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Finding the upper and lower bounds of the principal eigenvalue of a positive uncertain matrix

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1 Introduction

Several mathematical models such as Leontief's Input-Output analysis and Analytic Hierarchy Process involve positive matrices whose entries are empirically observed or estimated. The key index of these models is the principal eigenvalue of the matrices. It is important to estimate the lower and upper bounds of the index when the entries of the matrix are perturbed or estimated. This study considers that some entries of the matrix are uncertain but row vectors are contained in a given polytope. We show that the problem of estimating the principal eigenvalue of such a uncertain positive matrix reduces to a pair of multifractional programs and can be solved in a finite time.

2 Multifractional Problems

The largest eigenvalue is often referred to as the principal eigenvalue. Let $I = \{1, 2, \ldots, n\}$. The i^{th} row vector of X of order n is denoted by x_i for all $i \in I$. We consider a perturbation of the i^{th} row vector x_i of X as $x_iC^i \leq b^i$ for all $i \in I$, where C^i is some matrix and b^i is some row vector. The polyhedron $\{x \,|\, xC^i \leq b^i\}$ is denoted by S^i for all $i \in I$. We put the following assumption in order to to keep the matrix X consisting of the rows $\{x_1, \ldots, x_n\}$ positive.

Assumption 2.1 *Assume that the polyhedron S^i is a nonempty bounded set in the positive orthant of R^n for all $i \in I$.*

Since S^i is a polytope for all $i \in I$, the product set $\prod_{i\in I} S^i$ is a polytope. For every positive matrix X, the principal eigenvalue of X is denoted by $\Lambda(X)$. From Frobenous' theorem [1], $\Lambda(X)$ is a real function on $\prod_{i\in I} S^i$. By $X \in \prod S^i$, $x_i \in S^i$ for all $i \in I$. We consider a pair of the following two problems:

$$\min_{X\in\prod S^i} \Lambda(X) \tag{1}$$

and

$$\max_{X\in\prod S^i} \Lambda(X) \tag{2}$$

as the problem of finding the bounds for the principal eigenvalue of X with perturbed rows. Then we have the existence of optimal solutions of Problem (1) and Problem (2) as follows:

Theorem 2.2 *Each of Problem (1) and Problem (2) has a real optimal solution.*

From Frobenius' Theorem [1],

$$\begin{aligned}\Lambda(X) &= \max_{w>0} \min \left\{ \frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n} \right\} \\ &= \min_{w>0} \max \left\{ \frac{x_1 w}{w_1}, \ldots, \frac{x_n u}{w_n} \right\}\end{aligned}$$

for all $X \in \prod S^i$. Therefore, we have we can transform Problem (1) and Problem (2) into into the following two multifractional problems:

$$\begin{array}{ll} \min & \max\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{x_n}\right\} \\ \text{s.t.} & w > 0 \text{ and } X \in \prod S^i \end{array} \tag{3}$$

and

$$\begin{array}{ll} \max & \min\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n}\right\} \\ \text{s.t.} & w > 0 \text{ and } X \in \prod S^i, \end{array} \tag{4}$$

respectively. Both Problem (3) and Problem (4) have n homogeneous ratios of a single variable w_i to a bilinear term $x_i w$.

Let V^i be the vertex set of S^i for all $i \in I$, then the polytope S^i is the convex hull of V^i. Then we have the following theorem:

Theorem 2.3 *The optimal value of Problem (1) is equal to that of the following problem:*

$$\min_{w>0} \max \left\{ \min_{x_1 \in V^1} \frac{x_1 w}{w_1}, \ldots, \min_{x_n \in V^n} \frac{x_n w}{w_n} \right\}. \tag{5}$$

Theorem 2.4 *The optimal value of Problem (2) is equal to that of the following problem:*

$$\max_{w>0} \min \left\{ \max_{x_1 \in V^1} \frac{x_1 w}{w_1}, \ldots, \max_{x_n \in V^n} \frac{x_n w}{w_n} \right\}. \tag{6}$$

3 Coloring matrix and algorithm

Firstly, we will define "coloring matrix" for matrix of order n which is similar to a basis matrix of a linear programming as follows: A matrix A of order n is called a coloring matrix if the i^{th} row vector a_i of A is a vertex of S^i for all $i \in I$, i.e., $a_i \in V^i$ for all $i \in I$. By solving Problem (5) and Problem (6), we will find an optimal solution of Problem (1) and Problem (2), respectively. The following lemma states that a coloring matrix attaining the optimal value of Problem (5) is also an optimal solution of Problem (1).

Lemma 3.1 *There is a coloring matrix whose principal eigenvalue is the optimal solution of Problem (1). Let $(\bar{w}, \bar{X})$ be an optimal solution of Problem (5). Then $\bar{w}$ is a positive principal eigenvector of $\bar{X}$.*

If a coloring matrix has the principal eigenvalue that is equal to the optimal value of Problem (1), it is called the optimal coloring matrix for Problem (1). We have the following property for an optimal solution of Problem (5):

Theorem 3.2 *Let $\bar{X}$ be an optimal coloring matrix for Problem (1). Then, $(\bar{w}, \bar{X})$ is an optimal solution of Problem (5) if and only if $\bar{w}$ is a positive principal eigenvector of $\bar{X}$.*

From Lemma 3.1 and Theorem 3.2 we have only to find the least principal eigenvalue of a coloring matrix among those of all the coloring matrices. We develop the following algorithm:

Algorithm for Problem (1)

Step 0 Choose a positive vector w^0 and set $k = 1$.

Step 1 Find an optimal solution $x_i^k \in V^i$ and the optimal value γ_i^k of $\min_{x_i \in S^i}(x_i w^{k-1})/w_i^{k-1}$ for every $i \in I$. Let $X^k = \left[x_1^{kT}, \ldots, x_n^{kT}\right]^T$.

Step 2 Find the principal eigenvalue λ and a principal eigenvector $\hat{w}$ of X^k.

Step 3 If $\lambda \geq \max_{i \in I} \gamma_i^k$, then X^k and λ are an optimal coloring matrix and the optimal value of Problem (1), respectively and stop. Otherwise let $\lambda_k = \lambda$, $w^k = \hat{w}$ and $k = k+1$ and go to Step 1.

The algorithm has the following properties:

Lemma 3.3 $\lambda_{k+1} < \lambda_k$ *for* $k = 1, 2, \ldots$.

Lemma 3.4
$\min_{i \in I} \gamma_i^k \leq \lambda \leq \max_{i \in I} \gamma_i^k$ *for* $k = 1, 2, \ldots$.

Lemma 3.5 *Suppose that $\lambda \geq \max_{i \in I} \gamma_i^k$ in Step 3, then $\min_{i \in I} \gamma_i^k = \lambda = \max_{i \in I} \gamma_i^k$ and λ is the least principal eigenvalue among those of all the coloring matrices.*

Theorem 3.6 *The coloring algorithm for Problem (1) provides its optimal solution after a finite number of iterations.*

Replacing min / max with max / min in the definition of γ_i^k of Step 1 and the stopping criteria of Step 3, we obtain the same algorithm for Problem (2) as the above one and we can show the similar prpoerties to the above lemmas and theorem.

4 Duality

We define

$$\Phi(w) = \min_{X \in \prod S^i} \left\{ \frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n} \right\} \text{ and}$$

$$\Psi(w) = \max_{X \in \prod S^i} \left\{ \frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n} \right\}$$

and consider the following two multifractional problem:

$$\max_{w > 0} \quad \Phi(w) \text{ and} \qquad (7)$$

$$\min_{w > 0} \quad \Psi(w) \qquad (8)$$

The following theorem states the weak duality between Problem (1) and Problem (7).

Theorem 4.1 *For every positive vector w and for every matrix $X \in \prod S^i$, we have $\Phi(w) \leq \Lambda(X)$. Furthermore, there are a matrix $\bar{X} \in \prod S^i$ and a positive vector $\bar{w}$ such that $\Phi(\bar{w}) = \Lambda(\bar{X})$.*

We call Problem (7) a dual problem of Problem (1).

Corollary 4.2 *An optimal solution of Problem (7) is unique up to scalar multiplication.*

The following theorem provides the same results between Problem (2) and Problem (8) as that between Problem (1) and Problem (7).

Theorem 4.3 *For every positive vector w and for every matrix $X \in \prod S^i$, we have $\Psi(w) \geq \Lambda(X)$. Furthermore, there are a matrix $\bar{X} \in \prod S^i$ and a positive vector $\bar{w}$ such that $\Psi(\bar{w}) = \Lambda(\bar{X})$.*

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References

[1] Takayama. A. (1985): Mathematical Economics., Cambridge University Press,Cambridge.