# OPTIMAL STOPPING PROBLEMS RELATED TO THE URN

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## 1. Introduction

Suppose that we have an urn containing $m$ minus balls and $p$ plus balls in it, where the value $-1$ is attached to minus ball and the value $+1$ to plus ball. We draw balls one at a time randomly, without replacement until we wish to stop. We know the values of $m$ and $p$ and are also allowed not to draw at all. Let $X_k$ be the value of the ball chosen at the $k$-th draw, $1 \le k \le m+p$, and define

$$Z_0 = 0, \quad Z_n = \sum_{k=1}^{n} X_k, \quad 1 \le n \le m+p.$$

$Z_n$ can be interpreted, for example, as a profit we can earn if we choose to stop after the $n$-th draw. Each time a ball is drawn, we observe the value of the ball and decide either to stop or continue drawing. In Section 2, we consider a problem where the trial is regarded as successful if we could attain the largest value of $\{Z_n\}_{n=0}^{m+p}$ upon stopping, the objective being to find a stopping policy that will maximize the probability of success. As a simple modification of this problem, we consider in Section 3 a problem where the trial is regarded as successful if we could attain either the largest or the second largest value of $\{Z_n\}_{n=0}^{m+p}$. These problems were first posed by Sakaguchi[1]. For later use, we introduce a random variable $T(m,p)$ that represents the first time $Z_n$ becomes non-negative, namely

$$T(m,p) = \min\{n : Z_n \ge 0, \quad 1 \le n \le m+p\}.$$

$T(m,p)$ takes the values of $1, 2, 4, \cdots, 2p$ for $m \ge p$ and if we denote by $p_j(m,p)$ the probability mass function of $T(m,p)$, i.e., $p_j(m,p) = P_r\{T(m,p) = j\}$, then these are given in the following lemma.

**Lemma 1** The probability mass function of $T(m,p)$ is given by

$$p_1(m,p) = \frac{p}{m+p} \qquad p_{2i}(m,p) = \frac{1}{2(2i-1)} \frac{\binom{m}{i}\binom{p}{i}}{\binom{m+p}{2i}}, \quad i = 1, 2, \cdots, p.$$

## 2. Stopping at the largest

Suppose that we have drawn $k$ balls and recognized $Z_1, \cdots, Z_k$ through the observed values of $X_1, \cdots, X_k$. Also suppose that we know there still remain $m$ minus balls and $p$ plus balls in the urn. If $Z_k < \max\{Z_0, Z_1, \cdots, Z_k\}$, we do not stop drawing because $Z_k$ cannot be the largest among all and so the immediate stop cannot lead to a success. If $m < p$, evidently we do not stop then and continue drawing until the remaining number of minus balls exceeds that of plus balls. Thus the serious decision of either stop or continue takes place only when $Z_k = \max\{Z_0, Z_1, \cdots, Z_k\}$ and $m \ge p$. Let this state be described as $(m,p)$ regardless of $k$ because, as a bit of consideration shows, the decision depends only on the remaining numbers of minus balls and plus balls in the urn but not on the number of balls already drawn.

Let $v(m,p)$ be the probability of success starting from state $(m,p)$, and let $s(m,p)$ and $c(m,p)$ be respectively the probability of success when we stop drawing and continue drawing in an optimal manner in state $(m,p)$; then, from the principle of optimality,

$$v(m,p) = \max\{s(m,p), c(m,p)\}, \qquad m \ge p \ge 0,$$

where

$$s(m,p) = \frac{m+1-p}{m+1}, \qquad m \geq p \geq 0,$$

and

$$c(m,p) = p_1(m,p)v(m,p-1) + \sum_{i=1}^{p} p_{2i}(m,p)v(m-i,p-i),$$

for $m, p \geq 1$ with the boundary condition $c(m,0) \equiv 0$ for $m \geq 0$.

Let $B$ be the stopping region derived by the one-stage look-ahead (OLA) stopping policy, that is, the set of states for which stopping immediately is at least as good as continuing one more transition and then stopping ; then,

$$B = \{(m,p) : m \geq m^*(p)\}, \qquad \text{where} \quad m^*(p) = p + \frac{\sqrt{8p+1}-1}{2}.$$

The following theorem states that the OLA stopping region $B$ in fact gives the optimal stopping region.

**Theorem 1** The optimal policy stops drawing as soon as the state enters the set $B$.

## 3. Stopping at the largest or the second largest

Suppose that, as in section 3, we have just observed $Z_1, \cdots, Z_k$ and know that the urn contains $m$ minus balls and $p$ plus balls in it. Since the objective is now to maximize the probability of attaining either the largest or the second largest value of $\{Z_n\}_{n=0}^{m+p}$, we easily see that serious decision of stop or continue takes place only when $Z_k \geq \max\{Z_0, Z_1, \cdots, Z_k\} - 1$ and $m+1 \geq p$. We denote this state by $(m,p\,;i)$ or $(m,p\,;2)$ regardless of $k$, depending on whether $Z_k = \max\{Z_0, Z_1, \cdots, Z_k\}$ or $Z_k = \max\{Z_0, Z_1, \cdots, Z_k\} - 1$.

Let $v_i(m,p), i = 1,2$, be the probability of success starting from state $(m,p\,;1)$, and let $s_i(m,p)$ and $c_i(m,p)$ be respectively the probability of success when we stop drawing and continue drawing in an optimal manner in state $(m,p\,;i)$; then, from the principle of optimality,

$$v_i(m,p) = \max\{s_i(m,p), c_i(m,p)\}, \qquad m+1 \geq p, \quad i = 1,2,$$

where

$$s_1(m,p) \;=\; s_2(m,p) = \frac{(m+1+p)(m+2-p)}{(m+1)(m+2)}, \qquad m+1 \geq p,$$

$$c_1(m,p) \;=\; \frac{p}{m+p}v_1(m,p-1) + \frac{m}{m+p}v_2(m-1,p),$$

and

$$c_2(m,p) = p_1(m,p)v_1(m,p-1) + \sum_{i=1}^{p} p_{2i}(m,p)v_2(m-i,p-i),$$

for $m+1 \geq p \geq 1, m \geq 1$ with the boundary conditions $c_1(0,0) = 0, c_1(0,1) = c_1(m,0) = 1$ for $m \geq 1$ and $c_2(0,1) = 1, c_2(m,0) = 0$ for $m \geq 0$.

Let $B$ be the stopping region derived by the OLA stopping region. Then,

$$B = \{(m,p\,;2) : m \geq m^*(p) - 1\}.$$

**Theorem 2** The optimal policy stops drawing as soon as the state $(m,p\,;2)$ enters the set $B$.

## 参考文献

[1] M.Sakaguchi : Secretary problems and their related Areas, *J.of Economics and Manegement,* vol.42, NUCBA (1998),85-137. (in Japanese)