

A Pincer Randomization Method for Valuing American Options

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1 Introduction

Valuation of American options written on dividend-paying assets is an important issue in the actual market, since they have a much broader range of applications. However, there have been no closed-form formulas and analytical solutions. Many researchers have directed their effort to developing accurate and quick approximations for valuing American options.

An accurate approximation is the randomization method proposed by Carr [2], which is based on an American option with a random maturity. The random maturity follows the n -stage Erlangian distribution with mean equal to the pre-specified maturity. Although the idea is easy to understand, the *pdf* of Erlangian distribution is not suitable for obtaining a simple formula for the n -th approximation. Actually, Carr's formula for the n -th approximation of the American put value is given by a recursion of complex triple sums. To improve this shortcoming, an alternative randomization method has been recently developed by Kimura [3], which used an order statistic for the random maturity. Kimura's randomization not only has a much simpler expression than Carr's one, but also its numerical results have almost the same accuracy as Carr's. However, computational results sometimes behave unstably under a certain condition. Improving this inadequacy is a principal goal of our new randomization method, which we call a *pincer randomization*. The primal focus of this paper is on the American put option because the call case can be analyzed by put-call symmetry relations.

2 Preliminaries

Assume that the stock price $(S_t)_{t \geq 0}$ is a risk-neutral process governed by the stochastic differential equation

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t, \quad t \geq 0$$

where $W \equiv (W_t)_{t \geq 0}$ is a standard Brownian motion process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration corresponding to W and the probability measure \mathbb{P} is chosen so that the stock has mean rate of return r . Here, r is the risk-free rate of interest, δ is the dividend rate, and σ is the

volatility coefficient of the asset price. We consider an American *put* option written on the stock price process $(S_t)_{t \geq 0}$, which has maturity date $T > 0$ and strike price $K > 0$. Let $P \equiv P(t, S_t) = P(t, S_t; K, r, \delta)$ ($0 \leq t \leq T$) denote the value of the American put option at time t . It is sometimes convenient to work with the equations where the current time t is replaced by the remaining time until maturity $\tau = T - t$. Let $\hat{P}(\tau, S_\tau) = P(T - \tau, S_{T-\tau})$ and for $\lambda > 0$ let

$$P^* \equiv P^*(\lambda, S) = \int_0^\infty \lambda e^{-\lambda\tau} \hat{P}(\tau, S) d\tau$$

be the Laplace-Carson transform (LCT) of $\hat{P}(\tau, S)$. A simple and explicit expression for P^* can be obtained by Kimura [3, Theorem 1].

3 Kimura's Randomization

Let X_1, \dots, X_{n+m} be *iid* random variables with an *exponential* distribution with parameter $\alpha (> 0)$, and let $X_{(i)}$ denote the i -th smallest of these random variables ($i = 1, \dots, n+m$). Then, the *pdf* of $X_{(n+1)}$ is

$$f_{n,m}(t) = \frac{(n+m)!}{n!(m-1)!} (1 - e^{-\alpha t})^n \alpha e^{-\alpha t}, \quad t \geq 0,$$

where α is determined to satisfy either (i) $\mathbb{E}[X_{(n+1)}] = T$ or (ii) $\mathbb{M}[X_{(n+1)}] \equiv \arg \max_t f_{n,m}(t) = T$. The case (i) is called *mean matching*, and the case (ii) is *mode matching*. For a continuous function $g(t)$ ($t \geq 0$), define

$$g_{n,m}^*(T) = \mathbb{E}[g(X_{(n+1)})] = \int_0^\infty g(t) f_{n,m}(t) dt.$$

Kimura [3, Proposition 3] showed that the sequence $(g_{n,m}^*)_{n,m \geq 1}$ satisfies the recursion

$$g_{0,m}^*(T) = \int_0^\infty m \alpha e^{-\alpha t} g(t) dt$$

$$g_{n,m}^*(T) = \frac{n+m}{n} g_{n-1,m}^*(T) - \frac{m}{n} g_{n-1,m+1}^*(T), \quad n \geq 1,$$

and that $\lim_{n,m \rightarrow \infty} g_{n,m}^*(T) = g(T)$. If we set $g_{0,m}^*(T) = P^*(m\alpha, S)$, then the recursion above generates a sequence of approximations for $\hat{P}(\tau, S_\tau)$. This is just mathematical essence of Kimura's randomization.

4 A Pincer Randomization Method

Kimura's randomization method is not only much simpler than Carr's one, but also as accurate as Carr's one. However, the method shows unstable behavior near the expiry under certain conditions. The reasons for this instability are considered as

- the algorithm is sensitive to the precision of numbers used in calculations;
- the (n, m) -th approximation $g_{n,m}^*(T)$ cannot appropriately satisfy the value matching condition in the way of recursion.

Taking account of these points, we propose a new randomization scheme named a *pincer randomization* (PR) method. The PR method is an *interpolation* approximation based on a pair of *lower and upper bounds* for a true value. This methods reflect some fundamental properties of the order statistic $X_{(n+1)}$ and the option Greek *Theta* that the shorter the remaining time to expiration, the cheaper the option value.

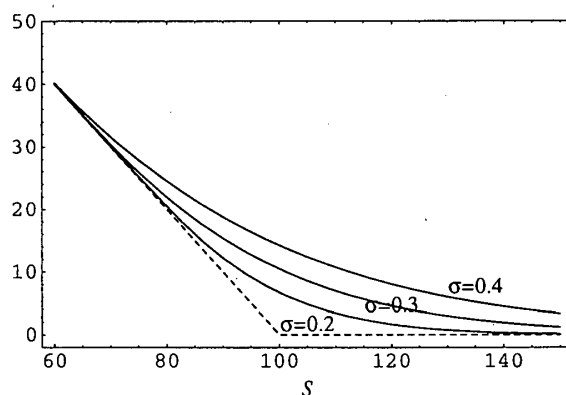
Assume that the maturity T is a random variable \tilde{T} distributed as the order statistic $X_{(n+1)}$ with mean $\mathbb{E}[\tilde{T}] = T$. From some numerical experiments, we can certainly observe that the mean-matching approximation for the option value always underestimates the true value when n, m is not large enough, *i.e.*, it gives a lower bound. Similarly, the mode-matching approximation gives an upper bound. It should be noted that each approximation generates an opposite-side bound for the early exercise boundary. In those experiments, the arithmetic average of the 1000- and 1001-step CRR binomial values is used as a benchmark of the true value. The experiments show that the true values are appropriately sandwiched in between the bounds, and that the lower bound derived by the mean matching is a good approximation. From these observations and other numerical experiments, we employ three methods for valuing American put options: (1) arithmetic average of the bounds; (2) geometric average of the bounds; and (3) lower bound itself.

5 Computational Results

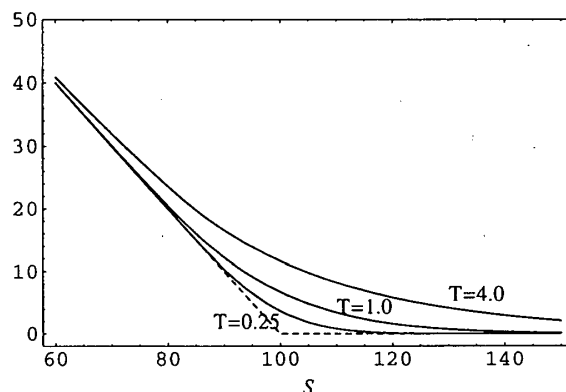
We can summarize the performance of our randomization methods as follows:

- The PR methods (1) and (2) perform very well and competes with the previous randomization methods. In addition, both methods are more accurate than LBA and LUBA developed by Broadie and Detemple [1], which are also lower-bound and interpolation approximations, respectively.
- The PR methods (1) and (2) become accurate as the initial price S increases, because the early exercise premium relatively constitutes a smaller portion of the value for such cases.

- The PR methods (1) and (2) become accurate as the remaining time becomes long.
- The lower-bound approximation (3) is less accurate than other approximations, and it performs well only if $\delta = 0$, for which $P^*(m\alpha, S)$ can be computed without using Newton's method. This implies that the accuracy of the lower-bound (or mean-matching) approximation would be highly sensitive to the computational precision.



(a) $T = 1.0, \sigma = 0.2, 0.3, 0.4$



(b) $T = 0.25, 1.0, 4.0, \sigma = 0.2$

Figure 1: Values of put options ($K = 100, t = 0, r = 0.05, \delta = 0.02$)

References

- [1] Broadie, M. and J. Detemple, "American option valuation: new bounds, approximations, and a comparison of existing methods," *Review of Financial Studies*, 9 (1996) 1211-1250.
- [2] Carr, P., "Randomization and the American put," *Review of Financial Studies*, 11 (1997) 597-626.
- [3] Kimura, T., "Alternative randomization for valuing American options," The 2004 Daiwa International Workshop on Financial Engineering, Kyoto, Japan, 2004.