective function of the TFMSP is to minimize the average total costs occurred per unit of time. Therefore, we divide the cost terms of each vehicle by its cycle time respectively to obtain their corresponding terms in the objective function. By the derivation above, the mathematical model for the TFMSP can be expressed as problem (P_0).

$$(P_0) \min Z((k_1, k_2, \ldots, k_m), T) = \frac{S}{T} + \sum_{i=1}^{m} \Phi_i(k_i, T) + u \quad (1.2)$$

where $\Phi_i(k_i, T) = \frac{n_iC_{1i}}{k_iT} + n_iC_{2i}k_iT$, $C_{1i} = s_i - X_iY_i(a_i - 0.5b_iX_iY_i)$ and $C_{2i} = 0.5b_iY_i^2$. Also, $u = \sum_{i=1}^{m} n_iY_i(a_i - b_iX_iY_i)$ is a constant since all the parameters are given in its expression.

Then, solving the problem (P_0) is equivalent to obtaining the optimal solution for the problem (P) as follows.

$$(P)\ \Psi(k_1, k_2, \ldots, k_m), T) = \inf_{T>0}\{\frac{S}{T} + \sum_{i=1}^{m} \Phi_i(k_i, T) | k_i \in \mathbb{N}^+, i = 1, \ldots, m)\}. \quad (1.3)$$

In the TFMSP, the decision maker needs to determine T (*i.e.*, the basic period) and $(k_1, k_2, ..., k_m)$ (*i.e.*, the frequency of maintenance for vehicles in each group) so as to minimize the total costs incurred per unit of time.

We outline the organization of this paper as follows: We will review the studies in the literature for the Transportation Fleet Maintenance Scheduling Problem in the second section. Then, in Section 3, we present a full theoretical analysis on the optimal cost curve of the problem (P). Based on our theoretical results, we derive an effective search algorithm that efficiently solves the TFMSP in Section 4. In the first part of Section 5, we employ a numerical example to demonstrate the implementation of the proposed algorithm. Then, we use randomly generated examples to show that the proposed algorithm significantly outperforms Goyal and Gunasekaran's search procedure in the second part of Section 5. Finally, we address our concluding remarks in Section 6.

2. Literature Review

In this section, we review the studies in the literature for the Transportation Fleet Maintenance Scheduling Problem.

We first review the solution approach proposed in Goyal and Gunasekaran's [3] paper. The algorithm is based on two equations that are derived by setting the first derivative of $Z((k_1, k_2, \ldots, k_m), T)$ with respect to the decision variables to zero:

$$T(k_1, k_2, \ldots, k_m) = \sqrt{\frac{S + \sum_{i=1}^{m} \frac{n_i C_{1i}}{k_i}}{\sum_{i=1}^{m} n_i C_{2i} k_i}} \tag{2.1}$$

$$k_i^*(T) = \sqrt{\frac{C_{1i}}{C_{2i}} \frac{1}{T}} \tag{2.2}$$

Goyal & Gunasekaran's search procedure

1. For the first iteration, assume $k_i = k_i^{(0)} = 1$ for all i, and obtain the first estimate of $T = T^{(1)}$ from (2.1). At $T = T^{(1)}$, determine $k_i = k_i^{(1)}$ from (2.2) for all i. If $k_i^{(1)}$ values are not integers, then select the nearest non-zero integer.

2. Using $k_i = k_i^{(1)}$ from (2.2) for $i = 1, \ldots, m$, obtain $T = T^{(2)}$ from (2.1) and then $k_i = k_i^{(2)}$ from (2.2) using $T = T^{(2)}$. Repeat the process until the r^{th} iteration and stop when $k_i^{(r)} = k_i^{(r-1)}$ for $i = 1, \ldots, m$. The economic policy is obtained at $T^* = T^{(r)}$ and $k_i^* = k_i^{(r)}$.

Later, van Egmond, Dekker & Wildeman [9] discussed Goyal and Gunasekaran's search procedure. They indicated that the function $Z((k_1, k_2, \ldots, k_m), T)$ is not convex as Goyal and Gunasekaran assumed in [3]. Since the values of k_i need to be integers, the determination of the global optimization is not as easy as Goyal and Gunasekaran suggested. They also showed that it is not necessarily the k_i minimizing Z when one rounds (2.2) to the nearest non-zero integer. Finally, they indicated that Goyal and Gunasekaran's search procedure often stops after its first iteration without obtaining an optimal solution. These three problems explain why Goyal and Gunasekaran's solution does not always obtain an optimal solution. In fact, it is often stuck in a local optimal solution.

However, van Egmond, Dekker & Wildeman [9] only mentioned that one needs to try different starting values to find an optimal solution, but without proposing a new solution approach to solve the TFMSP.

To the best of the authors' knowledge, there exists no solution approach that can find an optimal solution for the TFMSP. Therefore, we are motivated to propose a new solution approach towards this aim in this study.

3. Theoretical Analysis

In this section, we discuss some theoretical analyses that provide insights into the optimal cost function of $\Psi((k_1, k_2, \ldots, k_m), T)$.Our theoretical analyses facilitate the derivation of the search algorithm presented in Section 3.

By observing the right-side of (1.3), we learn that the terms are separable. Therefore, we are motivated to study the properties of $\Phi_i(k_i, T)$ since they will establish the foundation for further investigation of the function $\Psi((k_1, k_2, \ldots, k_m), T)$.

Proposition 3.1. *For any given $k_i \in \mathbb{N}^+$, the function $\Phi_i(k_i, T)$ satisfies the following properties for $T > 0, i \in \{1, \ldots, m\}$.*

1. $\Phi_i(k_i, T)$ is strictly convex;
2. $\Phi_i(k_i, T)$ has a minimum for $T = x_i^*/k_i$ with x_i^* given by

$$x_i^* = \sqrt{C_{1i}/C_{2i}} \tag{3.1}$$

3. The function $\Phi_i(k_i, T)$ obtains its minimal objective function value by

$$2n_i\sqrt{C_{1i}C_{2i}} \tag{3.2}$$

Proof. We may prove these assertions using simple algebra. □

Let us define a new function $g_i(T)$ by taking the optimal value of k_i at any value $T' > 0$ for the function $\Phi_i(k_i, T)$ as follows.

$$g_i(T) := \inf_{k_i \in \mathbb{N}^+} \{\Phi_i(k_i, T') | T = T' \in \mathbb{R}^+\} \tag{3.3}$$

Consequently, the problem (P) can be re-written as

$$(P_1)\ \Gamma(T) = \inf_{T>0} \{\frac{S}{T} + \sum_{i=1}^{m} g_i(T)\} \tag{3.4}$$

where the function $\Gamma(T)$ is the optimal objective function value of problem (P_1) with respect to T.

Before further analyzing problem (P_1), we first graphically display the curves of the $\Phi_i(k_i, T)$ function with $k_i = (1, 2, 3, 4)$ in Figure 1. Note that the curve of the $g_i(T)$ function is actually the lower envelope of the $\Phi_i(k_i, T)$ functions.

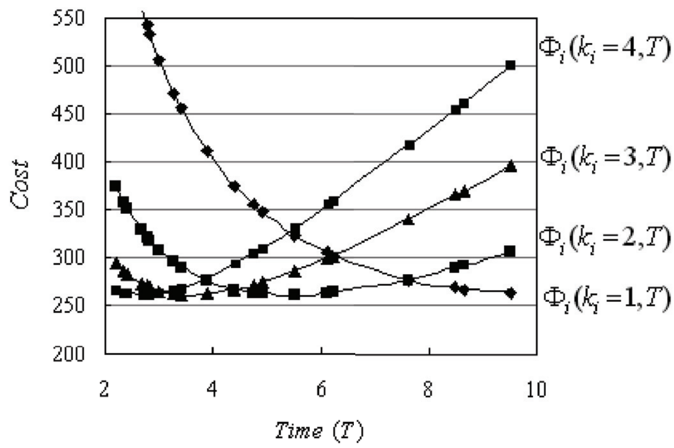

Figure 1: The curves of the $\Phi_i(k_i, T)$ function with $k_i = (1, 2, 3, 4)$

Importantly, Figure 1 shows two interesting observations on the $g_i(T)$ function:

1. The function $g_i(T)$ is piece-wise convex with respect to T.
2. Suppose that $k^*(w^-)$ and $k^*(w^+)$, respectively, are the optimal multipliers of the left-side and right-side convex curves with regard to a junction point w in the plot of the $g_i(T)$ function. Then, $k^*(w^-) = k^*(w^+)+1$, where $w^- = w-\varepsilon$, $w^+ = w+\varepsilon$ and $\varepsilon \to 0^+$.

In the following discussion, we will have further analysis on these two observations and will formally prove them as the basis for deriving the theoretical properties for problem (P_1).

3.1. The junction point

Next, we define a "junction point" for $g_i(T)$ as a particular value of T where two consecutive convex curves $\Phi_i(k_i, T)$ and $\Phi_i(k_i + 1, T)$ concatenate. These junction points determine at "what value of T" where one should change the value of k_i so as to obtain the optimal value for the $g_i(T)$ function.

We first derive a closed-form for the location of the junction points. We define the difference function $\Delta_i(k, T)$ by

$$\begin{aligned}\Delta_i(k,T) &= \Phi_i(k_i+1,T) - \Phi_i(k_i,T) \\ &= \frac{n_iC_{1i}}{(k+1)T} + n_iC_{2i}(k+1)T - \frac{n_iC_{1i}}{kT} - n_iC_{2i}kT \\ &= -\frac{n_iC_{1i}}{k(K+1)T} + n_iC_{2i}T\end{aligned} \tag{3.5}$$

We note that w is the point where two neighboring convex curves $\Phi_i(k_i + 1, T)$ and $\Phi_i(k_i, T)$ meet. Importantly, such a junction point w provides us with the information on at "what value of T" where one should change the value of k so as to secure the optimal value for the $g_i(T)$ function.

By the rationale discussed above, we derive a closed form to locate the junction points by letting $\Delta_i(k, T) = 0$ as follows.

$$\delta_i(k) = \sqrt{\frac{C_{1i}}{C_{2i}(k+1)k}} = \sqrt{\frac{2(s_i - X_iY_i(a_i - 0.5b_iX_iY_i)}{b_iY_i^2(k+1)k}} \tag{3.6}$$

Note that $\delta_i(k)$ indicates the location of the k^{th} junction point of the function $g_i(T)$ (from its right-side). By (3.6), the following inequality (3.7) holds

$$\delta_i(v) < ... < \delta_i(k+1) < \delta_i(k) < ... < \delta_i(2) < \delta_i(1) \tag{3.7}$$

where v is an (unknown) upper bound on the value of k.

Theorem 3.1 is an immediate result from (3.6) and (3.7).

Theorem 3.1. *Suppose that $k^*(w^-)$ and $k^*(w^+)$ are the optimal multipliers of the left-side and right-side convex curves with regard to a junction point w of the $g_i(T)$ function, then $k^*(w^-) = k^*(w^+) + 1$.*

The following corollary is also a by-product of (3.6) and (3.7), and it provides an easier way to obtain the optimal multiplier: $k_i^*(T) \in \mathbb{N}^+$ for the $g_i(T)$ function for any given $T > 0$.

Corollary 3.1. *For any given $T > 0$, an optimal value of $k_i^*(T) \in \mathbb{N}^+$ for the $g_i(T)$ function is given by*

$$k_i^*(T) = \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4C_{1i}}{C_{2i}T^2}} \right\rceil \tag{3.8}$$

with $\lceil\cdot\rceil$ denoting the upper-entier function.

Proof. For any given $T > 0$, an optimal value of $k \in \mathbb{N}^+$ for the $g_i(T)$ function is such that $\delta_i(k) \leq T < \delta_i(k-1)$. Equivalently, the value of k must satisfy $\sqrt{\frac{C_{1i}}{C_{2i}(k+1)k}} \leq T$ and $T < \sqrt{\frac{C_{1i}}{C_{2i}(k-1)k}}$. Therefore, we have $k^2 + k - \frac{C_{1i}}{C_{2i}T^2} \geqslant 0$ and $k^2 - k - \frac{C_{1i}}{C_{2i}T^2} < 0$. Since k must be positive, we have $-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4C_{1i}}{C_{2i}T^2}} \leq k < \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4C_{1i}}{C_{2i}T^2}}$. Thus, we complete the proof. □

3.2. Some insights into the optimal cost function

Utilizing our theoretical analyses on the $\Phi_i(k_i, T)$ and $g_i(T)$ functions one can gain more insight into the $\Gamma(T)$ function.

First, Propositions 3.2 and 3.3 characterize the $\Gamma(T)$ function as follows.

Proposition 3.2. *The $\Gamma(T)$ function is piece-wise convex with respect to T.*

Proposition 3.3. *All the junction points for each group i, will be inherited by the $\Gamma(T)$ function. In other words, if w is a junction point for a group i, w must also show as a junction point on the piece-wise convex curve of the $\Gamma(T)$ function.*

To make our notation simpler, we define $\mathbf{k} = (k_1, \ldots, k_m)$ to represent a set of maintenance frequency. Theorem 3.2 is an immediate result of Theorem 3.1 and Proposition 3.3.

Theorem 3.2. *Suppose that $\mathbf{k}(w^-)$ and $\mathbf{k}(w^+)$, respectively, are the set of optimal multipliers for the left-side and right-side convex curves with regard to a junction point w in the plot of the $\Gamma(T)$ function. Then, $\mathbf{k}(w^-)$ is secured from $\mathbf{k}(w^+)$ by changing at least one of k_i by $k_i^*(w^-)$ and $k_i^*(w^+)$.*

4. The Proposed Search Algorithm

In this section, we propose a search algorithm that solves the optimal solution for the problem (P_1) in (3.4).

Our theoretical analyses in Section 3 encourage us to solve the problem (P_1) by searching along the T-axis. To design such a search algorithm, we first need to define the search range by a lower and an upper bound on the T-axis, which are denoted by $T_{\min}$ and $T_{\max}$, respectively. We note that the bounds $T_{\min}$ and $T_{\max}$ are derived by asserting that the optimal solution in $[T_{\min}, T_{\max}]$ must be no worse than any solution outside of $[T_{\min}, T_{\max}]$. Also, we must utilize our theoretical analyses on the optimality structure, especially, the properties of the junction points on the $\Gamma(T)$ function.

In the following discussions, we first discuss how to find initial lower and upper bounds of the search range and how to use an iterative procedure to improve the initial lower and upper bounds. Also, we demonstrate how to use the junction points to proceed with the search. Then, we propose an approach to further improve the lower bound on the search range. Finally, we summarize our proposed search algorithm.

4.1. The initial lower and upper bounds of the search range

First, we present an upper bound on the search range by the Common Cycle (CC) approach in which it requires that $k_i = 1$ for all i, *i.e.*, all the vehicle groups share a common maintenance cycle. We set

$$T_{CC} = \sqrt{\left(S + \sum_i n_i C_{1i}\right) / \sum_i n_i C_{2i}} \tag{4.1}$$

where T_{CC} is the optimal maintenance cycle for the CC approach. Next, we will show that it is appropriate to set $T_{\max} = T_{CC}$ in the following lemma.

Lemma 4.1. *For the $\Gamma(T)$ function, there exist no local minima for $T > T_{cc}$.*

Denote the optimal objective function value of (P_1) and the optimal value of the basic period by $\Psi^*$and T^*. Next, we derive an initial lower bound on the search range in the following lemma.

Lemma 4.2. *The value*

$$\beta_1 = \frac{2S}{\Psi^U} \tag{4.2}$$

serves as a lower bound for T^* where Ψ^U is an upper bound on the optimal objective function value of the problem (P_1).

Proof. For any given set of $\mathbf{k}$, its local minimum $\breve{T}(\mathbf{k})$ is given by $\breve{T}(\mathbf{k}) = \sqrt{\left(S + \sum_{i=1}^{m} \frac{n_i C_{1i}}{k_i}\right) / \sum_{i=1}^{m} n_i C_{2i} k_i}$. Substituting $\breve{T}(\mathbf{k})$ into the objective function of the problem (P) in (1.3), one shall obtain its optimal objective function value by

$$\Psi(\mathbf{k}, T) = 2\sqrt{\left(S + \sum_{i=1}^{m} \frac{n_i C_{1i}}{k_i}\right)\left(\sum_{i=1}^{m} n_i C_{2i} k_i\right)}. \tag{4.3}$$

By the expressions of $\breve{T}(\mathbf{k})$ and (4.3), it follows that $\Psi^* T^* > 2S$, so $T^* > 2S/\Psi^*$. Given Ψ^U is an upper bound on the optimal objective function value of the problem (P_1), it obviously holds that $T^* > 2S/\Psi^*$ since $\Psi^U \geq \Psi^*$.
Note that we need an upper bound Ψ^U to obtain β_1 as indicated in eq. (4.2). The lower the value of Ψ^U, the tighter the lower bound β_1. Here, we have an easy way to obtain a good value of Ψ^U. First, we shall locate $T_0 = \min_i \sqrt{0.5 C_{1i}/C_{2i}}$. Denote $\mathbf{k}^*(T') \equiv (k_1^*(T'), k_2^*(T'), \ldots, k_m^*(T'))$ as the set of optimal maintenance frequencies with respect to a given value of T'. Then, we obtain the optimal $\mathbf{k}^*(T_0)$ corresponding to T_0 by (3.8). Since the objective function value of any feasible solution serves as an upper bound on Ψ^*, we have an upper bound by $\Psi^U = \Psi(\mathbf{k}^*(T_0), T_0)$ from eq. (4.3). Consequently, an initial lower bound is obtained by $T_{\min} = 2S/\Psi^U$. □

Intuitively, if we may shorten the search range on the T-axis, we may reduce the computational efforts in the proposed search algorithm. Therefore, we are motivated to improve the initial lower and upper bounds.

Based on our numerical experiments in our study, we have an interesting observation on the $\Gamma(T)$ function: the $\Gamma(T)$ function is monotonically decreasing from $T_{\min}$ to a value T_A, and monotonically increasing from a value T_B to $T_{\max}$. Therefore, if we may determine the values of T_A and T_B, then we could efficiently confine our search range to $[T_A, T_B]$. Before presenting our iterative procedures to determine the values of T_A and T_B, we discuss their theoretical foundations, *i.e.*, Lemma 4.3, Theorem 4.1 and Corollary 4.1, as follows.

Lemma 4.3. *Let $\mathbf{k}^a$ be the set of optimal maintenance frequencies that minimizes $\Psi(\cdot, T)$ in the range $[T_l^a, T_u^a]$. Let the optimal value of T corresponding to $\mathbf{k}^a$ be T_a^*. If $T_l^a > T_u^a$, then the function is monotonically decreasing in $[T_l^a, T_u^a]$.*

Next, Theorem 4.1 and Corollary 4.1 lay important foundations for our iterative procedures to improve the bounds of the search range.

Theorem 4.1. *Let* $\mathbf{k}^a$ *be the set of optimal maintenance frequencies that minimizes* $\Psi(\cdot, T)$ *in the range* $[T_l^a, T_u^a]$. *Let the optimal value of* T *corresponding to* $\mathbf{k}^a$ *be* T_a^*. *If* $T_a^* > T_u^a$, *then the* $\Gamma(T)$ *function is monotonically decreasing in* $[T_l^a, T_a^*]$.

Proof. From Lemma 4.3, it is clear that when $T_a^* \geq T_u^a$, the function $\Psi(\mathbf{k}^a, T)$, which is the same as the $\Gamma(T)$ function, is monotonically decreasing in $[T_l^a, T_a^*]$. Now, consider those sub-intervals of T between T_u^a and T_a^*. Within each of them the optimal set of optimal maintenance frequencies remains unchanged. Let $\mathbf{k}^b$ be the set of optimal maintenance frequencies in $\left[T_l^b, T_u^b\right]$, where $\left[T_l^b, T_u^b\right] \subseteq [T_l^a, T_a^*]$, *i.e.*, $T_u^a \leq T_l^b < T_u^b \leq T_a^*$, We assert that there exists at least one group i such that $k_i^b < k_i^a$ from (3.8) because of $T_u^a \leq T_l^b$. Let the optimal value of T corresponding to $\mathbf{k}^b$ be T_b^*. As $k_i$decreases, the numerator of $\breve{T}(k) = \sqrt{\left(S + \sum_{i=1}^{m} \frac{n_i C_{1i}}{k_i}\right) / \sum_{i=1}^{m} n_i C_{2i} k_i}$ increases and the denominator decreases. Therefore, we have $T_u^b < T_a^* \leq T_b^*$. Therefore, the $\Gamma(T)$ function is monotonically decreasing in $\left[T_l^b, T_u^b\right]$ by Lemma 4.3. We can repeat the same argument for all the convex sub-intervals of T in $[T_u^a, T_a^*]$. Therefore, the $\Gamma(T)$ function is monotonically decreasing in $[T_l^a, T_a^*]$. □

Corollary 4.1. *Let* $\mathbf{k}^a$ *be the set of optimal maintenance frequencies that minimizes* $\Psi(\cdot, T)$ *in the range* $[T_l^a, T_u^a]$. *Let the optimal value of* T *corresponding to* $\mathbf{k}^a$ *be* T_a^*. *If* $T_a^* \leq T_l^a$, *then the* $\Gamma(T)$ *function is monotonically increasing in* $[T_a^*, T_u^a]$.

Proof. The proof is similar to that of Theorem 4.1. □

The basic idea of our iterative improving procedures is summarized as follows. Let $\breve{T}^1(\mathbf{k}^1) > T_{\min}$ be the local minimum for the set of optimal maintenance frequencies $\mathbf{k}^1$. By Theorem 4.1, we assert that the $\Gamma(T)$ function is monotonically decreasing in $\left(T_{\min}, \breve{T}^1(\mathbf{k}^1)\right)$. Therefore, one may alternatively find the local minimum $\breve{T}(\mathbf{k})$ and the set of optimal maintenance frequencies $\mathbf{k}$ iteratively to reach the first local minimum to the right of $T_{\min}$, *i.e.*, the value of T_A mentioned above. In a similar fashion, the first local minimum to the left of $T_{\max}$, *i.e.*, T_B , can be determined. Once the improved bounds of T_A and T_B are determined, we use the procedure presented in the next section to proceed with the search within the search range.

4.2. Proceed with the search by the junction points

How can one proceed with the search from our initial point T_{CC} to lower values of T? We do this by obtaining $\mathbf{k}^*(T_{CC})$ by (3.8) in Corollary 3.1 where $\mathbf{k}^*(T)$ denotes as the set of optimal multipliers at given T. Then, by Propositions 3.2 and 3.3, each junction point $\delta_i(k_i)$ provides the information whereby one should change the optimal multiplier of group i from k_i to $k_i + 1$ at $\delta_i(k_i)$ to obtain the optimal value for the $\Gamma(T)$ function. Therefore, during the search, we need to keep an m-dimensional array $(\delta_1(k_1), \delta_2(k_2), \ldots, \delta_m(k_m))$ in which each value of $\delta_i(k_i)$ indicates the location of the next junction point of group i where the optimal multiplier of group i should be changed. Since the algorithm searches toward lower values of T, one should change the multiplier for the particular group i with the largest value of $\delta_i(k_i)$ to correctly update the set of optimal multipliers. Let T_c be the current value of T where the search algorithm reaches. Denote π as the group index for the group i with the largest value of $\delta_i(k_i)$, *i.e.*,

$$\pi = \arg\max_i \{\delta_i(k_i) < T_c\}. \tag{4.4}$$

To proceed with the search from T_c, we need to update the set of optimal multipliers at $\delta_i(k_i)$ by

$$\mathbf{k}^*(\delta_\pi(k_\pi)) \equiv (\mathbf{k}^*(T_c) \setminus \{k_\pi\}) \cup \{k_\pi + 1\} \tag{4.5}$$

where "$\setminus$ " denotes set subtraction.

Note that Theorem 3.2 implies that the set of optimal multipliers $\mathbf{k}^*$ is invariant in each convex sub-interval (*i.e.*, between a pair of consecutive junction points) on the $\Gamma(T)$ function. Hence, this step actually obtains the set of optimal multipliers for all the values of $T \in (\delta_\pi(k_\pi), T_c)$. Then, we should check if the local minimum for $\mathbf{k}^*(T_c)$ exists in the convex sub-interval $(\delta_\pi(k_\pi), T_c)$, since such a local minimum could be a candidate for the optimal solution. For any given set of $\mathbf{k}$, one may obtain its local minimum, $\breve{T}(\mathbf{k})$, by first taking the derivative of the $\Gamma(T)$ function with respect to T and then, equating it to zero. Therefore, $\breve{T}(\mathbf{k})$ is given by eq. (4.6) as follows.

$$\breve{T}(k) = \sqrt{\left(S + \sum_{i=1}^{m} \frac{n_i C_{1i}}{k_i}\right) / \sum_{i=1}^{m} n_i C_{2i} k_i} \tag{4.6}$$

4.3. Further improvement on the lower bound

On the other hand, we derive another lower bound β_2 in Lemma 4.4, which is usually tighter than van Ejis' lower bound β_1. We note that our lower bound β_2 is derived by asserting that there exists no solution that obtains a lower objective value than $Z\left(\mathbf{k}\left(\breve{T}\right), \breve{T}\right)$ for $T < \beta_2$.

Lemma 4.4. *At a local minimum $\breve{T}$, one may secure a lower bound β_2 on the search range by*

$$\beta_2 = \frac{S}{S/\breve{T} + \sum_{i=1}^{m} \phi_i\left(k_i^*\left(\breve{T}\right), \breve{T}\right)} \tag{4.7}$$

where $\phi_i\left(k_i^*\left(\breve{T}\right), \breve{T}\right) = \left\{ \begin{array}{ll} \frac{n_i C_{1i}}{\breve{T}} + n_i C_{2i}\breve{T} - 2n_i\sqrt{C_{1i}C_{2i}}, & k_i^*\left(\breve{T}\right) = 1 \\ 2n_i\sqrt{C_{1i}C_{2i}}\left[\left(\sqrt{(k_i+1)/k_i} + \sqrt{k_i/(k_i+1)}\right) - 1\right], & k_i^*\left(\breve{T}\right) > 1 \end{array} \right\}.$

Proof. We note that the function $\phi_i\left(k_i^*\left(\breve{T}\right), \breve{T}\right)$ indicates the maximum magnitude of decrement in $\Phi(k_i, T)$ from $\breve{T}$ to any value of $T < \breve{T}$ for group i. Recall that Proposition 3.1 asserts that the function $\Phi(k_i, T)$ is bounded from below by $2n_i\sqrt{C_{1i}C_{2i}}$. If the optimal multiplier for group i is $k_i^*\left(\breve{T}\right) = 1$, then the maximum magnitude of decrement in $g_i(k_i, T)$ from $\breve{T}$ to any value of $T < \breve{T}$ is bounded by $n_i C_{1i}/\breve{T} + n_i C_{2i}\breve{T} - 2n\sqrt{C_{1i}C_{2i}}$. If $k_i^*\left(\breve{T}\right) > 1$, then

$$g_i(k_i, \breve{T}) \le \max\left\{\Phi_i\left(k_i^*(\breve{T}) - 1, \delta_i(k_i^*(\breve{T}) - 1)\right), \Phi_i\left(k_i^*(\breve{T}), \delta_i(k_i^*(\breve{T}))\right)\right\} \tag{4.8}$$

by the piece-wise convexity of the $\Phi_i(k_i^*(\breve{T}), \breve{T})$ function.

Since one can easily prove that $\Phi_i\left(k_i^*(\breve{T}), \delta_i(k_i^*(\breve{T}))\right) < \Phi_i\left(k_i^*(\breve{T}) - 1, \delta_i(k_i^*(\breve{T}) - 1)\right)$, it leads to the fact $\Phi_i\left(k_i^*(\breve{T}), \breve{T}\right) \le \Phi_i\left(k_i^*(\breve{T}), \delta_i(k_i^*(\breve{T}))\right)$. By plugging $k_i^*(\breve{T})$ and $\delta_i(k_i^*(\breve{T}))$ into the function $\Phi_i(k_i, T)$, we have the following concise expression for $\Phi_i\left(k_i^*(\breve{T}), \delta_i(k_i^*(\breve{T}))\right)$ after some simplification.

$$\Phi_i\left(k_i^*(\breve{T}), \delta_i(k_i^*(\breve{T}))\right) = 2n_i\sqrt{C_{1i}C_{2i}}(\sqrt{(k_i+1)/k_i} + \sqrt{k_i/(k_i+1)}) \tag{4.9}$$

In other words, if $k_i^*(\breve{T}) > 1$, the maximum magnitude of decrement in $g_i(k_i, T)$ from $\breve{T}$ to any value of $T < \breve{T}$ is bounded by $2n_i\sqrt{C_{1i}C_{2i}}\left\lfloor\left(\sqrt{(k_i+1)/k_i}+\sqrt{k_i/(k_i+1)}\right)-1\right\rfloor$.
On the other hand, the major setup cost would increase from $S/\breve{T}$ to S/T from $\breve{T}$ to any value of $T < \breve{T}$.
The lower bound is derived by asserting that for $T \le \beta_2$, the increment in the major setup cost, *i.e.*, $S/T - S/\breve{T}$, must exceed the maximum magnitude of decrement, *i.e.*, $\sum_{i=1}^{m}\phi_i\left(k_i^*\left(\breve{T}\right),\breve{T}\right)$; or, $S/T - S/\breve{T} \ge \sum_{i=1}^{m}\phi_i\left(k_i^*\left(\breve{T}\right),\breve{T}\right)$, which gives exactly eq. (4.7). □

The revision of $T_{\min}$ at the newly-obtained, best-on-hand, local minimum can be summarized as follows. Denote $\mathbf{K}^*$ and T^* as the set of optimal maintenance frequencies and the optimal value of the basic period obtained by the proposed search algorithm. After securing a new local minimum $\breve{T}$, if $\Psi(\mathbf{k}^*(\breve{T}),\breve{T}) < \Psi(\mathbf{K}^*, T^*)$, then one should try to revise $T_{\min} = \max\{T_{\min}, \beta_1, \beta_2\}$, where β_1 is secured by plugging $\Psi^U = \Psi(\mathbf{k}^*(\breve{T}),\breve{T})$ in eq. (4.3) and β_2 is obtained from eq. (4.7), respectively.

4.4. The algorithm

We are now ready to enunciate the proposed search algorithm.

1. Obtain the initial lower and upper bounds of the search range using the following steps:
   (a) Find an initial upper bound: Calculate $T_{\max} = T_{CC}$ by (4.1).
   (b) Find an initial lower bound: Compute $T_0 = \min_i \sqrt{0.5C_{1i}/C_{2i}}$, obtain the optimal $\mathbf{k}^*(T_0)$ corresponding to T_0 by (3.8). Also, we calculate $\Psi^U = \Psi(\mathbf{k}^*(T_0), T_0)$ by eq. (4.3). Then, obtain an initial lower bound by $T_{\min} = 2S/\Psi^U$.
   (c) Let $\mathbf{K}^* = \mathbf{k}^*(T_{\min}), T^* = T_{\min}$ and $\Psi^* = \Psi^U$.
2. Improve the bounds of the search range by the following iterative procedures:
   (a) Improve the upper bound: Set $T_0 = T_{\max}$ and $T_{old} = T_{\max}$. Calculate $\mathbf{k}^*(T_0)$ corresponding to T_0 by (3.8). Compute $\breve{T}(\mathbf{k}^*(T_0))$ by eq. (4.6). Set $T_{\max} = T_0$ (Repeat the above steps until $T_{old}/T_{\min} = 1$.)
   (b) Improve the lower bound: Set $T_0 = T_{\min}$ and $T_{old} = T_{\min}$ Calculate $\mathbf{k}^*(T_0)$ corresponding to T_0 by (3.8). Compute $\breve{T}(\mathbf{k}^*(T_0))$ by eq. (4.6). Set $T_{\min} = T_0$ (Repeat the above steps until $T_{\min}/T_{old} = 1$.)
3. Set $T_c = T_{\max}$. If $T_c \le T_{\min}$, then go to step 5.
4. Proceed to the next convex sub-interval:
   (a) Set $\pi = \arg\max_i\{\delta_i(k_i) < T_c\}$, $\mathbf{k}^*(\delta_\pi(k_\pi)) \equiv (\mathbf{k}^*(T_c)\backslash\{k_\pi\}) \cup \{k_\pi + 1\}$. Then, let $T_c = \delta_\pi(k_\pi)$.
   (b) Calculate $\breve{T}(\mathbf{k}^*(T_c))$ by (4.6) and compute $\Psi(\mathbf{k}^*(T_c), \breve{T}(\mathbf{k}^*(T_c)))$.
   (c) If $\Psi^* \ge \Psi(\mathbf{k}^*(T_c), \breve{T}(\mathbf{k}^*(T_c)))$, set $\Psi^* = \Psi(\mathbf{k}^*(T_c), \breve{T}(\mathbf{k}^*(T_c)))$, $\mathbf{K}^* = \mathbf{k}^*(T_c)$, $T^* = T_c$. Also, try to revise the lower bound $T_{\min}$ by β_2 in eq. (4.7).
   (d) Go to Step 3.
5. The optimal solution is given by $(\mathbf{K}^*, T^*)$ with the corresponding minimal cost Ψ^*.

In many real-world applications, the basic period must take a discrete value; for instance, a day or a week. Still, we could apply the proposed algorithm for those cases after some modifications. First, we should locate the lower bound $T_{\min}$ and the upper bound $T_{\max}$ using the first two steps in the proposed algorithm. Next, instead of using Steps 3 and 4 in the proposed algorithm, for each discrete value of T in the interval $[T_{\min}, T_{\max}]$, we obtain the corresponding vector of optimal maintenance frequencies $\mathbf{k}(T)$ and its optimal objective function value, *i.e.*, $\Psi(\mathbf{k}(T), T)$. Then, we could find an optimal solution by picking the one with the minimum objective function value. Note that since the objective function value may significantly change after taking the discrete values of $\lfloor T^* \rfloor$ or $\lfloor T^* \rfloor + 1$ from the proposed algorithm directly, we suggest examining the optimal objective function value for each discrete value of T (in the interval $[T_{\min}, T_{\max}]$).

5. Numerical Experiments

In the first part of this section, we employ a numerical example to demonstrate the implementation of the proposed search algorithm. Then, we use randomly generated instances to show that the proposed search algorithm outperforms Goyal and Gunasekaran's [3] search procedure.

5.1. A demonstrative example

In this section, we use the five-group example presented in Goyal and Gunasekaran's [3] paper (pp. 658) to demonstrate the implementation of the proposed search algorithm. The data set of this five-group example is shown in Table 1.

Table 1: The data set of the five-group example

$m = 5$	$S = 50$				
n_i	X_i	Y_i	a_i	b_i	s_i
10	0.8	0.90	80	3	198
24	0.6	0.95	50	2	192
30	0.4	0.85	90	1	193
16	0.6	0.95	85	1.5	205
12	0.5	0.94	95	2.5	204

In the first step, we first find the initial bounds by $T_{\min} = 0.051$ and $T_{\max} = 14.620$. We note that $T_{\min}$ is secured by $T_{\min} = 2S/\Psi^U$ where $\Psi^U = \Psi(\mathbf{k}^*(T_0), T_0) = \$1,972.43$ and $T_0 = \min_i \sqrt{0.5C_{1i}/C_{2i}} = 7.622$. Let $\mathbf{K}^* = \mathbf{k}^*(T_0)$, $T^* = T_0$ and $\Psi^* = \Psi^U$.

In the second step, we use the iterative procedures to improve the bounds of the search range. We note that it takes only one iteration to reach the latest updated upper bound at $T_{\max} = 12.410$. On the other hand, the iterative procedure improves the lower bound to 0.071 after the first iteration, and after 50 iterations, we finally obtain the latest updated lower bound by $T_{\min} = 1.255$.

Now we set $T_c = T_{\max} = 12.410$. Since $T_c > T_{\min}$, we proceed with the search to the next convex sub-interval by setting $\pi = \arg\min_i\{\delta_i(k_i) < T_c\} = 4$. We locate the next junction point at $w_1 = \delta_\pi(k_\pi) = 10.762$. We calculate the local minimum $\breve{T}(k^*(T_c)) = 12.410$ corresponding to $\mathbf{k}^*(T_c)$ by (4.6). Since $\breve{T}(\mathbf{k}^*(T_c)) \in (\delta_\pi(k_\pi), T_{\max}]$,we obtain the first local minimum. Consequently, we compute the optimal objective function value $\Psi(\mathbf{k}^*(T_c), \breve{T}(\mathbf{k}^*(T_c))) = \$1,974.93$ (without including the constant term u here) corresponding to this local minimum. Also, by Lemma 4.4, we update the lower bound by $T_{\min} = \beta_2 = 0.1087$ using eq. (4.7).

Now, we move to the junction point by letting $T_c = w_1 = 10.762$ with the set of optimal maintenance frequency as $\mathbf{k}^*(\delta_\pi(k_\pi)) \equiv (\mathbf{k}^*(T_c)\backslash\{k_4\}) \cup \{k_4 + 1\} = (1, 1, 2, \underline{\mathbf{2}}, 1)$. Again, since $T_c > T_{\min}$, we proceed with the search to the next convex sub-interval by setting $\pi = \arg\min_{\bar{\imath}}\{\delta_{\bar{\imath}}(k_{\bar{\imath}}) < T_c\} = 4$. Next, we locate the next junction point at $w_2 = \delta_\pi(k_\pi) = 9.527$. We calculate the local minimum $\breve{T}(\mathbf{k}^*(T_c)) = 11.031$ corresponding to $\mathbf{k}^*(T_c)$ by (4.6). Since $\breve{T}(\mathbf{k}^*(T_c)) = 11.031 \notin (w_2, w_1] = (9.527, 10.762)$, we obtain no local minimum at this convex sub-interval.

Next, we move to $w_2 = 9.527$ with $\mathbf{k}^*(\delta_\pi(k_\pi)) \equiv (\mathbf{k}^*(T_c)\backslash\{k_4\}) \cup \{k_4+1\} = (1,1,2,\underline{\mathbf{2}},1)$. We continue the search, but find no local minimum in $(w_5, w_2]$, either. We note that $w_3 = \delta_3(3) = 8.658$, $w_4 = \delta_5(2) = 8.501$, and $w_5 = \delta_1(2) = 7.622$. The next local minimum is secured in the interval $(w_6, w_5] = (6.214, 7.622]$. We have $\mathbf{k}^*(T_c) = (2,2,3,2,2)$, $\breve{T}(\mathbf{k}^*(T_c)) = 6.650$ and $\Psi(\mathbf{k}^*(T_c), \breve{T}(\mathbf{k}^*(T_c))) = \$1,972.43$. Since we obtain a local minimum with an improved objective function value, we try to revise the lower bound by Lemma 4.4 with $T_{\min} = \beta_2 = 1.530$ using eq. (4.7). Then, we continue our search by moving to the next junction point again.

In this example, we visit 29 convex sub-intervals in total, and secure 21 local minima before the search algorithm terminates. When the search algorithm meets the eighth local minimum at $\breve{T} = 3.634$, which is located in $(w_{15}, w_{14}]$, we have the lowest cost \$1,971.09. We tried to revise the lower bound by Lemma 4.4 with $T_{\min} = \beta_2 = 2.174$. (We note that it was the last time we revised the lower bound there.) The search algorithm stops when it encounters the largest junction point that is less than $T_{\min}$, that is $w_{29} = 2.079$. The optimal solution is obtained at $T^* = 3.634$ (*i.e.*, the eighth local minimum) and K^* is given by $(3,4,6,4,3)$. The optimal annual total cost is given by $Z^* = \Psi^* + u = \$8,409.33$.

One might be interested in the effectiveness of employing the lower bound revising technique in Lemma 4.4 on shortening the range of the search. In this example, the value of $T_{\min}$ was revised from 1.255 to 2.174 during the search process, which in turn reduces the number of the convex sub-intervals to visit from 53 to 29. It helps to save almost half of the run time by employing the lower bound revising technique.

On other hand, we note that Goyal and Gunasekaran's [3] search procedure solves this example with the following solution $Z^* = \$8,451.41$, $T^* = 14.645$, and K^* is given by $(1,1,1,1,1)$. Obviously, the proposed search algorithm obtains a better solution than Goyal and Gunasekaran's [3] search procedure in this example.

If the basic period must take a discrete value, we should obtain the corresponding vector of optimal maintenance frequencies $\mathbf{k}(T)$ and its optimal objective function value, *i.e.*, $\Psi(\mathbf{k}(T), T)$ for $T \in [0.1087, 12.410]$ and $T \in \mathbb{N}^+$. By evaluating the optimal objective function values of each $T \in \{1, ..., 12\}$, we obtain the optimal solution by $T^* = 4$, $\mathbf{k}(T) = (3,3,5,4,3)$ and $\Psi(\mathbf{k}(T), T) = 1,971,693$. The optimal objective function value $\Psi(\mathbf{k}(T), T)$ increases by only 0.03%, which is not significant at all, in this example.

5.2. Random experiments

In this subsection, we present a summary of our random experiments. We designed our experimental settings by referring to the settings in Table 1 brought by [3]. We selected six different values for the number of groups of vehicles ($m = 3,5,7,10,25,50$), and seven different values for the fixed cost in each basic period T ($S = 10,50,100,200,500,750,1,000$). This yielded 42 combinations from these parameter settings. Then, for each combination, we randomly generated 1,000 instances by randomly choosing the values for X_i, Y_i, a_i, b_i and s_i by using uniform distribution functions. Table 2 indicates the ranges of these uniformly distributed random variables.

After randomly generating 42,000 instances in total, we solved each one of them by the proposed search algorithm as well as Goyal and Gunasekaran's [3] search procedure on a Pentium-III PC with a 736M-CPU. Our experimental results for the smaller-size (with $m = 3,5,7$) and larger-size (with $m = 10,25,50$) are summarized in Tables 3 and 4, respectively.

One may observe that the run time of Goyal and Gunasekaran's [3] search procedure is extremely short. On the other hand, the proposed search algorithm solves the TFMSP with an optimal solution very efficiently. (We note that the third and the fourth columns

Table 2: The settings of the parameters in our random experiments

m	$3, 5, 7, 10, 25, 50$
S	$10, 50, 100, 200, 500, 750, 1000$
n_i	$U[10 - 30]$
X_i	$U[0.4 - 0.8]$
Y_i	$U[0.9 - 0.95]$
a_i	$U[5 - 10]$
b_i	$U[1 - 3]$
s_i	$U[25 - 40]$

Table 3: Our experimental results for the smaller-size problems

		The proposed algorithm		Goyal and Gunasekaran's [3] search procedure			
m	S	Total Run time (seconds)	Avg. no. of convex sub-intervals	Total Run time (seconds)	Non-optimal instances	Max. Error (%)	Avg. Error (%)
3	10	0.722	280	0.070	457/1000	3.673	0.268
	50	0.320	80	0.080	174/1000	2.803	0.114
	100	0.260	42	0.060	115/1000	2.138	0.056
	200	0.191	23	0.060	65/1000	2.079	0.031
	500	0.150	11	0.080	10/1000	0.318	0.014
	750	0.130	8	0.060	1/1000	0.247	0.000
	1000	0.130	7	0.060	2/1000	0.361	0.000
5	10	1.252	433	0.100	741/1000	2.363	0.366
	50	0.571	152	0.091	288/1000	2.735	0.129
	100	0.440	112	0.110	212/1000	1.796	0.089
	200	0.321	67	0.090	148/1000	1.488	0.043
	500	0.260	30	0.100	43/1000	1.244	0.009
	750	0.211	18	0.100	20/1000	0.803	0.004
	1000	0.190	15	0.100	6/1000	0.373	0.001
7	10	1.903	471	0.131	889/1000	1.862	0.449
	50	0.891	295	0.140	470/1000	1.602	0.167
	100	0.651	202	0.130	236/1000	1.381	0.070
	200	0.491	104	0.150	203/1000	1.405	0.054
	500	0.360	56	0.140	84/1000	0.723	0.015
	750	0.311	37	0.141	46/1000	0.485	0.006
	1000	0.270	28	0.140	32/1000	0.395	0.004

Table 4: Our experimental results for the larger-size problems

		The proposed algorithm		Goyal and Gunasekaran's [3] search procedure			
m	S	Total Run time (seconds)	Avg. no. of convex sub-intervals	Total Run time time (seconds)	Non-optimal instances	Max. Error (%)	Avg. Error (%)
10	10	3.154	960	0.190	979/1000	1.635	0.502
	50	1.443	635	0.190	628/1000	1.440	0.193
	100	1.041	449	0.191	252/1000	1.063	0.082
	200	0.771	224	0.190	288/1000	1.725	0.062
	500	0.541	100	0.190	163/1000	0.654	0.024
	750	0.471	72	0.190	121/1000	0.850	0.014
	1000	0.420	51	0.191	76/1000	0.441	0.009
25	10	11.817	4120	0.470	1000/1000	1.605	0.665
	50	5.548	2601	0.511	985/1000	1.067	0.398
	100	3.756	1975	0.581	867/1000	0.942	0.229
	200	3.074	955	0.471	592/1000	0.700	0.071
	500	1.913	539	0.470	461/1000	0.596	0.034
	750	1.482	392	0.461	435/1000	0.382	0.030
	1000	1.352	310	0.471	355/1000	0.308	0.021
50	10	36.152	15201	0.931	1000/1000	1.247	0.775
	50	15.813	9728	0.941	1000/1000	1.152	0.529
	100	10.995	7564	0.942	999/1000	0.870	0.391
	200	7.782	3521	0.921	958/1000	0.998	0.211
	500	5.007	2064	0.941	747/1000	0.387	0.040
	750	4.476	1501	0.922	707/1000	0.411	0.033
	1000	3.946	1022	0.931	712/1000	0.257	0.030

of Tables 3 and 4 indicate the total run time for all of the 1,000 instances for a particular parameter setting.) It takes less than **36** seconds for 1,000 instances of larger-size problems with $m = 50$. (Or, we could solve each instance within 0.04 seconds on average.) For the readers' further information on the run time, we also include the data of the average number of convex sub-intervals visited by the proposed algorithm in Tables 3 and 4.

On the aspect of solution quality, the proposed search algorithm significantly outperforms Goyal and Gunasekaran's search procedure. In the fifth column of Tables 3 and 4, we indicate the number of instances out of the 1,000 instances in the combination of m and S that Goyal and Gunasekaran's search procedure is not able to obtain an optimal solution. The number of instances that Goyal and Gunasekaran's search procedure obtains non-optimal solutions increases as the size of the problem m increases, and it decreases as the value of S increases. In the last two columns of Tables 3 and 4, we present the maximum error and the average error of Goyal and Gunasekaran's search procedure in percentages, respectively. We observe that the smaller the values of m and S, the larger the maximum error and the average error. For the same value of S, the maximum error and the average error increase as the value of m increases. Also, for the same value of m, the average error decreases as the value of S increases.

6. Concluding Remarks

In this study, we presented a full analysis on the mathematical model for the Transportation Fleet Maintenance Scheduling Problem (TFMSP). We showed that the optimal objective function value of the mathematical model is piece-wise convex with respect to T. By utilizing our theoretical results, we proposed an efficient search algorithm that solves the optimal solution for the problem (P) within a very short run time. Based on our random experiments, we conclude that the proposed search algorithm out-performs Goyal and Gunasekaran's [3] search procedure. Therefore, the proposed search algorithm provides the decision makers in transportation industries an excellent decision-support tool for their planned maintenance operations.

On the other hand, as one may notice it from our numerical results, the run time of the proposed algorithm grows significantly when the number of groups of vehicles (i.e., m) is large and the fixed maintenance cost (*i.e.*, S) is small. The authors are currently devoting their efforts to improve the efficiency of the proposed algorithm for these cases.

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