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An Economic Premium Principle in Multiperiod Economy

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1 Introduction

This paper considers a multiperiod economic equilibrium model to derive the economic premium principle of Bühlmann (1980, 1983). To do this, we construct a consumption/portfolio model in which each agent characterized by his/her utility function and endowments can invest his/her wealth into insurance market as well as financial market to maximize the expected, discounted total utility from consumption. The state price density in equilibrium is obtained in terms of the Arrow–Pratt index of absolute risk aversion for the representative agent. As special cases, power and exponential utility functions are examined, and some comparative statics results are derived.

2 The Multiperiod Model

Trading dates; $\mathcal{T} \equiv \{0, 1, \cdots, T\}$ with $T > 0$.
Unncertainty is described by $(\Omega, \mathcal{F}, \mathbb{F}, P)$;
$\mathbb{F} \equiv \{\mathcal{F}_t; t \in \mathcal{T}\}$; $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$ with $\mathcal{F}_0 = (\Omega, \emptyset)$. Agent i, $i = 1, 2, \cdots, n$.

$w_i(t)$: endowments for agent i at time t.

$X_i(t)$: *risk* for agent i at time t.

$Z_i(t) \equiv w_i(t) - X_i(t)$: the *net endowment* for agent i at time t.

$$Z(t) \equiv \sum_{i=1}^{n} Z_i(t) > 0 :$$

the *aggregated net endowment*.

$p(t)$: premium-per-share of an *insurance* at time t;

$$\begin{aligned} p(t+1) &= p(t)(1 + \xi(t+1)), \\ & \qquad t \in \mathcal{T}_- \equiv \{0, 1, \cdots, T-1\}, \\ p(0) &= p > 0 \end{aligned}$$

for some $\mathbb{F}$-adapted process $\xi(t) > -1$.

$S_0(t)$: time t price of the money market account;

$$\begin{aligned} S_0(t+1) &= S_0(t)(1 + r(t+1)), \quad t \in \mathcal{T}_-, \\ S_0(0) &= 1, \end{aligned}$$

for some $\mathbb{F}$-adapted, positive process $r(t)$.

$S_j(t)$: time t price of security j, $j = 1, 2, \cdots, m$;

$$\begin{aligned} S_j(t+1) &= S_j(0)(1 + \xi^{(j)}(t+1)), \quad t \in \mathcal{T}_-, \\ S_j(0) &= s_j > 0, \end{aligned}$$

for some $\mathbb{F}$-adapted process $\xi^{(j)} > -1$.

The market is referred to as

$$\mathcal{M} = (\{Z_i(t)\}_{i=1}^{n}, r(t), \{\xi^{(j)}(t)\}_{j=1}^{m}, \xi(t)).$$

$Y_i(t)$: # of shares of insurance,
$\theta_i^{(0)}(t)$: # of shares in the money market,
$\theta_i^{(j)}(t)$: # of shares in security j,
carried by agent i from time t to $t+1$.
$(\boldsymbol{\theta}_i(t), Y_i(t)) \equiv$ a *portfolio* of agent i at time t,

$$\boldsymbol{\theta}_i(t) \equiv (\theta_i^{(0)}(t), \theta_i^{(1)}(t), \cdots, \theta_i^{(m)}(t))^\top.$$

$\phi(t)$: the *state price density*;

$$\begin{aligned} &\phi(0) = 1, \qquad 0 < \phi(t) < \infty, \\ &E_{t-1}[\phi(t)p(t)] = \phi(t-1)p(t-1); \\ &E_{t-1}[\phi(t)S_j(t)] = \phi(t-1)S_j(t-1), \\ &\qquad j = 0, 1, \cdots, m, \quad t = 1, 2, \cdots, T. \end{aligned}$$

The problem that each agent faces in the market $\mathcal{M}$:

(MP) Find an optimal $(\hat{c}_i(t), (\hat{\boldsymbol{\theta}}_i(t), \hat{Y}_i(t)))$;

$$\begin{aligned} \max \quad & E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)} U_i(c_i(t))\right] \\ \text{s.to} \quad & E\left[\sum_{t=0}^{T} \phi(t) c_i(t)\right] \le E\left[\sum_{t=0}^{T} \phi(t) Z_i(t)\right], \\ & E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)} U_i^-(c_i(t))\right] < \infty. \end{aligned}$$

$\beta(t)$: a common *discount process*,
$U_i(\cdot)$: a *utility function* for agent i.

Theorem 2.1 *The optimal $\hat{c}_i(t)$ is given by*

$$\hat{c}_i(t) = I_i\left(y_i e^{\sum_{u=0}^{t}\beta(u)}\phi(t)\right), \qquad t \in \mathcal{T},$$

where

$$\begin{aligned} I_i(\cdot) &\equiv \text{the inverse of } U_i', \\ y_i &= \mathcal{Y}_i\left(E\left[\sum_{t=0}^{T}\phi(t)Z_i(t)\right]\right), \qquad (1) \\ \mathcal{Y}_i(\cdot) &\equiv \text{the inverse of} \\ \mathcal{X}_i(y) &\equiv E\left[\sum_{t=0}^{T}\phi(t)I_i\left(y\, e^{\sum_{u=0}^{t}\beta(u)}\phi(t)\right)\right]. \end{aligned}$$

Given $\mathbf{\Lambda} = (\lambda_1, \cdots, \lambda_n) \in (0,\infty)^n$,

$$U(c;\mathbf{\Lambda}) \equiv \max\left\{\sum_{i=1}^{n}\lambda_i U_i(c_i);\ c_i > 0,\ i = 1,\cdots,n,\ \sum_{i=1}^{n} c_i = c\right\}:$$

a utility function of the *representative agent.*

$$\rho(x,\mathbf{\Lambda}) \equiv -\frac{U''(x,\mathbf{\Lambda})}{U'(x,\mathbf{\Lambda})}:$$

the Arrow-Pratt index of ARA (absolute risk aversion).

Theorem 2.2 *If $\mathcal{M}$ is an equilibrium market, then*

$$\phi(t) = \frac{\exp\left\{-\sum_{u=0}^{t}\beta(u) - \int_0^{Z(t)}\rho(x,\mathbf{\Lambda}^*)\,dx\right\}}{E\left[S_0(t)\exp\left\{-\sum_{u=0}^{t}\beta(u) - \int_0^{Z(t)}\rho(x,\mathbf{\Lambda}^*)\,dx\right\}\right]}. \qquad (2)$$

with $\mathbf{\Lambda}^* = (y_1^{-1},\cdots,y_n^{-1})$, where $y_1,\cdots,y_n$ are given by (1).

3 Some Special Cases

3.1 Power Utility Functions

$$U_i(x) = \gamma_i^{1-\alpha}\frac{x^\alpha}{\alpha},$$
$$0 < x < \infty, \quad \gamma_i > 0, \quad \alpha \in (-\infty, 1).$$

From (2)

$$\phi(t) = e^{-\sum_{u=0}^{t}\beta(u)}\left(\frac{Z(t)}{Z(0)}\right)^{-(1-\alpha)}$$

Especially, if $S_0(t) = e^{\sum_{u=0}^{t}\beta(u)}$,

$$\mathcal{P}(X(t)) = \frac{E[X^*(t)Z^{-(1-\alpha)}(t)]}{E[Z^{-(1-\alpha)}(t)]},$$

where $X^*(t) \equiv X(t)/S_0(t)$.

3.2 Exponential Utility Functions

$$U_i(x) = \frac{1-e^{-\gamma_i x}}{\gamma_i},$$
$$0 < x < \infty, \quad \gamma_i > 0.$$

From (2)

$$\phi(t) = \exp\left\{-\sum_{u=0}^{t}\beta(u) - \gamma(Z(t) - Z(0))\right\},$$

where $\frac{1}{\gamma} \equiv \sum_{i=1}^{n}\frac{1}{\gamma_i}$.

In the case that $S_0(t) = e^{\sum_{u=0}^{t}\beta(u)}$,

$$\mathcal{P}(X(t)) = \frac{E[X^*(t)e^{-\gamma Z(t)}]}{E[e^{-\gamma Z(t)}]}.$$

Especially, if $X^*(t) = Z(t)$,

$$\mathcal{P}(S_0(t)Z(t)) = \frac{E[Z(t)e^{-\gamma Z(t)}]}{E[e^{-\gamma Z(t)}]},$$

which is called the *Esscher principle.*

4 Some Comparative Statics

Assume $S_0(t) = e^{\sum_{u=0}^{t}\beta(u)}$ and $X^*(t) = f(Z(t))$ for a certain function f.

The present value of $X(t)$ is given by

$$\mathcal{P}_t(f;U) = \frac{E[f(Z(t))U'(Z(t);\mathbf{\Lambda}^*)]}{E[U'(Z(t);\mathbf{\Lambda}^*)]}.$$

Consider two economies, E_1 and E_2.
$U_k \equiv$ the utility function of the representative agent of E_k.

Theorem 4.1 *Suppose that the representative agent in E_2 is more risk averse than the one in E_1. If f is nonincreasing (nondecreasing, respectively) then,*

$$\mathcal{P}_t(f;U_1) \le\ (\ge)\ \mathcal{P}_t(f;U_2).$$

References

[1] Bühlmann, H. (1980). An Economic Premium Principle. *Astin Bulletin* 11, 52–60.

[2] Bühlmann, H. (1983). The General Economic Premium Principle. *Astin Bulletin* 14, 13–21.