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New bounds on the minimum calls of failure-tolerant gossiping

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1 Introduction

A gossiping problem and its various variations have been extensively studied for several decades (See, for example, [4] for survey). In the gossiping problem, first proposed by A. Boyd in 1971, there are n ladies, each of whom knows a unique message that is not known by any of the others. They communicate by telephone. Whenever two ladies make a call, they pass on to each other all information they know at that time. The gossiping problem is to find the minimum number of calls required for all ladies to know all massages. It has been proven that the solution to the problem is $2n-4$ for $n \geq 4$.

Gossiping is a fundamental task in network communication. As communication networks grow in size, they become increasingly vulnerable to component failures. Berman and Hawrylaycz [1] introduced the additional feature that as many as k of the calls may fail in the sense that no information is exchanged, where k is a second parameter of the problem. We assume that the ladies cannot attempt different calls depending on which ones have failed previously. Berman and Hawrylaycz [1] have sought bounds on $\tau(n,k)$, the number of telephone calls needed to ensure that all n ladies possess all n messages even if some arbitrary k calls fail. They established a lower bound of $\tau(n,k)$:

$$\tau(n,k) \geq \begin{cases} \left\lceil \left(\frac{k+4}{2}\right)(n-1) \right\rceil - 2\lceil\sqrt{n}\rceil + 1 & (k \leq n-2) \\ \left\lceil \left(\frac{k+3}{2}\right) n \right\rceil - 2\lceil\sqrt{n}\rceil & (k \geq n-2) \end{cases}$$

and an upper bound

$$\tau(n,k) \leq \left\lfloor \left(k+\frac{3}{2}\right)(n-1) \right\rfloor.$$

Haddad, Roy and Schäffer[3] showed the following upper bound:

$$\tau(n,k) \leq \frac{nk}{2} + \mathrm{O}(k\sqrt{n} + n\log_2 n)$$

which was an improvement for almost all k. Recently, Berman and Paul[2] proved that

$$2n-2+\left\lceil \frac{k(n-1)}{2} \right\rceil - \lfloor \log_2 n \rfloor \leq \tau(n,k).$$

In this paper, we propose a new upper bound on $\tau(n,k)$, which is improved the previous bounds for a sufficiently large k. Moreover, we give tighter bounds on $\tau(n,k)$ for small n.

Gossiping is modeled by an ordered multigraph $G=(V,E)$, where V is a vertex set with $|V|=n$ and E is an edge set on which a linear ordering is imposed. The vertices of G represent ladies, the edges represent telephone calls between pairs of ladies, and the linear order determines the turn of telephone calls. A message from $u \in V$ to $v \in V$ must proceed along a u-v *ascending path*, i.e., a path from u to v such that for any two edges in the path the edge closer to u is smaller in the linear order. We say an ordered multigraph G is *gossiping* if there is a u-v ascending path for every ordered pair of distinct vertices $u, v \in V$. If an ordered multigraph G is still gossiping whenever any k edges are deleted, G is called *k-failure tolerant gossiping.* Note that G is k-failure tolerant gossiping if and only if G has at least $k+1$ edge disjoint u-v ascending paths for every ordered pair of distinct vertices $u, v \in V$.

2 A construction of k-failure tolerant gossiping

Let a *class* be a subset of edges whose calls can be made in any order among themselves. Edges in the same class are ordered arbitrarily, but, for $i<j$, all the edges in the class i are ordered before any edge in the class j.

We now show a construction that establishes $\tau(n,k) \leq n(n-1)/2 + \lceil (nk)/2 \rceil$. At first, prepare a complete graph K_n. All edges of this complete graph belong to the class 0. Obviously, this complete graph is (0-failure tolerant) gossiping of redundancies. We next form edge sets of classes from 1 to $n-1$. When n is even, there are edge disjoint 1-factors, i.e., subgraphs induced by perfect matchings, $G_1, \ldots, G_{n-1}$ such that $\bigcup_{i=1}^{n-1} G_i$ is a complete graph K_n. When n is odd, we can decompose the complete graph K_n into $n-1$ spanning subgraph $G_1, \ldots, G_{n-1}$, where the degree of each vertex of G_i excepted to a specified vertex v_0 is exactly one, and, if i is odd, the degree of v_0 is two, otherwise, it is zero. We then parcel out the edge set of G_i, denoted by $E(G_i)$, to the class i for $1 \leq i \leq n-1$. Since each vertex $v \in V$ is adjacent to at least k vertices by edges belonging to the classes from 1 to k, there are at least k edge disjoint ascending paths form any vertex $u \in V$ to v. These u-v ascending paths have at most two edges: the first edge belongs to the class 0 and the second edge to one of the classes from 1 to k; or u and v are adjacent in the classes 1 to k. In addition, there is an edge incident to u and v in the class 0. Therefore, for $0 \leq k \leq n-1$, the ordered multigraph obtained by the collection of all the edges in the classes from 0 to k is k-failure tolerant gossiping.

For any $k \geq n$, we only repeat the above process. For example, in the case of even n and $nr \leq k \leq 2nr-1$, we make r copies of the 1-factors decomposition $(G_1^t, G_2^t, \ldots, G_{n-1}^t)$ $(1 \leq t \leq r)$ and parcel out $E(G_i^t)$ to the class $nt+i$ for $1 \leq i \leq n-1$ and $1 \leq t \leq r$. If n is odd, we can also construct k-failure tolerant gossiping by the same way.

Theorem 1 *For any k, the ordered multigraph obtained by the collection of all the edges in the classes from 0 to k is k-failure tolerant gossiping. Therefore, we obtain*

$$\tau(n,k) \leq \frac{n(n-1)}{2} + \left\lceil \frac{nk}{2} \right\rceil.$$

This result improves the previous upper bounds for a sufficiently large k.

3 Tighter bounds for small graphs

If G is k-failure tolerant gossiping, since there are at least $k+1$ edge disjoint ascending paths between every pair of vertices, the degree of each vertex is at least $k+1$. Moreover, we can see that if there are more than two vertices whose degrees are exactly $k+1$, then the graph becomes disconnected. Thus, we have the following result.

Theorem 2 *It holds that $\lfloor n(k+2)/2 \rfloor \leq \tau(n,k)$.*

This lower bound is tighter than the previous lower bounds when $n < 20$.

We also give tighter upper bounds on $\tau(n,k)$ for $n = 4, 6, 8, 10, 12, 14$ and 16 by constructing a k-failure tolerant gossiping for each case. Table 1 shows our upper bounds for these small cases, which, together with Theorem 2, implies $\tau(4,k) = 2k+4$. Unfortunately, we have not found a generalized rule for constructing k-failure tolerant gossipings for small case. So this paper shows only rules for $n = 8$ and 12

Table 1: Tighter upper bounds for small graphs

n	4	6	8	10, 12, 14, 16
$\tau(n,k) \leq$	$2k+4$	$3k+9$	$4k+12$	$nk/2+2n$

in Figures 1 and 2, respectively. For convenience, in these figures, we draw one edge to present multiedges. For example, in Figure 1, the edge (v_1, v_2) means the multiedges ordered with $1, 4, 7, 10 \ldots$. When $n = 8$, we can verify that the collection of all the edges in the classes 1 to 3 forms a 0-failure tolerant gossiping, and edges in the classes 1 to $k+3$ forms a k-failure tolerant gossiping. In the case of $n = 12$, edges in the classes 1 to 4 make a 0-failure tolerant gossiping, and edges in the classes 1 to $k+4$ form a k-failure tolerant gossiping.

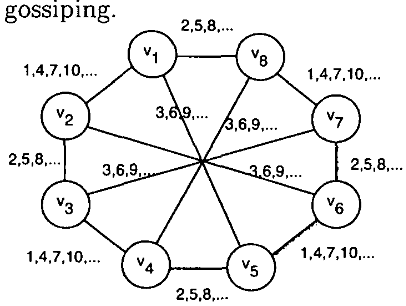

Figure 1: Construction for a k-failure tolerant gossiping with $n = 8$

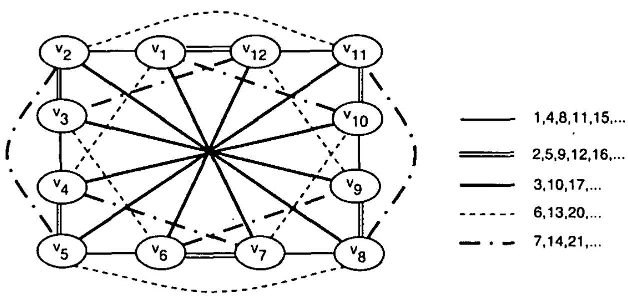

Figure 2: Construction for a k-failure tolerant gossiping with $n = 12$

References

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