A Decomposition of Cost Efficiency under Weight Restrictions in DEA

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1. Introduction

In the optimal weight of DEA models for inefficient DMUs, we may see many zeros — showing that the DMU has a weakness in the corresponding items compared with other (efficient) DMUs. Also, large differences in weights from item to item may be a concern.

The imposition of weight restrictions in DEA has been recognized as one of the important factors when applying DEA to actual situations and several models are developed for this purpose. See [2, 3, 4, 5, 6].

As in statistics or other empirically oriented methodologies, there is a problem involving degrees of freedom, which is compounded in DEA because of its orientation to *relative* efficiency. The number of degrees of freedom will increase with the number of DMUs and decrease with the number of inputs and outputs. A rough rule of thumb which can provide guidance is as follows [1]:

$$n \geq \max\{m \times s, 3(m+s)\},$$

where n = number of DMUs, m = number of inputs and s = number of outputs.

The weight restriction approach helps to discriminate among DMUs in efficiency even if the above inequality is hard to hold. The purpose of this paper is to develop a decomposition of cost efficiency into scale, pure technical and allocative ones when weight restrictions are imposed.

2. Weight Restrictions

We consider the following extension of the CCR model:

$$
\begin{aligned}
(DWR_o) \quad & \max\ z = uy_o & (1)\\
\text{st.} \quad & vx_o = 1 & (2)\\
& -vX + uY \leq 0 & (3)\\
& vP \leq 0 & (4)\\
& uQ \leq 0 & (5)\\
& v \geq 0,\ u \geq 0, & (6)
\end{aligned}
$$

where the matrices P and Q are associated with weight restrictions as described below.

A simple restriction such as $v_1 \leq 2v_2$ can be expressed using P as well as a more complicated constraint such as $v_1 + v_2 \leq 3v_3$ can be included in P. As for the output, the matrix Q has a similar role for restricting the region of output weights. Here, we deal with homogenous inequality constraints (4) and (5), i.e., constraints with the right-hand being zero. Further, we assume that the restrictions (4) and (5) are consistent and (DWR_o) has a finite positive optimum. The dual of (DWR_o) is:

$$
\begin{aligned}
(WR_o) \quad & \min\ \theta & (7)\\
\text{st.} \quad & \theta x_o - X\lambda + P\pi - s^- = 0 & (8)\\
& Y\lambda + Q\tau - s^+ = y_o & (9)\\
& \lambda \geq 0,\ s^- \geq 0,\ s^+ \geq 0, & (10)
\end{aligned}
$$

where the vectors π and τ are nonnegative dual variables corresponding to the constraints (4) and (5), respectively.

3. Production Possibility Set of WR Model

The production possibility set P_W of the WR model is defined as a set of semipositive (x, y) as follows:

$$P_W = \{(x,y) \mid x \geq X\lambda - P\pi,\ y \leq Y\lambda + Q\tau\}, \quad (11)$$

where $\lambda \geq 0, \pi \geq 0$, and $\tau \geq 0$.

4. Several Efficiency Measures

We define the *technical efficiency* of DMU (x_o, y_o) under WR as the optimal solution θ^* to (WR_o). The *pure technical efficiency* θ_P^* of (x_o, y_o) is defined as the optimal solution of the above LP under the variable returns-to-scale assumption, i.e,

$$e\lambda = 1.$$

Evidently, we have

Proposition 1 $1 \geq \theta_P^* \geq \theta^*$.

The *scale efficiency* σ^* under weight restrictions is defined as

$$\sigma^* = \theta^* / \theta_P^*. \quad (12)$$

Let the cost vector of input x_o for DMU (x_o, y_o) be c_o. Solve the following LP:

$$\min \quad c_o x \quad (13)$$

$$\text{st.} \quad x \geq X\lambda - P\pi \quad (14)$$
$$y_o \leq Y\lambda + Q\tau \quad (15)$$
$$\lambda \geq 0,\ \pi \geq 0,\ \tau \geq 0. \quad (16)$$

Using the optimal value $c_o x^*$ for the above LP, we define the *cost efficiency* γ^* of DMU (x_o, y_o) under WR by

$$\gamma^* = c_o x^* / c_o x_o. \quad (17)$$

We have a proposition

Proposition 2 *The cost efficiency γ^* under WR never exceeds the technical efficiency θ^*.*

The *allocative efficiency* α^* under WR is defined as the ratio of cost vs. technical efficiencies as

$$\alpha^* = \gamma^* / \theta^*. \quad (18)$$

5. A Decomposition

Using the above definitions we have a decomposition of the cost efficiency under WR as follows:

$$\gamma^* = \theta^* \times \alpha^* \quad (19)$$
$$= \theta_P^* \times \sigma^* \times \alpha^*. \quad (20)$$

Verbally we have

Cost Eff = PTE × Scale Eff × Allocative Eff.

6. Implication

The above-mentioned decomposition, which is unique, depicts the sources of inefficiency, i.e., whether it is caused by inefficient operation (PTE) or by disadvantageous condition displayed by the scale efficiency or by allocative inefficiency.

Also, we can interpret the value defined by (Cost Eff) / (Scale Eff) as "scale-adjusted" cost efficiency which is the product of pure technical efficiency and allocative efficiency.

7. An Example

Table 1 exhibits a sample of decomposition. We will show detailed data at the Conference.

8. Conclusion

As an extension of researches in DEA-efficiency under weight restrictions we have demonstrated a cost efficiency model under WR circumstances along with its decomposition. This will enhance practical usage of DEA results. See also [7] for characterizations of returns to scale under WR.

Table 1: A Sample Decomposition

DMU	Cost-Eff	PTE	Scale	Allocative
H1	0.97	1	0.99	0.98
H2	1	1	1	1
H3	0.86	0.90	0.96	0.98
H4	0.50	0.53	0.99	0.95
H5	0.89	1	0.96	0.93
H6	0.81	0.97	0.86	0.96
H7	0.81	0.82	0.99	0.98
H8	1	1	1	1
H9	0.42	0.47	0.94	0.95
H10	0.98	1	1	0.98
H11	0.75	0.88	0.85	0.99
H12	0.96	1	0.97	0.98
H13	0.89	1	0.91	0.97
H14	0.85	1	0.86	0.98
H15	0.74	0.87	0.89	0.95
H16	0.81	1	0.82	0.99
H17	0.97	1	1	0.97

References

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