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STOCHASTIC ANALYSIS OF NUMBER OF CORPORATIONS IN A MARKET DERIVED FROM STRATEGIC POLICIES OF INDIVIDUAL CORPORATIONS FOR MARKET ENTRY AND RETREAT

Ushio Sumita Tsunehisa Ise
University of Tsukuba

Koichi Yonezawa
University of Wisconsin-Madison

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Abstract A stochastic model is developed for describing a market lifecycle expressed in terms of the number of corporations N in the market. Each corporation independently determines the probability of market entry if it is not in the market yet or the probability of market retreat if it is already in the market. These probabilities may depend on time t, the number of corporations in the market at time t and the number of corporations which have retreated from the market by time t. Of interest is the number of corporations in the market at time t, thereby enabling one to analyze the market lifecycle in terms of strategic actions of individual corporations.

Rigorous analysis of this process becomes numerically intractable since the corresponding state space explodes as N increases. In order to overcome this difficulty, we propose temporally inhomogeneous marginal processes describing the states of individual corporations. The process of interest is then approximated as the independent sum of such marginal processes. An algorithmic procedure is developed for computing the probability distribution of the number of corporations in the market based on spectral analysis of the temporally inhomogeneous marginal processes combined with a bivariate generating function approach.

Corporations are classified into three groups: RT(Risk-Taking), RN(Risk-Neutral), and RA(Risk-Aversive), where these groups are characterized by specifying the transition probabilities of the underlying marginal processes. It is numerically observed that any class alone is not sufficient to form a market and a typical market lifecycle emerges only through the presence of an appropriate combination of corporations from the three classes.

Keywords: Marketing, stochastic modeling, market lifecycle

1. Introduction

For understanding the growth and decline of a market, a traditional approach has been to model a product lifecycle based on analysis of consumer behavior. Bass[1969], for example, developed a diffusion model by assuming that the conditional probability of a consumer purchasing a product under consideration at time t given that he/she has not purchased the product by time t would depend only on the number of consumers who have purchased the product by time t. Horsky and Simon[1983] extended this model by incorporating the level of the advertisement expenditure in addition to the number of consumers who have purchased the product by time t in the dependency structure of the conditional probabilities. Horsky[1990] further strengthened the analysis by introducing the utility structure and incomes of consumers as well as the price of the product into the model, which enabled one to combine a decision mechanism of consumers for purchasing the product with the product lifecycle analysis for the first time.

The diffusion process approach for modeling a product lifecycle through analysis of consumer behavior can be justified simply because there exist sufficiently many consumers despite their discrete nature. In order to analyze the growth and decline of a market from corporate side, however, the diffusion process approach is inappropriate due to the limited number of corporations which are potentially interested in entering into the market. The principal tool employed for this type of the market analysis is an econometric approach where the number of corporations in the market is expressed as a time series governed by the total product sales in the market, technological progress, etc. Many extended models have been developed and the reader is referred to Geroski and Mazzucato[2001] for an extensive summary of the literature.

A major pitfall of the econometric approach above can be found in that it cannot directly connect strategic policies of individual corporations with the market state. The purpose of this paper is to fill this gap by modelling individual corporations as temporally inhomogeneous descrete time processes and then approximating the market by the independent sum of such marginal processes. Despite this rather simple model structure, the temporal inhomogeneity present makes analysis fairly complicated. We conquer this difficulty via spectral analysis of the underlying marginal process combined with a bivariate generating function approach. By capturing sophisticated interactions among individual corporations with different strategic policies, our model will provide an insight into processes of how the market as a whole would be constructed through separate decisions by individual corporations.

In this paper, the market state is defined in terms of the number of corporations in the market. In parallel with a product lifecycle, we introduce a market lifecycle consisting of the following four stages: the introduction stage; the growth stage; the maturity stage, and the decline stage. Actual data on the automobile industry and the tire industry in the United States are extracted from Simons[1995] and are depicted in Figures 1 and 2 respectively.

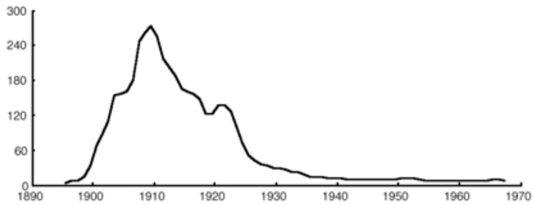

Figure 1: Number of corporations in the US automobile industry

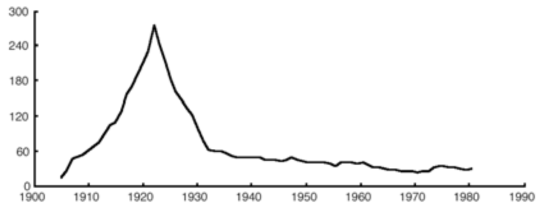

Figure 2: Number of corporations in the US tire industry

The model proposed here is limited in that the market lifecycle is captured only through the number of corporations in the market, ignoring the total sales and other important market features. For example, the decline of the market in number does not necessarily imply the decline of the sales volume. However, this approach enables one to understand the structural relationship between strategic policies of individual corporations and the market lifecycle. For the future research, the numerical tractability of this model opens a new path toward development of more sophisticated market growth-decline models by incorporating additional features in the construction of the transition probabilities of the underlying temporally inhomogeneous marginal processes.

In Section 1, an analytical Model is formally introduced, where strategic policies of N individual corporations are expressed in terms of conditional probabilities of entry into and retreat from the market. These conditional probabilities may depend on time t, the number of corporations in the market at time t, $X(t)$, and the number of corporations which have retreated from the market by time t, $Y(t)$. This interdependence is the key to the potential usefulness of our model. The state of a corporation is described as a temporally inhomogeneous discrete time process involving the conditional probabilities above. Rigorous analysis of this stochastic process $\{X(t), Y(t)\}$ requires the joint probability of the states of all corporations, which soon becomes numerically intractable as N increases since the size of the corresponding state space explodes as a function of N. In order to overcome this difficulty, we assume that the state of the whole market can be approximated by the independent sum of the individual marginal processes. Section 2 is devoted to spectral analysis of the underlying temporally inhomogeneous marginal process and the transition probability matrix at time t is derived in a closed form. Based on a bivariate generating function approach, the joint probabilities of $\{X(t), Y(t)\}$ at time t are evaluated through the above approximation procedure. A computational algorithm is summarized in Section 3, and finally numerical results are presented in Section 4. The set of corporations potentially interested in entering the market is decomposed into three categories: RT(Risk-Taking) corporations, RN(Risk-Neutral) corporations, and RA(Risk-Aversive) corporations. The three classes are characterized in terms of transition probabilities of the underlying temporally inhomogeneous marginal processes. Numerical experiments reveal that any class alone is not sufficient to form a market and a typical market lifecycle emerges only through the presence of an appropriate combination of corporations from the three classes. Some concluding remarks are given in Section 5.

2. Model Description

We consider a situation that N corporations are potentially interested in entering into a new product market. Of interest is to develop a stochastic model which captures the market lifecycle consisting of the four stages discussed in Section 1 through analysis of strategic actions of individual corporations. More specifically, at time t ($t = 0, 1, 2, \cdots$) any corporation is in one of the following three states:

$$\begin{cases} 0 & \text{The corporation has not entered the market yet.} \\ 1 & \text{The corporation is in the market.} \\ 2 & \text{The corporation has retreated from the market.} \end{cases} \tag{2.1}$$

It is assumed that if any corporation retreats from the market, it never enters the market again. At time t, each corporation makes an independent decision so as to determine its state at time $t + 1$. However, the decision parameters may be time-dependent or may depend on the market state at time t involving all other corporations. Consequently each corporation is modelled to follow a discrete time marginal process on $\mathcal{S}_{\mathcal{C}} = \{0, 1, 2\}$ which is temporally inhomogeneous having state 2 as the absorbing state. Despite this structural simplicity, the temporal inhomogeneity presents considerable analytical complexity as we will see.

Let $\mathcal{CP} = \{1, \cdots, N\}$ be a set of corporations under consideration and let $\{N_i(t) : t = 0, 1, 2, \cdots\}$ be a stochastic process describing the state of corporation i at time t. We define two stochastic processes $\{X(t) : t = 0, 1, 2, \cdots\}$ and $\{Y(t) : t = 0, 1, 2, \cdots\}$ where

$$X(t) = \sum_{i \in \mathcal{CP}} \delta_{\{N_i(t)=1\}}; \ Y(t) = \sum_{i \in \mathcal{CP}} \delta_{\{N_i(t)=2\}}. \tag{2.2}$$

Here $\delta_{\{P\}} = 1$ if the statement P holds and $\delta_{\{P\}} = 0$ otherwise. We note that $X(t)$ is the number of corporations in the market at time t, while $Y(t)$ is the number of corporations which have retreated from the market by time t. Consequently the bivariate stochastic process $\{X(t), Y(t)\}$ represents the state of the whole market at time t. The corresponding state space $\mathcal{S}_M$ is then defined as

$$\mathcal{S}_M = \{(x, y) : 0 \le x + y \le N, \text{ for any nonnegative integers } x, y\}. \tag{2.3}$$

The corresponding state probabilities and the bivariate generating functions are defined respectively by

$$\underline{\underline{m}}(t) = [m(x,y,t)]_{(x,y)\in\mathcal{S}_M};\ m(x,y,t) = P[X(t)=x, Y(t)=y] \tag{2.4}$$

and

$$\psi(u,v,t) = E[u^X v^Y] = \sum_{(x,y)\in\mathcal{S}_M} m(x,y,t)u^x v^y. \tag{2.5}$$

In order to analyze the market excluding corporation i, we introduce the followings in parallel with (2.2) through (2.5):

$$X_i(t) = \sum_{j\in\mathcal{CP}\setminus\{i\}} \delta_{\{N_j(t)=1\}};\ Y_i(t) = \sum_{j\in\mathcal{CP}\setminus\{i\}} \delta_{\{N_j(t)=2\}}; \tag{2.6}$$

$$\mathcal{S}_{M\setminus\{i\}} = \{(x,y) : 0 \le x+y \le N-1,\ \text{for any nonnegative integers } x, y\}; \tag{2.7}$$

$$\underline{\underline{m}}_i(t) = [m_i(x,y,t)]_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}};\ m_i(x,y,t) = P[X_i(t)=x, Y_i(t)=y]; \tag{2.8}$$

and

$$\psi_i(u,v,t) = E[u^{X_i} v^{Y_i}] = \sum_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}} m_i(x,y,t)u^x v^y. \tag{2.9}$$

Rigorous analysis of the joint process $\{X(t), Y(t)\}$ requires the joint probability of the vector process $[N_1(t),\cdots,N_N(t)]$ defined on S_C^N of size 3^N. This state space explodes as a function of N. In what follows, we assume that the sum in (2.2) can be approximated by the independent sum of the individual marginal processes $N_i(t),\ 1 \le i \le N$. In order to understand the gap between the exact process and the approximated process, the case of two corporations ($N = 2$) is discussed in detail in Appendix, which should be read after going through the approximation procedure discussed in this section.

Let $\underline{p}_i^T(t)$ be the state probability vector of $\{N_i(t) : t = 0, 1, \cdots\}$, that is,

$$\underline{p}_i^T(t) = [p_{i,0}(t), p_{i,1}(t), p_{i,2}(t)];\ p_{i,j}(t) = P[N_i(t)=j],\ 0 \le j \le 2. \tag{2.10}$$

The corresponding bivariate generating function is defined by

$$\varphi_i(u,v,t) = p_{i,0}(t) + p_{i,1}(t)u + p_{i,2}(t)v. \tag{2.11}$$

We assume that $\{N_i(t) : t = 0,1,2,\cdots\}$ is a temporally inhomogeneous discrete time process governed by one step transition probability matrix $\underline{\underline{a}}_i(t)$ at time t specified in the following manner. At time $t = 0$, no corporation is assumed to be in the market so that one has for all $j \in \mathcal{CP}$

$$\underline{p}_j^T(0) = [1,0,0];\ m_j(x,y,0) = \delta_{\{x=y=0\}}\ \mathit{for}\ (x,y) \in \mathcal{S}_{M\setminus\{j\}}. \tag{2.12}$$

Suppose that $\underline{p}_j^T(t)$ and $\underline{\underline{m}}_j(t)$ are known for all $j \in \mathcal{CP}$. Then $\underline{\underline{a}}_i(t)$ is determined by

$$\underline{\underline{a}}_i(t) = \begin{bmatrix} 1-\alpha_i(t) & \alpha_i(t) & 0 \\ 0 & \beta_i(t) & 1-\beta_i(t) \\ 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

where

$$\alpha_i(t) = \sum_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}} m_i(x,y,t)\eta_i(t|x,y) \tag{2.14}$$

and

$$\beta_i(t) = \sum_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}} m_i(x-1,y,t)\xi_i(t|x,y). \tag{2.15}$$

Here $\eta_i(t|x,y)$ is the probability that corporation i enters the market at time $t+1$ given that it is not in the market at time t, $X(t)=x$ and $Y(t)=y$. Similarly $\xi_i(t|x,y)$ is the probability that corporation i remains in the market at time $t+1$ given that it is in the market at time t, $X(t)=x$ and $Y(t)=y$. More formally, we define;

$$\eta_i(t|x,y) = P[N_i(t+1)=1|N_i(t)=0, X(t)=x, Y(t)=y] \tag{2.16}$$

and

$$\xi_i(t|x,y) = P[N_i(t+1)=1|N_i(t)=1, X(t)=x, Y(t)=y]. \tag{2.17}$$

When corporation i is not in the market, both $X(t)$ and $Y(t)$ are contributed by other corporations. Accordingly $\alpha_i(t)$ in (2.14) is expressed as a probability mixture of $\eta_i(t|x,y)$ with corresponding weights $m_i(x,y,t)$ over $(x,y)\in\mathcal{S}_{M\setminus\{i\}}$. For evaluation of $\beta_i(t)$ in (2.15), the mixing weights become $m_i(x-1,y,t)$ over $(x,y)\in\mathcal{S}_{M\setminus\{i\}}$ since corporation i is already in the market.

It can be seen that

$$\underline{p}_i^T(t+1) = \underline{p}_i^T(t)\underline{\underline{a}}_i(t). \tag{2.18}$$

Equation (2.18) enables one to specify $\varphi_i(u,v,t+1)$ through (2.11) for all $i\in\mathcal{CP}$. Once $\underline{\underline{a}}_i(t)$ of (2.13) is given, under the assumption that $X(t)$ and $Y(t)$ can be approximated by the independent sum of the individual marginal processes, one has for each $i\in\mathcal{CP}$

$$\psi_i(u,v,t+1) = \prod_{j\in\mathcal{CP}\setminus\{i\}} \varphi_j(u,v,t+1). \tag{2.19}$$

In summary, the state transition diagram is depicted in Figure 3. Because of dependence of individual entry and retreat probabilities on time t, the number of corporations in the market and the number of corporations which have retreated from the market, the model enables one to understand how strategic policies of individual corporations affect the market state, as we will see in Section 4.

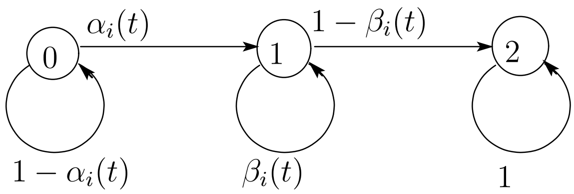

Figure 3: State transition diagram

By specifying the coefficients of $u^x v^y$ of (2.19), one can see that $\underline{p}_i^T(t)$ and $\underline{\underline{m}}_i(t)$ generate $\underline{p}_i^T(t+1)$ and $\underline{\underline{m}}_i(t+1)$ for all $i\in\mathcal{CP}$ via (2.13) through (2.19). We note that if we define

$$\underline{\underline{P}}_i(t) = \prod_{k=0}^{t} \underline{\underline{a}}_i(k), \tag{2.20}$$

then

$$\underline{p}_i^T(t+1) = \underline{p}_i^T(0)\underline{\underline{P}}_i(t). \tag{2.21}$$

Since $\underline{p}_i^T(0) = [1,0,0]$, $\underline{p}_i^T(t+1)$ is actually the first row of $\underline{\underline{P}}_i(t)$.

3. Spectral Analysis of Market Entry/Retreat Decisions by Individual Corporations

In this section, we analyze the spectral representation of the stochastic matrices $\underline{\underline{a}}_i(t)$ of (2.13) and $\underline{\underline{P}}_i(t)$ of (2.20), which in turn enables one to capture the stochastic structure of market entry/retreat decisions by individual corporations. A few preliminary lemmas are needed.

For $0 \leq \alpha, \beta \leq 1$ with $\alpha + \beta \neq 1$, we define

$$f(\alpha, \beta) = \frac{\alpha}{\alpha + \beta - 1}; \; g(\alpha, \beta) = \frac{1 - \beta}{\alpha + \beta - 1}. \tag{3.1}$$

We also introduce $\underline{\underline{J}}_1(\alpha, \beta)$, $\underline{\underline{J}}_2(\alpha, \beta)$ and $\underline{\underline{J}}_3(\alpha, \beta)$ as follows:

$$\underline{\underline{J}}_1(\alpha, \beta) = \underline{u}_1 \underline{v}_1^T \text{ where } \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \underline{v}_1^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}; \tag{3.2}$$

$$\underline{\underline{J}}_2(\alpha, \beta) = \underline{u}_2(\alpha, \beta) \underline{v}_2^T \tag{3.3}$$
$$\text{where } \underline{u}_2(\alpha, \beta) = \begin{bmatrix} f(\alpha, \beta) \\ 1 \\ 0 \end{bmatrix} \text{ and } \underline{v}_2^T = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix};$$

and

$$\underline{\underline{J}}_3(\alpha, \beta) = \underline{u}_3 \underline{v}_3^T(\alpha, \beta) \tag{3.4}$$
$$\text{where } \underline{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \underline{v}_3^T(\alpha, \beta) = \begin{bmatrix} 1 & -f(\alpha, \beta) & g(\alpha, \beta) \end{bmatrix}.$$

The case $\alpha + \beta = 1$ will be treated separately soon. When no ambiguity is present, we omit (α, β) and write $\underline{u}_2 = \underline{u}_2(\alpha, \beta)$, $\underline{\underline{J}}_i = \underline{\underline{J}}_i(\alpha, \beta)$, etc. The following lemma then holds true.

Lemma 3.1 *Suppose* $0 \leq \alpha, \; \beta \leq 1$ *with* $\alpha + \beta \neq 1$. *Then:*

a) $\underline{\underline{J}}_i(\alpha, \beta)$, $1 \leq i \leq 3$, *are dyadic and idempotent, i.e.* $\underline{\underline{J}}_i^2(\alpha, \beta) = \underline{\underline{J}}_i(\alpha, \beta)$, $1 \leq i \leq 3$.

b) $\underline{\underline{J}}_i(\alpha, \beta)$, $1 \leq i \leq 3$, *are matrix orthogonal to each other, i.e.* $\underline{\underline{J}}_i(\alpha, \beta)\underline{\underline{J}}_j(\alpha, \beta) = \underline{\underline{0}}$ *if* $i \neq j$, $1 \leq i, j \leq 3$.

c) $\underline{\underline{J}}_1(\alpha_1, \beta_1)\underline{\underline{J}}_j(\alpha_2, \beta_2) = \underline{\underline{J}}_j(\alpha_1, \beta_1)\underline{\underline{J}}_1(\alpha_2, \beta_2) = \underline{\underline{0}}$ *for* $j = 2, 3$.

d) $\underline{\underline{J}}_2(\alpha_1, \beta_1)\underline{\underline{J}}_2(\alpha_2, \beta_2) = \underline{\underline{J}}_2(\alpha_1, \beta_1)$.

e) $\underline{\underline{J}}_3(\alpha_1, \beta_1)\underline{\underline{J}}_3(\alpha_2, \beta_2) = \underline{\underline{J}}_3(\alpha_2, \beta_2)$.

f) $\underline{\underline{J}}_2(\alpha_1, \beta_1)\underline{\underline{J}}_3(\alpha_2, \beta_2) = \underline{\underline{0}}$.

g) $\underline{\underline{J}}_3(\alpha_1, \beta_1)\underline{\underline{J}}_2(\alpha_2, \beta_2) = \{f(\alpha_2, \beta_2) - f(\alpha_1, \beta_1)\}\underline{u}_3\underline{v}_2^T$.

h) $\underline{u}_3\underline{v}_2^T\underline{\underline{J}}_1 = \underline{\underline{0}}$.

i) $\underline{u}_3\underline{v}_2^T\underline{\underline{J}}_2(\alpha_2, \beta_2) = \underline{u}_3\underline{v}_2^T$.

j) $\underline{u}_3\underline{v}_2^T\underline{\underline{J}}_3(\alpha_2, \beta_2) = \underline{\underline{0}}$.

Proof *We first note that* $\underline{v}_i^T\underline{u}_i = 1$, $1 \leq i \leq 3$, *while* $\underline{v}_i^T(\alpha, \beta)\underline{u}_j(\alpha, \beta) = \delta_{ij}$, *where* $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$ *otherwise,* $1 \leq i, j \leq 3$. *Hence parts a), b), c), f), h),i),j) follow immediately. For part d), one sees that*

$$\begin{aligned} \underline{\underline{J}}_2(\alpha_1, \beta_1)\underline{\underline{J}}_2(\alpha_2, \beta_2) &= \underline{u}_2(\alpha_1, \beta_1)\underline{v}_2^T\underline{u}_2(\alpha_2, \beta_2)\underline{v}_2^T \\ &= \underline{u}_2(\alpha_1, \beta_1)\underline{v}_2^T \\ &= \underline{\underline{J}}_2(\alpha_1, \beta_1) \end{aligned}$$

since $\underline{v}_2^T \underline{u}_2(\alpha_2, \beta_2) = 1$. *Part e) follows similarly since* $\underline{v}_3^T(\alpha_1, \beta_1)\underline{u}_3 = 1$. *For part g), one has*

$$\begin{aligned}\underline{\underline{J}}_3(\alpha_1, \beta_1)\underline{\underline{J}}_2(\alpha_2, \beta_2) &= \underline{u}_3\underline{v}_3^T(\alpha_1, \beta_1)\underline{u}_2(\alpha_2, \beta_2)\underline{v}_2^T \\ &= \{f(\alpha_2, \beta_2) - f(\alpha_1, \beta_1)\}\underline{u}_3\underline{v}_2^T,\end{aligned}$$

completing the proof. □

Lemma 3.2 *Let* $\underline{\underline{b}}(\alpha, \beta)$ *be a* 3×3 *stochastic matrix given by*

$$\underline{\underline{b}}(\alpha, \beta) = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & \beta & 1-\beta \\ 0 & 0 & 1 \end{bmatrix}, \ 0 \le \alpha, \beta \le 1, \tag{3.5}$$

where $\alpha + \beta \neq 1$. *Let* $f(\alpha, \beta)$ *be as in (3.1). Then the following statements hold true.*
a) $\underline{\underline{b}}(\alpha, \beta) = \underline{\underline{J}}_1 + \beta\underline{\underline{J}}_2(\alpha, \beta) + (1-\alpha)\underline{\underline{J}}_3(\alpha, \beta)$.
b) $\underline{\underline{b}}(\alpha_1, \beta_1)\underline{\underline{b}}(\alpha_2, \beta_2) = \underline{\underline{J}}_1 + \beta_1\beta_2\underline{\underline{J}}_2(\alpha_1, \beta_1) + (1-\alpha_1)(1-\alpha_2)\underline{\underline{J}}_3(\alpha_2, \beta_2)$

$$+ (1-\alpha_1)\beta_2\{f(\alpha_2, \beta_2) - f(\alpha_1, \beta_1)\}\underline{u}_3\underline{v}_2^T.$$

Proof *It can be readily seen that* $\underline{u}_i$ *and* $\underline{v}_i^T$, $1 \le i \le 3$, *are right and left eigenvectors of* $\underline{\underline{b}}(\alpha, \beta)$ *associated with eigenvalues* 1, β, *and* $(1-\alpha)$ *respectively and part a) follows immediately. Part b) can be proven from a) and Lemma 3.1.* □

From (2.13) and (3.5), one sees that

$$\underline{\underline{a}}_i(t) = \underline{\underline{b}}(\alpha_i(t), \beta_i(t)). \tag{3.6}$$

Hence if temporal homogeneity is present, i.e. $\alpha_i(t) = \alpha_i(0)$ and $\beta_i(t) = \beta_i(0)$ for $t = 1, 2, \cdots$, one sees from (2.20) and Lemma 3.2 a) that

$$\begin{aligned}\underline{\underline{P}}_i(t) &= \underline{\underline{a}}_i^{t+1}(0) \\ &= \underline{\underline{J}}_1 + \beta_i^{t+1}(0)\underline{\underline{J}}_2(\alpha_i(0), \beta_i(0)) + (1-\alpha_i(0))^{t+1}\underline{\underline{J}}_3(\alpha_i(0), \beta_i(0)).\end{aligned}$$

Because of temporal inhomogeneity, however, this simple structure disappears. We overcome this difficulty by using Lemma 3.2 b), as shown in the main theorem of this section below.

Theorem 3.3 *Let* $f(\alpha, \beta)$, $\underline{\underline{J}}_1$, $\underline{\underline{J}}_2(\alpha, \beta)$ *and* $\underline{\underline{J}}_3(\alpha, \beta)$ *be as in (3.1) through (3.4) where* $0 \le \alpha,\ \beta \le 1$ *and* $\alpha + \beta \neq 1$. *Then* $\underline{\underline{P}}_i(t)$ *in (2.20) is given by*

$$\begin{aligned}\underline{\underline{P}}_i(t) &= \underline{\underline{J}}_1 + \prod_{k=0}^{t}\beta_i(k)\underline{\underline{J}}_2(\alpha_i(0), \beta_i(0)) \\ &\quad + \prod_{k=0}^{t}\{1-\alpha_i(k)\}\underline{\underline{J}}_3(\alpha_i(t), \beta_i(t)) + C_i(t)\underline{u}_3\underline{v}_2^T\end{aligned} \tag{3.7}$$

where $\alpha_i(k)$ *and* $\beta_i(k)$ *are as in (2.14) and (2.15) respectively, and*

$$\begin{aligned}C_i(t) &= \beta_i(t)C_i(t-1) + \prod_{k=0}^{t-1}\{1-\alpha_i(k)\}\beta_i(t) \\ &\quad \times\{f(\alpha_i(t), \beta_i(t)) - f(\alpha_i(t-1), \beta_i(t-1))\}, t = 1, 2, \cdots\end{aligned} \tag{3.8}$$

starting with $C_i(0) = 0$.

Proof *The theorem can be proven by induction as follows. For* $t = 0$*, one sees from (3.6) that* $\underline{\underline{P}}_i(0) = \underline{\underline{a}}_i(0) = \underline{\underline{b}}(\alpha_i(0), \beta_i(0))$ *and (3.7) holds true by Lemma 3.2 a). Suppose it is true for* t *and consider* $t + 1$*. One sees that*

$$\underline{\underline{P}}_i(t+1) = \underline{\underline{P}}_i(t)\underline{\underline{a}}_i(t+1) = \underline{\underline{P}}_i(t)\underline{\underline{b}}(\alpha_i(t+1), \beta_i(t+1)).$$

Using the induction hypothesis and Lemmas 3.1 and 3.2, the above equation leads to

$$\begin{aligned}\underline{\underline{P}}_i(t+1) &= \Big[\underline{\underline{J}}_1 + \prod_{k=0}^{t} \beta_i(k)\underline{\underline{J}}_2(\alpha_i(0), \beta_i(0)) \\ &\quad + \prod_{k=0}^{t}\{1-\alpha_i(k)\}\underline{\underline{J}}_3(\alpha_i(t), \beta_i(t)) + C_i(t)\underline{u}_3\underline{v}_2^T\Big] \\ &\quad \times \Big[\underline{\underline{J}}_1 + \beta_i(t+1)\underline{\underline{J}}_2(\alpha_i(t+1), \beta_i(t+1)) \\ &\quad + \{1-\alpha_i(t+1)\}\underline{\underline{J}}_3(\alpha_i(t+1), \beta_i(t+1))\Big]\end{aligned}$$

and the theorem follows from Lemma 3.1. □

Remark 3.4 *When* $\alpha + \beta = 1$*,* $\underline{\underline{b}}(\alpha, \beta)$ *in (3.5) is reduced to* $\underline{\underline{a}}(\alpha)$ *below having only one parameter* α*.*

$$\underline{\underline{a}}(\alpha) = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\alpha & \alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

In this case, $\underline{\underline{a}}(\alpha)$ *has the eigenvalues 1 of multiplicity 1 and* $1 - \alpha$ *of multiplicity 2. Accordingly, one sees that, for* $\underline{\underline{J}}_1$ *given in (3.2),*

$$\underline{\underline{a}}(\alpha) = \underline{\underline{J}}_1 + \underline{\underline{\Delta}}(\alpha); \quad \underline{\underline{\Delta}}(\alpha) = (1-\alpha)\begin{bmatrix} 1 & \frac{\alpha}{1-\alpha} & -\frac{1}{1-\alpha} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.10}$$

where

$$\underline{\underline{J}}_1\underline{\underline{\Delta}}(\alpha) = \underline{\underline{\Delta}}(\alpha)\underline{\underline{J}}_1 = \underline{\underline{0}}. \tag{3.11}$$

It can be readily seen that, for $0 < \alpha_i < 1,\ i = 1, 2, \cdots, t,$ *one has*

$$\prod_{i=1}^{t} \underline{\underline{a}}(\alpha_i) = \underline{\underline{J}}_1 + \prod_{i=1}^{t}(1-\alpha_i)\begin{bmatrix} 1 & \sum_{i=1}^{t}\frac{\alpha_i}{1-\alpha_i} & -\frac{1}{1-\alpha_t} - \sum_{i=1}^{t-1}\frac{\alpha_i}{1-\alpha_i} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.12}$$

Furthermore, the followings hold true:

$$\underline{\underline{J}}_2(\alpha_1, \beta_1)\underline{\underline{\Delta}}(\alpha_2) = (1-\alpha_2)\underline{\underline{J}}_2(\alpha_1, \beta_1) \tag{3.13}$$

$$\underline{\underline{\Delta}}(\alpha_2)\underline{\underline{J}}_2(\alpha_1, \beta_1) = (1-\alpha_2)\underline{\underline{J}}_2(\alpha_1, \beta_1) + \alpha_2\underline{u}_3\underline{v}_2^T \tag{3.14}$$

$$\underline{\underline{J}}_3(\alpha_1, \beta_1)\underline{\underline{\Delta}}(\alpha_2) = (1-\alpha_2)\underline{\underline{J}}_3(\alpha_1, \beta_1) + \alpha_2\underline{u}_3\underline{v}_2^T \tag{3.15}$$

$$\underline{\underline{\Delta}}(\alpha_2)\underline{\underline{J}}_3(\alpha_1, \beta_1) = (1-\alpha_2)\underline{\underline{J}}_3(\alpha_1, \beta_1) \tag{3.16}$$

$$\underline{u}_3\underline{v}_2^T\underline{\underline{\Delta}}(\alpha) = \underline{\underline{\Delta}}_2(\alpha)\underline{u}_3\underline{v}_2^T = (1-\alpha_2)\underline{u}_3\underline{v}_2^T \tag{3.17}$$

It follows that both $\underline{\underline{b}}(\alpha_1, \beta_1)\underline{\underline{\Delta}}(\alpha_2)$ *and* $\underline{\underline{\Delta}}(\alpha_2)\underline{\underline{b}}(\alpha_1, \beta_1)$ *have the spectoral representation involving only* $\underline{\underline{J}}_j(\alpha_1, \beta_1),\ 1 \le j \le 3,$ *and* $\underline{u}_3\underline{v}_2^T$*. Hence, when the case* $\alpha + \beta = 1$ *happens, Theorem 3.3 can be modified using (3.12) through (3.17). In order to avoid notational awkwardness, we assume throughout the paper that* $\alpha + \beta \neq 1$*.*

4. Development of Algorithm

In this section, an algorithmic procedure is summarized for computing $\underline{p}_i^T(t)$, $\underline{\underline{m}}_i(t)$ and $\underline{\underline{m}}(t)$ of (2.4), (2.8) and (2.10) respectively.

[Input]

N: the number of corporations

T: the time periods for consideration

Strategies of individual corporations : $[\eta_i(t|x,y)]_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}}$, $0 \le t \le T-1$, $i \in \mathcal{CP}$
$[\xi_i(t|x,y)]_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}}$, $0 \le t \le T-1$, $i \in \mathcal{CP}$

[Output]

$\underline{p}_i^T(t)$, $\underline{\underline{m}}_i(t)$, $\underline{\underline{m}}(t)$, $i \in \mathcal{CP}$, $0 \le t \le T$

[Algorithm]

[0] $\underline{p}_i^T(0) = [1,0,0]$ for all i; $t \leftarrow 0$.

[1] LOOP: Find $\varphi_i(u,v,t)$ using (2.11) for all $i \in \mathcal{CP}$.

[2] Generate $\underline{\underline{m}}_i(t)$ by identifying the coefficients of $\psi_i(u,v,t) = \prod_{j\in\mathcal{CP}\setminus\{i\}} \varphi_j(u,v,t)$ for all $i \in \mathcal{CP}$.

[3] Generate $\underline{\underline{m}}(t)$ by identifying the coefficients of $\psi(u,v,t) = \prod_{j\in\mathcal{CP}} \varphi_j(u,v,t)$.

[4] Compute $(\alpha_i(t), \beta_i(t))$ based on (2.14) and (2.15) for all $i \in \mathcal{CP}$.

[5] Compute $\underline{p}_i^T(t+1)$ as the first row of $\underline{\underline{P}}_i(t)$ based on Theorem 3.3.

[6] $\rightarrow (T > t \leftarrow t+1)$ / LOOP

5. Numerical Results

The purpose of this section is to demonstrate the usefulness of the market lifecycle model developed in the previous sections through numerical examples. In particular, we will see that the model enables one to capture how the market growth and decline would be affected by strategic policies of individual corporations.

For numerical experiments presented in this section, N corporations are decomposed into three categories, i.e. $\mathcal{CP} = \mathcal{CP}_1 \cup \mathcal{CP}_2 \cup \mathcal{CP}_3$, $\mathcal{CP}_i \cap \mathcal{CP}_j = \emptyset$ for $i \ne j$ where

$$\mathcal{CP}_1 : \text{the set of } N_1 = |\mathcal{CP}_1| \text{ RT(Risk-Taking) corporations;} \tag{5.1}$$

$$\mathcal{CP}_2 : \text{the set of } N_2 = |\mathcal{CP}_2| \text{ RN(Risk-Neutral) corporations;} \tag{5.2}$$

and

$$\mathcal{CP}_3 : \text{the set of } N_3 = |\mathcal{CP}_3| \text{ RA(Risk-Aversive) corporations,} \tag{5.3}$$

where $|\mathcal{CP}_i|$ denotes the cardinality of $\mathcal{CP}_i$, $1 \le i \le 3$. For computational simplicity, we assume that all corporations within one category have a common strategic policy.

RT corporations tend to enter the market when the market size $X(t)$ is small, but retreat from the market rather quickly when $X(t)$ becomes large. Since RT corporations play a key role only in the introduction stage and the growth stage, they are not affected by the number of corporations retreated from the market $Y(t)$. RN corporations incline to enter the market when $X(t)$ exceeds a certain level, continue to stay in the market for some time, and then retreat from the market. Their retreats are accelerated as $Y(t)$ increases, triggering the decline stage single-handedly. RA corporations do not enter the market easily. Even when they decide to enter the market, they do so only after $X(t)$ becomes sufficiently large. Once they enter, like RN corporations, they continue to stay in the market for some time, and then retreat from the market. However, they are risk aversive in that their retreats are accelerated by $Y(t)$ at a level lower than the level that prompts retreats of RN corporations. In other words, RA corporations tend to enter the market after and retreat from the market before RN corporations.

Concerning the dependency structure of $\eta_i(t|x,y)$ and $\xi_i(t|x,y)$ on t, x and y, for the sake of simplicity of presentation, we assume that both are independent of time t and depend only on $(x,y) \in \mathcal{S}_{M\setminus\{i\}}$. Let $H(A,B,x)$ be defined by

$$H(A,B,x) = e^{-\{A(x-B)\}^2}. \tag{5.4}$$

Then it may be appropriate to characterize the three categories RT, RN and RA by making $[\eta_i(t|x,y)]$ and $[\xi_i(t|x,y)]$ of the form $H(A,B,x) \times H(C,D,y)$ with different parameter values A, B, C and D where numbers for x and y are replaced by percentages against the whole population $N = 100$. These parameter values are summarized in Table 1 below, and the corresponding $\eta_i(t|x,y)$ and $\xi_i(t|x,y)$ are depicted in Figures 4 through 9.

Table 1: A, B, C and D

		A	B	C	D
RT	$\eta_i(t\|x,y)$	$4\sqrt{2}$	0.2	0	-
	$\xi_i(t\|x,y)$	$3\sqrt{2}$	0.2	0	-
RN	$\eta_i(t\|x,y)$	$8\sqrt{2}$	0.4	0	-
	$\xi_i(t\|x,y)$	$1.5\sqrt{2}$	0.4	$0.5\sqrt{2}$	0.4
RA	$\eta_i(t\|x,y)$	$8\sqrt{2}$	0.6	0	-
	$\xi_i(t\|x,y)$	$0.5\sqrt{2}$	0.6	$\sqrt{2}$	0.2

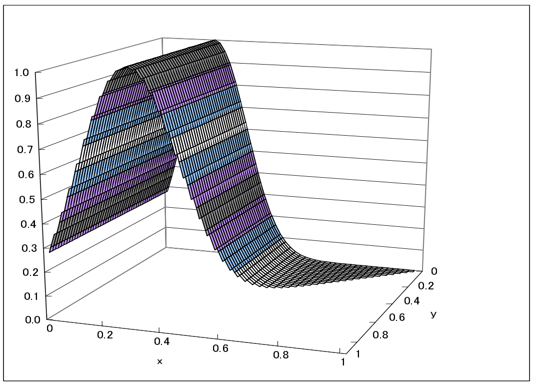

Figure 4: $[\eta_i(t|x,y)]$ of RT

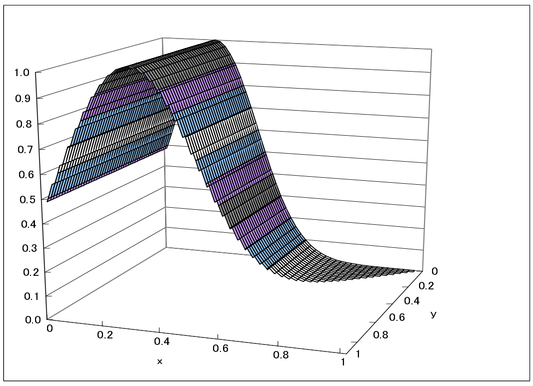

Figure 5: $[\xi_i(t|x,y)]$ of RT

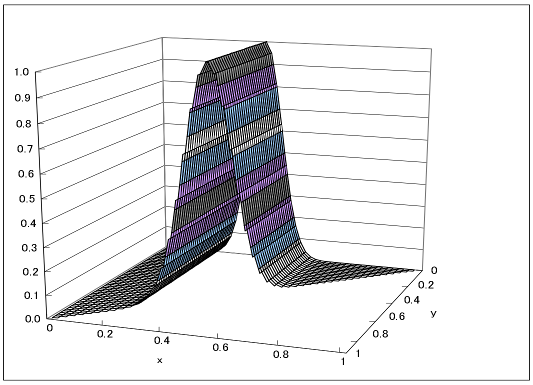

Figure 6: $[\eta_i(t|x, y)]$ of RN

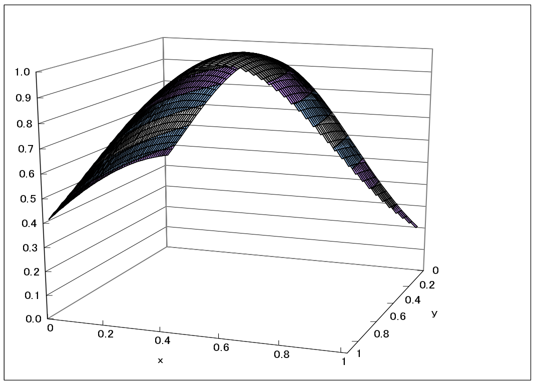

Figure 7: $[\xi_i(t|x, y)]$ of RN

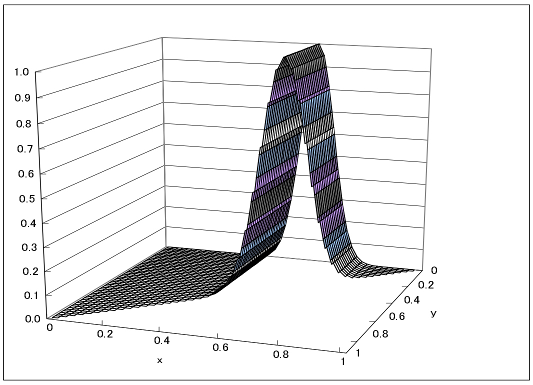

Figure 8: $[\eta_i(t|x, y)]$ of RA

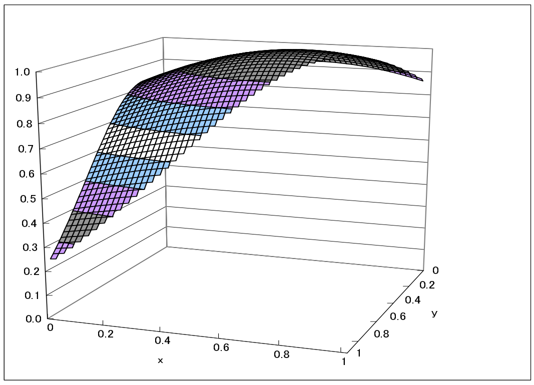

Figure 9: $[\xi_i(t|x, y)]$ of RA

In order to observe the characteristics of RT, RN and RA separately, we first consider three cases where corporations from only one category overwhelms corporations from other categories. Figures 10 through 12 exhibit $E[X(t)] = \frac{\partial}{\partial u}\psi(u, v, t)|_{u=1,v=1}$ for the three cases $(N_1, N_2, N_3) = (80, 10, 10)$, $(10, 80, 10)$ and $(10, 10, 80)$ where $\psi(u, v, t)$ is as given in (2.5). We observe that when RT corporations dominate, the market grows and declines very rapidly without having the maturity stage at all. On the other hand, when RN or RA corporations are present as the overwhelming majority, the market can hardly be formed.

Figure 10: $E[X(t)]$ for $(N_1, N_2, N_3) = (80, 10, 10)$

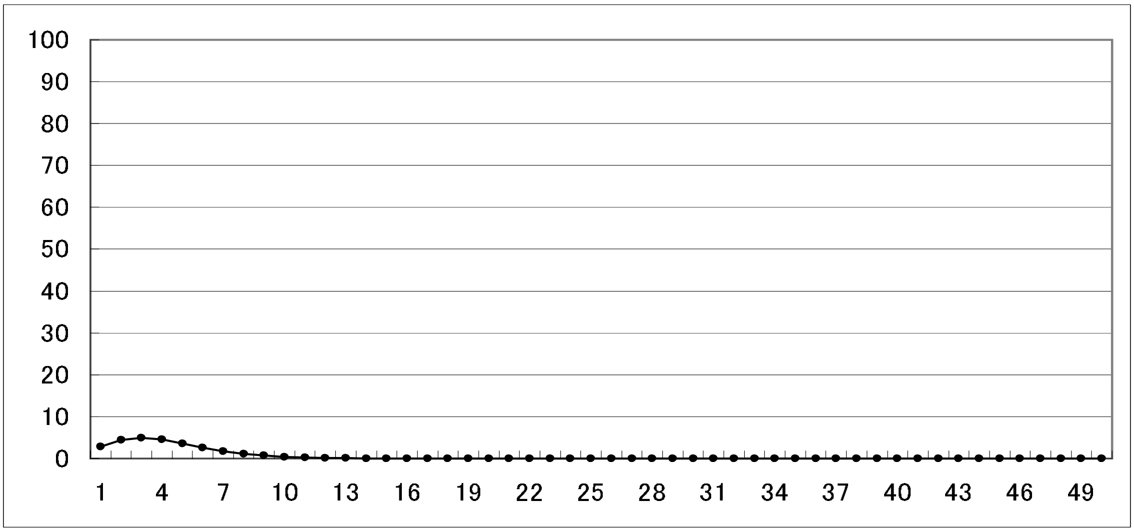

Figure 11: $E[X(t)]$ for $(N_1, N_2, N_3) = (10, 80, 10)$

Figure 12: $E[X(t)]$ for $(N_1, N_2, N_3) = (10, 10, 80)$

Figure 13 demonstrates the case $(N_1, N_2, N_3) = (30, 40, 30)$. It should be noted that the market lifecycle with four stages is clearly present. One can see that RT corporations trigger the first market growth, and then retreat from the market rather quickly, as the market growth is picked up next by RN corporations. Some of RA corporations then start to join the market. Both RN and RA corporations sustain the maturity stage. While RA corporations retreat from the market gradually, RN corporations tend to stay on and then begin to retreat rapidly. Consequently the decline stage is present largely due to RN corporations. As we saw in Figures 10 through 12, any category of corporations alone is incapable of creating the market lifecycle of this sort. It is remarkable to observe that interactions among the three categories change the market behavior so drastically.

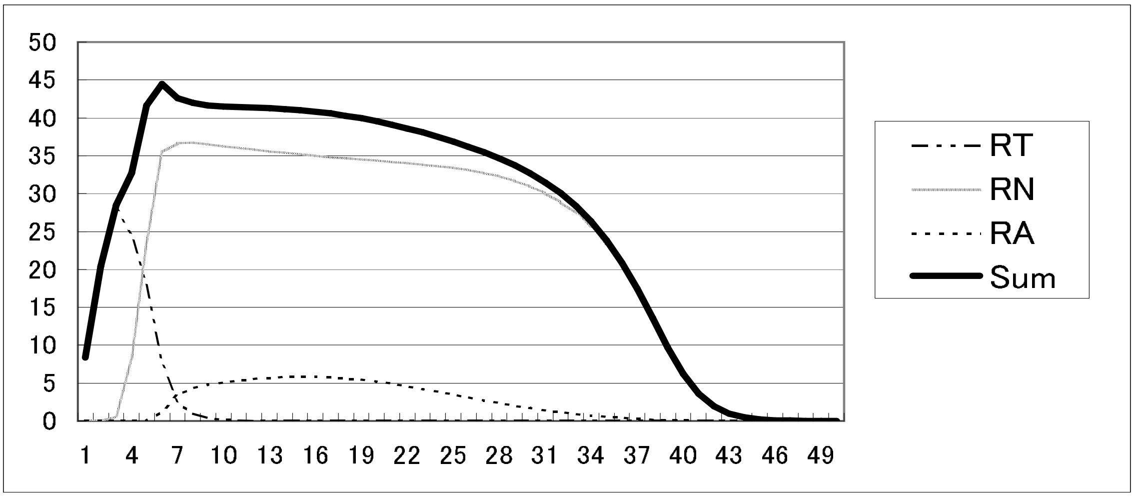

Figure 13: $E[X(t)]$ for $(N_1, N_2, N_3) = (30, 40, 30)$

We next conduct numerical experiments to understand the effect of interactions among the three categories in further detail. The total population $N = 100$ is fixed. In Figure 14, $E[X]$ is exhibited for $(N_1, N_2, N_3) = (80, 10, 10)$, $(70, 15, 15)$, $(60, 20, 20)$, $(50, 25, 25)$, $(40, 30, 30)$, $(30, 40, 30)$. It can be seen that the maturity stage starts to appear and becomes longer as N_1 decreases and two other classes increase from $(50, 25, 25)$ to $(30, 40, 30)$. However, beyond $N_1 = 60$ or more, the market rapidly loses its sustaining power after the peak.

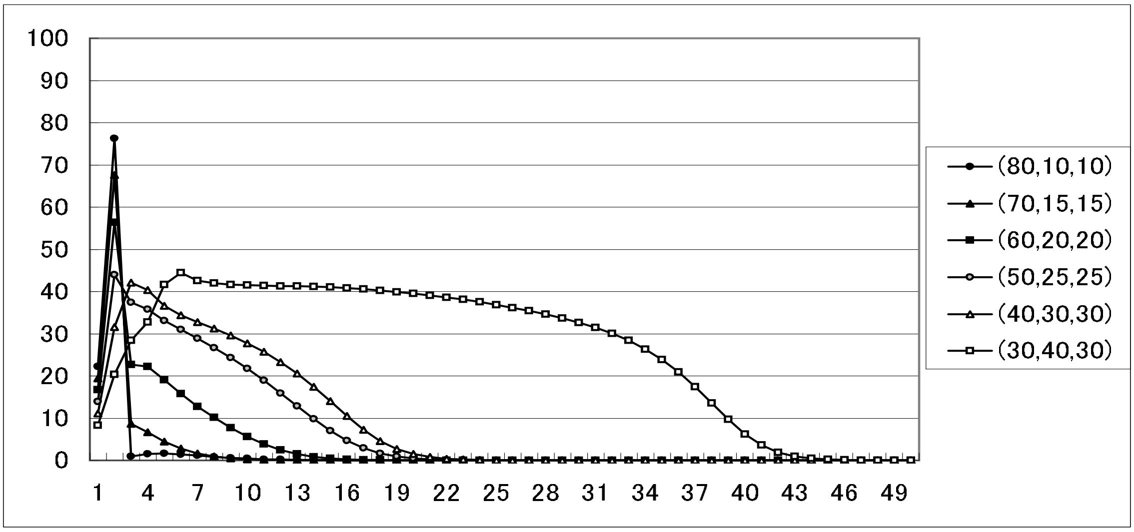

Figure 14: $E[X]$ for $(N_1, N_2, N_3) = (80, 10, 10), (70, 15, 15), (60, 20, 20), (50, 25, 25), (40, 30, 30), (30, 40, 30)$

In Figure 15, $E[X]$ is exhibied for $(N_1, N_2, N_3) = (10, 80, 10)$, $(15, 70, 15)$, $(20, 60, 20)$, $(25, 50, 25)$, $(30, 40, 30)$. It can be observed that the market lifecycle is clearly present for $N_2 = 50$ or less. However, at $N_2 = 60$, the market loses its growth momentum and almost disappears as N_2 increases further. Similar graphs are depicted in Figure 16 for $(N_1, N_2, N_3) = (10, 10, 80)$, $(15, 15, 70)$, $(20, 20, 60)$, $(25, 25, 50)$, $(30, 30, 40)$, $(30, 40, 30)$. As in Figure 15, one can observe that the market lifecycle is clearly present for $N_3 = 50$ or less. The market loses its growth momentum at $N_3 = 60$, and almost disappears as N_3 increases further.

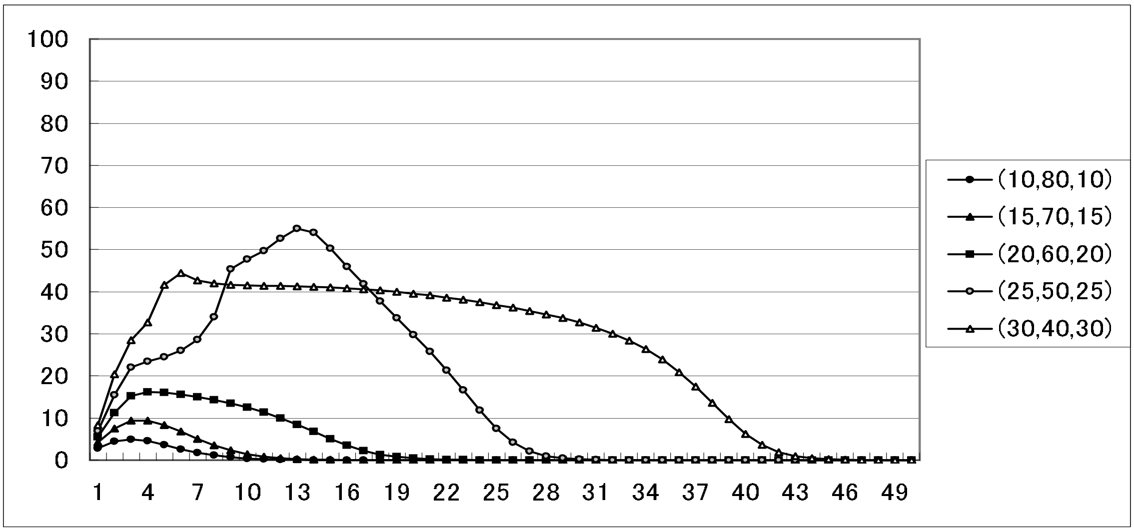

Figure 15: $E[X]$ for $(N_1, N_2, N_3) = (10, 80, 10), (15, 70, 15), (20, 60, 20), (25, 50, 25), (30, 40, 30)$

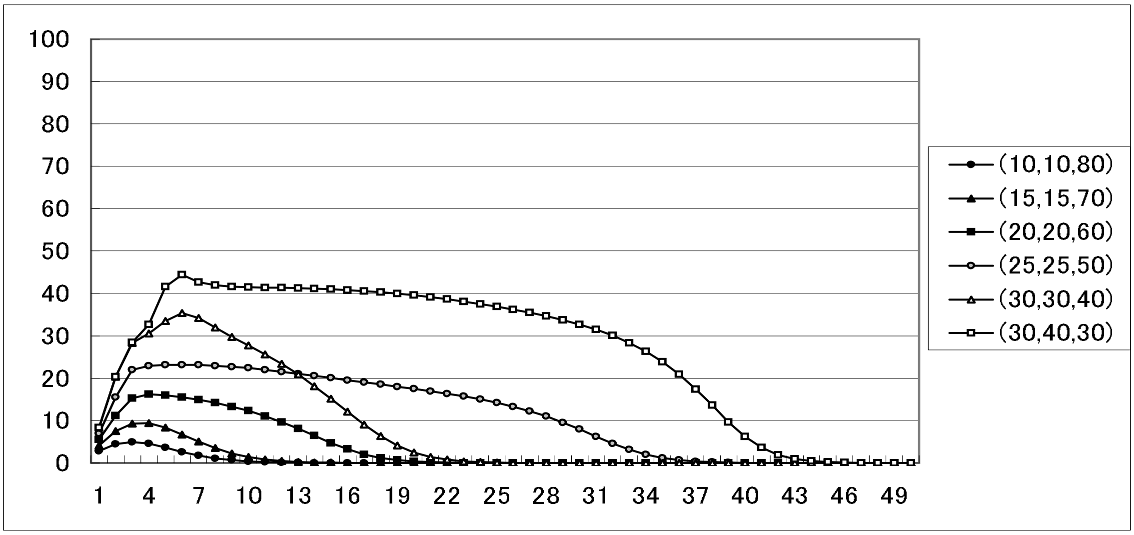

Figure 16: $E[X]$ for $(N_1, N_2, N_3) = (10, 10, 80), (15, 15, 70), (20, 20, 60), (25, 25, 50), (30, 30, 40), (30, 40, 30)$

6. Concluding Remarks

In this paper, an analytical model is developed for understanding the market lifecycle through strategic policies of individual corporations potentially interested in entering into the market. Strategic policies of individual corporations are expressed in terms of conditional probabilities of entry into and retreat from the market, which may depend on time t, the number of corporations in the market at time t, $X(t)$, and the number of corporations which have retreated from the market, $Y(t)$. Accordingly, each corporation is modelled as a temporally inhomogeneous discrete time margimnal process, and $\{\{X(t), Y(t)\} : t \geq 0\}$ is approximated by the independent sum of such marginal processes. Through spectral analysis of the underlying temporally inhomogeneous marginal process combined with a bivariate generating function approach, a numerical algorithm is developed for computing the joint probability distribution of $\{X(t), Y(t)\}$ for $t = 1, 2, \cdots$, capturing the characteristics of the market lifecycle in terms of $E[X(t)]$.

Corporations are classified into three groups: RT(Risk-Taking), RN(Risk-Neutral), and RA(Risk-Aversive), where these groups are characterized by specifying the transition probabilities of the underlying marginal processes. It is numerically observed that:

A) No category alone can constitute a typical market lifecycle with distinguishable four stages.
B) RT corporations trigger the creation of the market, motivating RN and RA corporations to join the market.
C) RN corporations play a major role in the growth stage and the maturity stage, stabilizing the market state, but take a leading role in initializing the decline stage.
D) RA corporations also contribute to form the maturity stage but only after the market reaches beyond a certain level.

In summary, the model developed in this paper enables one to understand how strategic policies of individual corporations collectively form the market lifecycle with four stages. While individual corporations make their own decisions separately, the market as a whole may emerge in a way that cannot be explained in terms of the characteristics of individual categories. Constructing this mechanism through an analytical model is the major contribution of this paper. The model proposed here is limited in that the market lifecycle is captured only through the number of corporations in the market, ignoring the total sales and other important market features. However, the numerical tractability of this model opens a new path toward development of more sophisticated market growth-decline models by incorporating additional features in the construction of the transition probabilities of the underlying temporally inhomogeneous marginal processes.

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Appendix

In this appendix, we analyze the case of two corporations, i.e. N=2, rigorously and compare the numerical results with those obtained by following the approximation procedure discussed in the paper.

For N=2, there are $3^2 = 9$ states $\{(m, n) : m, n = 0, 1, 2\}$. As in the paper, we assume that the strategic desicion parameters of two corporations are independent of time t and dependent only on $\{X(t), Y(t)\}$. For $i = 1, 2$, these parameters are denoted by $\eta_i(x, y)$ and $\xi_i(x, y)$ when $X(t) = x$ and $Y(t) = y$. Clearly the joint process $\{N_1(t), N_2(t)\}$ can be expressed as a temporally homogenuous Markov chain on $\{(m,n) : m,n = 0,1,2\}$ governed by the transition probability matrix $\underline{\underline{P}}$ given in Figure 17. We note that states (m,n) outside the matrix should not be confused with states (x,y) in the arguments of η_i and ξ_i.

	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
(0,0)	$1\text{-}\eta_1(0,0)$ $\times$ $1\text{-}\eta_2(0,0)$	$1\text{-}\eta_1(0,0)$ $\times$ $\eta_2(0,0)$	0	$\eta_1(0,0)$ $\times$ $1\text{-}\eta_2(0,0)$	$\eta_1(0,0)$ $\times$ $\eta_2(0,0)$	0	0	0	0
(0,1)	0	$1\text{-}\eta_1(1,0)$ $\times$ $\xi_2(1,0)$	$1\text{-}\eta_1(1,0)$ $\times$ $1\text{-}\xi_2(1,0)$	0	$\eta_1(1,0)$ $\times$ $\xi_2(1,0)$	$\eta_1(1,0)$ $\times$ $1\text{-}\xi_2(1,0)$	0	0	0
(0,2)	0	0	$1\text{-}\eta_1(0,1)$	0	0	$\eta_1(0,1)$	0	0	0
(1,0)	0	0	0	$\xi_1(1,0)$ $\times$ $1\text{-}\eta_2(1,0)$	$\xi_1(1,0)$ $\times$ $\eta_2(1,0)$	0	$1\text{-}\xi_1(1,0)$ $\times$ $1\text{-}\eta_2(1,0)$	$1\text{-}\xi_1(1,0)$ $\times$ $\eta_2(1,0)$	0
(1,1)	0	0	0	0	$\xi_1(2,0)$ $\times$ $\xi_2(2,0)$	$\xi_1(2,0)$ $\times$ $1\text{-}\xi_2(2,0)$	0	$1\text{-}\xi_1(2,0)$ $\times$ $\xi_2(2,0)$	$1\text{-}\xi_1(2,0)$ $\times$ $1\text{-}\xi_2(2,0)$
(1,2)	0	0	0	0	0	$\xi_1(1,1)$	0	0	$1\text{-}\xi_1(1,1)$
(2,0)	0	0	0	0	0	0	$1\text{-}\eta_2(0,1)$	$\eta_2(0,1)$	0
(2,1)	0	0	0	0	0	0	0	$\xi_2(1,1)$	$1\text{-}\xi_2(1,1)$
(2,2)	0	0	0	0	0	0	0	0	1

Figure 17 : Transition probability matrix $\underline{\underline{P}}$

The state probability vector $\underline{p}(t)^T$ at time t is given by

$$\underline{p}(t)^T = \underline{p}(0)^T \underline{\underline{P}}^t \tag{A.1}$$

where $\underline{p}(0)^T = [1,0,0,0,0,0,0,0,0]$. The joint probability generating function of $\{X(t), Y(t)\}$ at time t is then obtained as

$$\begin{aligned} \psi(u,v,t) &= E[u^{X(t)}v^{Y(t)}] \qquad \text{(A.2)} \\ &= p(t:0,0) + \{p(t:0,1) + p(t:1,0)\}u + \{p(t:0,2) + p(t:2,0)\}v \\ &\quad + p(t:1,1)u^2 + p(t:2,2)v^2 + \{p(t:1,2) + p(t:2,1)\}uv. \end{aligned}$$

This in turn yields the exact value

$$\begin{aligned} E[X(t)] &= \frac{\partial}{\partial u}\psi(u,v,t)|_{u=1,v=1} \qquad \text{(A.3)} \\ &= p(t:1,0) + p(t:0,1) + p(t:1,2) + p(t:2,1) + 2p(t:1,1) \end{aligned}$$

where $p(t:m,n)$ are obtained from (A.1).

The approximation procedure discussed in the paper for $N = 2$ can be summarized as follows. Perform the procedure below for $t = 0,1,\cdots$, starting with $\underline{p}_1^T(0) = \underline{p}_2^T(0) = [1,0,0]$.

1.

$$\alpha_i(t) = p_{3-i,0}(t)\eta_i(0,0) + p_{3-i,1}(t)\eta_i(1,0) + p_{3-i,2}(t)\eta_i(0,1); \tag{A.4}$$

$$\beta_i(t) = p_{3-i,0}(t)\xi_i(1,0) + p_{3-i,1}(t)\xi_i(2,0) + p_{3-i,2}(t)\xi_i(1,1), \ i = 1,2 \tag{A.5}$$

2.

$$\underline{\underline{a}}_i(t) = \begin{bmatrix} 1-\alpha_i(t) & \alpha_i(t) & 0 \\ 0 & \beta_i(t) & 1-\beta_i(t) \\ 0 & 0 & 1 \end{bmatrix},\ i=1,2 \tag{A.6}$$

3.

$$\underline{p}_i^T(t+1) = \underline{p}_i^T(t)\underline{\underline{a}}_i(t),\ \ i=1,2 \tag{A.7}$$

4.

$$\begin{aligned} \varphi_i(u,v,t) &= E[u^{X_i(t)}v^{Y_i(t)}] = p_{3-i,0}(t) + p_{3-i,1}(t)u + p_{3-i,2}(t)v,\ \ i=1,2 \\ \psi(u,v,t) &= \prod_{i=1}^{2}\varphi_i(u,v,t) \\ &= p_{1,0}(t)p_{2,0}(t) + \{p_{1,0}(t)p_{2,1}(t) + p_{1,1}(t)p_{2,0}(t)\}u \\ &\quad +\{p_{1,0}(t)p_{2,2}(t) + p_{1,2}(t)p_{2,0}(t)\}v \\ &\quad +p_{1,1}(t)p_{2,1}(t)u^2 + p_{1,2}(t)p_{2,2}(t)v^2 \\ &\quad +\{p_{1,1}(t)p_{2,2}(t) + p_{1,2}(t)p_{2,1}(t)\}uv \end{aligned}$$

5.

$$\begin{aligned} E[X(t)] &= \frac{\partial}{\partial u}\psi(u,v,t)|_{u=1,v=1} \\ &= p_{1,1}(t)p_{2,0}(t) + p_{1,0}(t)p_{2,1}(t) + p_{1,2}(t)p_{2,1}(t) \\ &\quad +p_{1,1}(t)p_{2,2}(t) + 2p_{1,1}(t)p_{2,1}(t) \end{aligned} \tag{A.8}$$

By setting the values of η_i and ξ_i as in Section 4, the exact values of $E[X(t)]$ computed via (A.3) are compared with the approximated values obtained from (A.8) in Table 2 for $1 \leq t \leq 30$. One sees that when two corporations are of the same type, (RT,RT), the approximation is excellent with relative errors contained within 1.6%. For the case of (RT,RN), the relative errors are within 6.0%, and they are within 1.3% for (RT,RA).

Table 2 : Ralative errors of $E[X(t)]$: exact values vs. approximated values

(Company 1, Company 2)=(RT,RT)

t	1	2	3	4	5	6	7	8	9	10
Exact	0.556	0.392	0.311	0.261	0.214	0.170	0.131	0.099	0.074	0.055
Approximated	0.556	0.392	0.328	0.264	0.209	0.163	0.125	0.095	0.071	0.053
Relative Error	0.0 %	0.0 %	1.6 %	0.3 %	0.5 %	0.7 %	0.6 %	0.4 %	0.2 %	0.1 %
t	11	12	13	14	15	16	17	18	19	20
Exact	0.040	0.029	0.021	0.015	0.011	0.008	0.006	0.004	0.003	0.002
Approximated	0.039	0.029	0.021	0.015	0.011	0.008	0.006	0.004	0.003	0.002
Relative Error	0.1 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %
t	21	22	23	24	25	26	27	28	29	30
Exact	0.002	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000
Approximated	0.002	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000
Relative Error	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %

(Company 1, Company 2)=(RT,RN)

t	1	2	3	4	5	6	7	8	9	10
Exact	0.278	0.333	0.321	0.302	0.284	0.268	0.253	0.239	0.226	0.215
Approximated	0.278	0.333	0.304	0.267	0.238	0.214	0.195	0.180	0.166	0.155
Relative Error	0.0 %	0.0 %	1.7 %	3.5 %	4.6 %	5.3 %	5.7 %	6.0 %	6.0 %	6.0 %
t	11	12	13	14	15	16	17	18	19	20
Exact	0.204	0.193	0.184	0.175	0.166	0.158	0.150	0.143	0.136	0.129
Approximated	0.144	0.135	0.127	0.120	0.113	0.107	0.102	0.096	0.092	0.087
Relative Error	5.9 %	5.8 %	5.6 %	5.5 %	5.2 %	5.0 %	4.8 %	4.6 %	4.4 %	4.2 %
t	21	22	23	24	25	26	27	28	29	30
Exact	0.123	0.117	0.111	0.106	0.100	0.096	0.091	0.086	0.082	0.078
Approximated	0.083	0.078	0.075	0.071	0.067	0.064	0.061	0.058	0.055	0.052
Relative Error	4.0 %	3.8 %	3.6 %	3.5 %	3.3 %	3.1 %	3.0 %	2.8 %	2.7 %	2.6 %

(Company 1, Company 2)=(RT,RA)

t	1	2	3	4	5	6	7	8	9	10
Exact	0.278	0.333	0.324	0.299	0.270	0.240	0.210	0.183	0.158	0.135
Approximated	0.278	0.333	0.316	0.287	0.257	0.229	0.203	0.178	0.155	0.134
Relative Error	0.0 %	0.0 %	0.7 %	1.3 %	1.3 %	1.0 %	0.8 %	0.5 %	0.3 %	0.2 %
t	11	12	13	14	15	16	17	18	19	20
Exact	0.115	0.098	0.083	0.070	0.059	0.050	0.042	0.035	0.029	0.025
Approximated	0.115	0.098	0.083	0.071	0.060	0.050	0.042	0.036	0.030	0.025
Relative Error	0.1 %	0.0 %	0.0 %	0.0 %	0.1 %	0.1 %	0.1 %	0.1 %	0.0 %	0.0 %
t	21	22	23	24	25	26	27	28	29	30
Exact	0.020	0.017	0.014	0.012	0.010	0.008	0.007	0.006	0.005	0.004
Approximated	0.021	0.017	0.015	0.012	0.010	0.008	0.007	0.006	0.005	0.004
Relative Error	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %	0.0 %

Ushio Sumita
Graduate school of Systems and Information Engineering
University of Tsukuba
1-1-1, Tennoudai, Tsukuba, Ibaraki, 305-8573 Japan
E-mail: sumita@sk.tsukuba.ac.jp