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Lexicographically Optimum Traffic Trees with Maximum Degree Constraints

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1 Introduction

We shall consider a problem of finding an 'optimum' tree which is closely related to the network flow problem proposed by Ford and Fulkerson [5], and regarded as a min-max problem.

Let V be a set of n vertices and $\binom{V}{2}$ the set of all pairs of distinct vertices in V. Also, let $\mathcal{T}_V$ be the set of undirected spanning trees on V. A tree $T \in \mathcal{T}_V$ with an edge set E ($|E| = n-1$) is denoted by $T = (V, E)$, and such E is sometimes denoted by E^T to emphasize that it is the edge set of T. Also, the edge $e \in E$ connecting two vertices v and w is denoted by $e = (v, w)$. Assume that a nonnegative value r_{vw} is assigned to each pair $\{v, w\} \in \binom{V}{2}$. For an edge $(v, w) \in E$ of a tree $T = (V, E) \in \mathcal{T}_V$, we define a subtree of T denoted by $T(v) = (V(v), E(v))$ as the connected component of $(V, E \setminus \{(v, w)\})$ containing v, while $T(w) = (V(w), E(w))$ is defined as the other connected component. Also, we define the *traffic* of the edge (v, w) by

$$t((v,w),T) = \sum_{x \in V(v), y \in V(w)} r_{xy}.$$

(In terms of the network flow problem, $t((v, w), T)$ is the capacity of the cut dividing V into $V(v)$ and $V(w)$ in the complete graph K_n on V with edge capacities r_{xy} ($x, y \in V$).)

The problem of minimizing

$$f(T) = \sum_{e \in E^T} t(e,T) = \sum_{\{v,w\} \in \binom{V}{2}} d(v,w;T) r_{vw}$$

can be regarded as a min-sum problem, and was discussed by Adolphson and Hu [1], Hu [4], Anazawa [2] and so on. Especially, Hu [4] showed that the solution is obtained by the Gomory-Hu algorithm [3] when the degrees of vertices are *not* restricted. On the other hand, Anazawa [2] considered a problem of minimizing f with maximum degree constraints and showed that the problem is explicitly solvable under a reasonable condition.

In this paper, we propose another problem regarded as a min-max problem. Let

$$\bar{t}^T = [t_1^T, t_2^T, \ldots, t_{n-1}^T]$$

be the sequence of traffics in which the traffics of edges in T are arranged in descending order, that is, $t_1^T \geq t_2^T \geq \cdots \geq t_{n-1}^T$ holds. For mathematical convenience, if $n = 1$ then we set $\bar{t}^T = [\,]$ (an empty sequence). The purpose of this paper is to find a tree $T \in \mathcal{T}_V$ which minimizes $\bar{t}^T$ lexicographically. We call such a tree a *lexicographically optimum traffic tree* (LOTT). If T is a LOTT, then it is obvious that T minimizes

$$t(T) = \begin{cases} \max_{e \in E^T} t(e,T) \ (= t_1^T) & \text{if } n \geq 2 \\ 0 & \text{if } n = 1. \end{cases}$$

Hence, we can regard the LOTT problem as a generalized min-max problem.

In this paper, we shall devote ourselves to a special case when

$$\text{every vertex } v \in V \text{ satisfies } \deg(v) \leq L \tag{1}$$

for a given integer L (≥ 2) and

$$r_{vw} = 1 \text{ holds for all } \{v,w\} \in \binom{V}{2}. \tag{2}$$

Note that, in this case, the solution to the LOTT problem for $n \leq 3$ or $L = 2$ is trivial.

2 Definitions

First, we add some notation and definitions. For a vertex set V with $|V| = n$ and an integer L (≥ 2), let $\mathcal{T}_{V,L}$ be the set of undirected spanning trees on V satisfying condition (1). For a tree $T = (V, E) \in \mathcal{T}_{V,L}$, we call the edges attaining $t(T)$ the *maximum traffic edges* of T. In general, a tree may have two or more maximum traffic edges. Also, for a subtree $T(v) = (V(v), E(v))$ of T defined for an edge $(v, \cdot) \in E$, let $\bar{V}(v) = V \setminus V(v)$. When condition (2) is satisfied, we find that $t((v, \cdot), T) = |V(v)| \cdot |\bar{V}(v)| = |V(v)|(n - |V(v)|)$ holds for any edge $(v, \cdot) \in E$.

We can easily verify that any tree $T \in \mathcal{T}_{V,L}$ $(n \geq 2)$ has a vertex v with $\deg(v) = m$ satisfying the following condition: When subtrees $T(u_i) = (V(u_i), E(u_i))$ $(i = 1, \ldots, m)$ of T are defined for edges (v, u_i) $(i = 1, \ldots, m)$, $|V(u_i)| \leq |\bar{V}(u_i)|$ holds for all $i = 1, \ldots, m$. We call such v the *maximum traffic vertex* of T. Let V^T be the set of all the maximum traffic vertices of T. Then $|V^T| \leq 2$ holds for all $T \in \mathcal{T}_{V,L}$. If T has two or more maximum traffic edges with the common end vertex v, then $V^T = \{v\}$ holds. On the other hand, $|V^T| = 2$ (say $V^T = \{v_1, v_2\}$) holds if and only if T has an edge (v_1, v_2) and $|V(v_1)| = |\bar{V}(v_1)|$ is satisfied for the subtree $T(v_1) = (V(v_1), E(v_1))$ defined for (v_1, v_2). For mathematical convenience, if $T = (\{v\}, \emptyset)$ then we define the maximum traffic vertex of T by v.

Further, for a tree $T = (V, E) \in \mathcal{T}_{V,L}$ and a vertex $v \in V$ $(m = \deg(v))$, we define a property called *(n, L, v)-balancedness* recursively as follows:

(i) If $n = 1$, then T is $(1, L, v)$-balanced.

(ii) If $n > 1$, then let $T(u_i)$ $(i = 1, \ldots, m)$ be subtrees of T defined for (v, u_i) $(i = 1, \ldots, m)$. If

$$m = \begin{cases} n-1 & \text{if } n-1 \leq L-1 \\ L-1 & \text{if } n-1 > L-1 \end{cases}$$

and

$$T(u_i) \text{ is } \begin{cases} (n'+1, L, u_i)\text{-balanced} & \text{for } i = 1, \ldots, r \\ (n', L, u_i)\text{-balanced} & \text{for } i = r+1, \ldots, m \end{cases}$$

for nonnegative integer n' and r satisfying $n - 1 = n'm + r$ $(0 \leq r < m)$, then T is (n, L, v)-balanced.

3 Main Result

From these definitions, we can describe the following :

Main Theorem *For a given set V with $|V| = n$ and a given integer L (≥ 2), let T be a tree belonging to $\mathcal{T}_{V,L}$ and v a vertex with $v \in V^T$. Also, let $m^T = \deg(v)$, and let $T(u_i)$ $(i = 1, \ldots, m^T)$ be subtrees of T defined for edges (v, u_i) $(i = 1, \ldots, m^T)$. Then T minimizes t^T lexicographically in $\mathcal{T}_{V,L}$ if and only if*

$$m^T = \begin{cases} n-1 & \text{if } n-1 \leq L \\ L & \text{if } n-1 > L \end{cases}$$

holds and

$$T(u_i) \text{ is } \begin{cases} (n'+1, L, u_i)\text{-balanced} & \text{for } i = 1, \ldots, r \\ (n', L, u_i)\text{-balanced} & \text{for } i = r+1, \ldots, m^T \end{cases}$$

for nonnegative integers n' and r satisfying $n - 1 = n'm^T + r$ $(0 \leq r < m^T)$.

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