

Dealing with Undesirable Outputs in DEA: A Slacks-based Measure (SBM) Approach

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1. Introduction

In accordance with the environmental conservation awareness in our modern society, undesirable outputs of productions and social activities, e.g., air pollutants and hazardous waste, have been widely recognized as societal evils. Thus, development of technologies with less undesirable outputs is an important subject of concern in every area of production. DEA (data envelopment analysis) usually assumes that producing more outputs relative to less input resources is a criterion of efficiency. In the presence of undesirable outputs, however, technologies with more good (desirable) outputs and less bad (undesirable) outputs relative to less input resources should be recognized as efficient. In the DEA literature, several authors have proposed methods for this purpose, e.g., Färe *et al.*(1989), Scheel (2001), and Seiford and Zhu (2002), among others.

This paper deals with the same problem by applying a slacks-based measure of efficiency (SBM) that the author proposed in Tone (2001). The SBM is non-radial and non-oriented, and utilizes input and output slacks directly in producing an efficiency measure. In this paper, SBM is modified so as to account for undesirable outputs.

2. An SBM with undesirable outputs

Suppose that there are n DMUs (decision making units) each having three factors: inputs, good outputs and bad (undesirable) outputs, as represented by three vectors $x \in R^m$, $y^g \in R^{s_1}$ and $y^b \in R^{s_2}$, respectively. We define the matrices X , Y^g and Y^b as follows. $X = [x_1, \dots, x_n] \in R^{m \times n}$, $Y^g = [y_1^g, \dots, y_n^g] \in R^{s_1 \times n}$, and $Y^b = [y_1^b, \dots, y_n^b] \in R^{s_2 \times n}$. We assume $X > 0$, $Y^g > 0$ and $Y^b > 0$.

The production possibility set (P) is defined by

$$P = \{(x, y^g, y^b) | x \geq X\lambda, y^g \leq Y^g\lambda, y^b \geq Y^b\lambda, \lambda \geq 0\}.$$

Definition 1 (Efficient DMU)

A DMU _{o} (x_o, y_o^g, y_o^b) is efficient in the presence of undesirable outputs if there is no vector $(x, y^g, y^b) \in P$ such that $x_o \geq x$, $y_o^g \leq y^g$ and $y_o^b \geq y^b$ with at least one strict inequality.

In accordance with this definition, we modify the SBM in Tone (2001) as follows.

$$\begin{aligned} \text{[SBM]} \rho^* = & \min \frac{1 - \frac{1}{m} \sum_{i=1}^m \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s_1 + s_2} \left(\sum_{r=1}^{s_1} \frac{s_r^g}{y_{ro}^g} + \sum_{r=1}^{s_2} \frac{s_r^b}{y_{ro}^b} \right)} \\ \text{subject to} & \quad x_o = X\lambda + s^- \\ & \quad y_o^g = Y^g\lambda - s^g \\ & \quad y_o^b = Y^b\lambda + s^b \\ & \quad s^- \geq 0, s^g \geq 0, s^b \geq 0, \lambda \geq 0. \end{aligned}$$

The objective function strictly decreases with respect to s_i^- ($\forall i$), s_r^g ($\forall r$) and s_r^b ($\forall r$) and the objective value satisfies $0 < \rho^* \leq 1$. Let an optimal solution of the above program be $(\lambda^*, s^{-*}, s^{g*}, s^{b*})$. Then, we have:

Theorem 1 *The DMU _{o} is efficient in the presence of undesirable outputs if and only if $\rho^* = 1$, i.e., $s^{-*} = 0$, $s^{g*} = 0$ and $s^{b*} = 0$.*

If the DMU _{o} is inefficient, i.e., $\rho^* < 1$, it can be improved and become efficient by deleting the excesses in inputs and bad outputs, and augmenting the shortfalls in good outputs via the following SBM-projection:

$$x_o \leftarrow x_o - s^{-*}, y_o^g \leftarrow y_o^g + s^{g*}, y_o^b \leftarrow y_o^b - s^{b*}.$$

Using the transformation by Charnes and Cooper (1962), we arrive at an equivalent linear program in t , Λ , S^- , S^g and S^b as displayed below.

$$\begin{aligned} \text{[LP]} \tau^* = & \min t - \frac{1}{m} \sum_{i=1}^m \frac{S_i^-}{x_{io}} \\ \text{subject to} & \quad 1 = t + \frac{1}{s_1 + s_2} \left(\sum_{r=1}^{s_1} \frac{S_r^g}{y_{ro}^g} + \sum_{r=1}^{s_2} \frac{S_r^b}{y_{ro}^b} \right) \\ & \quad x_o t = X\Lambda + S^- \\ & \quad y_o^g t = Y^g\Lambda - S^g \\ & \quad y_o^b t = Y^b\Lambda + S^b \\ & \quad S^- \geq 0, S^g \geq 0, \\ & \quad S^b \geq 0, \Lambda \geq 0, t > 0. \end{aligned}$$

Let an optimal solution of [LP] be $(t^*, \Lambda^*, S^{-*}, S^{g*}, S^{b*})$. Then we have an optimal solution of [SBM] as defined by

$$\begin{aligned}\rho^* &= \tau^*, \quad \lambda^* = \Lambda^*/t^*, \quad s^{-*} = S^{-*}/t^*, \\ s^{g*} &= S^{g*}/t^*, \quad s^{b*} = S^{b*}/t^*.\end{aligned}$$

The existence of $(t^*, \Lambda^*, S^{-*}, S^{g*}, S^{b*})$ with $t^* > 0$ is guaranteed by [LP].

3. Economic interpretations

We have the dual program of [LP] as follows:

$$\begin{aligned}[\text{DualLP}] \quad & \max \quad u^g y_o^g - v x_o - u^b y_o^b \\ \text{subject to} \quad & u^g Y^g - v X - u^b Y^b \leq 0 \\ & v \geq \frac{1}{m} [1/x_o] \\ & u^g \geq \frac{1 + u^g y_o^g - v x_o - u^b y_o^b}{s} [1/y_o^g] \\ & u^b \geq \frac{1 + u^g y_o^g - v x_o - u^b y_o^b}{s} [1/y_o^b].\end{aligned}$$

The dual variables v and u^b can be interpreted as the virtual prices (costs) of inputs and bad outputs, respectively, while u^g denotes the price of good outputs. The dual program aims at obtaining the optimal virtual costs and prices for DMU_o so that the profit $u^g y^g - v x - u^b y^b$ does not exceed zero for every DMU (including DMU_o) and maximizes the profit $u^g y_o^g - v x_o - u^b y_o^b$ for the DMU_o concerned. Apparently, the optimal profit is at best zero and hence $\xi^* = 1$ for the SBM efficient DMUs.

4. Returns to scale issue

Although we discussed our bad outputs model under the constant returns to scale (CRS) assumption, we can incorporate other RTS by adding the following constraint to [SBM] and hence to the definition of the production possibility set P ,

$$L \leq e\lambda \leq U,$$

where $e = (1, \dots, 1) \in R^n$ and, $L(\leq 1)$ and $U(\geq 1)$ are respectively the lower and upper bounds to the intensity λ .

The cases $(L = 1, U = 1)$, $(L = 0, U = 1)$ and $(L = 1, U = \infty)$ correspond to the variable (VRS), the decreasing (DRS) and the increasing (IRS) RTS, respectively.

The definition of the efficiency status is the same as described in [Definition 1] and Theorem 1 holds in these cases, too. The price interpretation of the role of the dual variables remains valid in this case, too.

5. Non-separable 'bad' and 'good' outputs

It is often observed that a certain 'bad' outputs are not separable from the corresponding 'good'

outputs. Hence, reducing bad outputs is inevitably accompanied by reduction in good outputs. In this section, we discuss this non-separable case. For this, we decompose the set of good and bad outputs (Y^g, Y^b) into (Y^{Sg}, Y^{Sb}) and (Y^{NSg}, Y^{NSb}) , where $(Y^{Sg} \in R^{s_{11} \times n}, Y^{Sb} \in R^{s_{12} \times n})$ and $(Y^{NSg} \in R^{s_{21} \times n}, Y^{NSb} \in R^{s_{22} \times n})$ denote the separable and non-separable good and bad outputs, respectively. For the separable outputs (Y^{Sg}, Y^{Sb}) , we have the same structure of production as (Y^g, Y^b) in P . However, the non-separable outputs (Y^{NSg}, Y^{NSb}) need handling differently. A reduction of the bad outputs y^{NSb} is designated by αy^{NSb} with $0 \leq \alpha \leq 1$, which is accompanied by a proportionate reduction in the good outputs y^{NSg} as denoted by αy^{NSg} . Although in this case we assume the same proportionate rate α in bad and good outputs, we can set other relationships between the two, e.g., αy^{NSb} and βy^{NSg} with $0 \leq \alpha, \beta \leq 1$.

An SBM with non-separable outputs can be implemented by the program in $(\lambda, s^-, s^{Sg}, s^{Sb}, \alpha)$ as below:

$$[\text{SBM-NS}]\rho^* = \min \left(1 - \frac{1}{m} \sum_{i=1}^m \frac{s_i^-}{x_{io}} \right) / \left(1 + \frac{1}{s} \left(\sum_{r=1}^{s_{11}} \frac{s_r^{Sg}}{y_{ro}^{Sg}} + \sum_{r=1}^{s_{12}} \frac{s_r^{Sb}}{y_{ro}^{Sb}} + (s_{21} + s_{22})(1 - \alpha) \right) \right)$$

subject to

$$\begin{aligned}x_o &= X\lambda + s^- \\ y_o^{Sg} &= Y^{Sg}\lambda - s^{Sg} \\ y_o^{Sb} &= Y^{Sb}\lambda + s^{Sb} \\ \alpha y_o^{NSg} &\leq Y^{NSg}\lambda \\ \alpha y_o^{NSb} &\geq Y^{NSb}\lambda \\ e\lambda &= 1\end{aligned}$$

$$s^- \geq 0, \quad s^{Sg} \geq 0, \quad s^{Sb} \geq 0, \quad \lambda \geq 0, \quad 0 \leq \alpha \leq 1,$$

where $s = s_{11} + s_{12} + s_{21} + s_{22}$.

The objective function is strictly monotone decreasing with respect to s_i^- ($\forall i$), s_r^{Sg} ($\forall r$), s_r^{Sb} ($\forall r$) and α . Let an optimal solution for [SBM-NS] be $(\rho^*, \lambda^*, s^{-*}, s^{Sg*}, s^{Sb*}, \alpha^*)$, then we have $0 < \rho^* \leq 1$ and the following theorem holds:

Theorem 2 *The DMU_o is NS-efficient if and only if $\rho^* = 1$, i.e., $s^{-*} = 0$, $s^{Sg*} = 0$, $s^{Sb*} = 0$ and $\alpha^* = 1$.*

We will compare our method with other methods proposed so far and exhibit numerical examples at the presentation.

References

1. Tone, K., 2001. A slacks-based measure of efficiency in data envelopment analysis. *European Journal of Operational Research* 130, 498-509.