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On Shapley Values and Cores of Cooperative Fuzzy Games

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1. Introduction

The Shapley value and the core are well-known solution concepts in cooperative game theory. They have been investigated by a number of researchers. Most of them treat games with crisp coalitions. However, there are some situations where some agents do not fully participate in a coalition, but to a certain extent. A coalition including some players who participate partially can be treated as a so-called fuzzy coalition. The fuzzy coalition, first introduced by Aubin [1,2], is a collection of players who transfer fractions of their representability to a specific coalition.

Butnariu [3] has introduced a class of fuzzy games and investigated the Shapley function on it. He has defined the Shapley function as a function which gives us the Shapley value of a given fuzzy game for a given fuzzy coalition. However, the class is unnatural since most games in the class are neither monotone nondecreasing nor continuous with respect to players' participation degree; and hence the Shapley values of them are also unnatural. He has also introduced the core of fuzzy games.

In this study, we define the Shapley function and the core function, which is applicable to any class of fuzzy games. The definitions are different from the Butnariu's. We introduce a particular class of fuzzy games $G_C(N)$ and show a Shapley function on it in an explicit form. Furthermore, we show a relationship between a Shapley function and the core function on a subclass composed of convex games in $G_C(N)$.

2. Notations and Definitions

In this paper, we consider cooperative fuzzy games with the set of players $N = \{1, \ldots, n\}$. A fuzzy coalition is a fuzzy subset of N identified with a function from N to $[0,1]$. Then for a fuzzy coalition S and a player i, $S(i)$ indicates the i-th player's participation degree to S. For a fuzzy coalition S, the level set is denoted by $[S]_h = \{i \in N \mid S(i) \geq h\}$ for any $h \in [0,1]$, and the support is denoted by Supp $S = \{i \in N \mid S(i) > 0\}$. The set of all fuzzy coalitions is denoted by $L(N)$. Particularly, $P(N)$ denotes the set of crisp subsets of N. For the sake of simplicity, let $L(U;N) = \{S \in L(N) \mid S \subseteq U \text{ for } U \in L(N)\}$ where $S \subseteq U$ iff $S(i) \leq U(i)$ for any $i \in N$.

A fuzzy game is a function v from $L(N)$ to $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ such that $v(\emptyset) = 0$. $G(N)$ denotes the set of all fuzzy games. $v(S)$ is often regarded as the least profit when the crisp coalition S is formed. It follows that any crisp game v is superadditive; and hence monotone nondecreasing with regard to set inclusion. This paper follows this interpretation.

In this paper, union and intersection of two fuzzy sets are defined as usual, i.e.,

$$(S \cup T)(i) = \max\{S(i), T(i)\}, \quad \forall i \in N,$$
$$(S \cap T)(i) = \min\{S(i), T(i)\}, \quad \forall i \in N.$$

Then we introduce convexity into fuzzy games as follows:

Definition 1 *v is said to be convex if*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad \forall S, T \in L(N).$$

3. A Shapley Function and A New Class of Fuzzy Games

We define C-carrier for U as an extension of carrier for $V \in P(N)$ in order to define the Shapley function.

Definition 2 *Let $v \in G(N)$ and $U \in L(N)$. $S \in L(U;N)$ is called a C-carrier for U if S satisfies:*

$$v(S \cap T) = v(T), \quad \forall T \in L(U;N).$$

Moreover, we make the following definitions to define the Shapley function.

Definition 3 *For $S \in L(U;N)$ and $i, j \in N$, let S_{ij}^U be defined by*

$$S_{ij}^U(k) = \begin{cases} \min\{S(i), U(j)\}, & \text{if } k = i, \\ \min\{S(j), U(i)\}, & \text{if } k = j, \\ S(k), & \text{otherwise.} \end{cases}$$

Definition 4 *For $S \in L(N)$, let $\mathcal{P}_{ij}[S]$ be a fuzzy coalition defined by*

$$\mathcal{P}_{ij}[S](k) = \begin{cases} S(j), & \textit{if} \quad k = i, \\ S(i), & \textit{if} \quad k = j, \\ S(k), & \textit{otherwise.} \end{cases}$$

We define the Shapley function as follows.

Definition 5 *Let $G'(N)$ be a class of fuzzy games, i.e. $G'(N) \subseteq G(N)$. A function f from $G'(N)$ to $(\mathbb{R}^n_+)^{L(N)}$ is said to be a Shapley function on $G'(N)$ if f satisfies the following four axioms.*

Axiom C_1: *If $v \in G'(N)$ and $U \in L(N)$, then*

$$\begin{cases} \sum_{i \in N} f_i(v)(U) = v(U), \\ f_i(v)(U) = 0, \quad \forall\, i \notin \text{Supp } U, \end{cases}$$

where $f_i(v)(U)$ is the i-th element of $f(v)(U)$.

Axiom C_2: *If $v \in G'(N)$, $U \in L(N)$ and T is a C-carrier for U, then*

$$f_i(v)(U) = f_i(v)(T), \quad \forall\, i \in N.$$

Axiom C_3: *Let $v \in G'(N)$ and $U \in L(N)$. If $v(S) = v(S^U_{ij}) = v(\mathcal{P}_{ij}[S^U_{ij}])$ for any $S \in L(U; N)$, then*

$$f_i(v)(U) = f_j(v)(U).$$

Axiom C_4: *Let $U \in L(N)$. For $v_1, v_2 \in G'(N)$, let $v_1 + v_2 \in G'(N)$ be defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for any $S \in L(N)$. Then*

$$f_i(v_1 + v_2)(U) = f_i(v_1)(U) + f_i(v_2)(U), \quad \forall\, i \in N.$$

Now, we shall define a particular class of fuzzy games. In the class, we can obtain a Shapley function in an explicit form.

Definition 6 *For $S \in L(N)$, let $Q(S) = \{S(i) \mid S(i) > 0, i \in N\}$ and let $q(S)$ be the cardinality of $Q(S)$. We rewrite the elements of $Q(S)$ in the increasing order as $h_1 < \cdots < h_{q(S)}$. Then $v \in G(N)$ is said to be a fuzzy game 'with Choquet integral form' iff the following holds:*

$$v(S) = \sum_{l=1}^{q(S)} v([S]_{h_l}) \cdot (h_l - h_{l-1}), \ \forall S \in L(N),$$

where $h_0 = 0$. We denote by $G_C(N)$ the set of all fuzzy games with Choquet integral forms.

The class defined above is more natural than Butnariu's, as shown in the next theorem.

Theorem 1 *Any $v \in G_C(N)$ is monotone nondecreasing and continuous with regard to players' participation degree.*

We obtain a Shapley function on $G_C(N)$.

Theorem 2 *Let f be a function from $G_C(N)$ to $(\mathbb{R}^n_+)^{L(N)}$ defined by*

$$f_i(v)(U) = \sum_{l=1}^{q(U)} f'_i(v)([U]_{h_l}) \cdot (h_l - h_{l-1}),$$

where $f'(v)(V)$ is the Shapley value of the crisp game v for $V \in P(N)$. Then f is a Shapley function on $G_C(N)$.

Theorem 3 *$f_i(v)$ is monotone nondecreasing and continuous with regard to players' participation degree.*

4. The Core Function

We define the core function which maps a pair of fuzzy game and a fuzzy coalition to the core.

Definition 7 *Let $G'(N) \subseteq G(N)$. A core function C from $G'(N)$ to $(P(\mathbb{R}^n_+))^{L(N)}$ is defined by*

$$C(v)(U) = \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i \in \text{Supp } S} x_i \geq v(S), \forall S \in L(U; N) \right\}$$

$$\forall\, v \in G'(N), \quad \forall\, U \in L(N).$$

If v is nondecreasing with regard to set inclusion, we have

$$\begin{aligned} & \textstyle\sum_{i \in \text{Supp } S} x_i \geq v(S), \quad \forall\, S \in L(U; N), \\ \Longleftrightarrow \quad & \textstyle\sum_{i \in \text{Supp } S} x_i \geq v(S), \\ & \quad \forall\, S \in L(U; N), \\ & \quad \text{s.t. } S(i) \in \{0, U(i)\}, \forall i \in N. \end{aligned}$$

Let $G_{CC}(N) = \{v \in G_C(N) \mid v : \text{convex}\}$. We have a relationship between a Shapley function and the core function on $G_{CC}(N)$.

Theorem 4 *For any $v \in G_{CC}(N)$ and for any $U \in L(N)$, $f(v)(U)$ defined in Theorem 2 coincides with the center of gravity of the extreme points of $C(v)(U)$.*

References

[1] J.-P. Aubin, Coeur et valeur des jeux flous á paiements latéraux, C. R. Acad. Sci. Paris, 279-A (1974) 891-894.

[2] J.-P. Aubin, Coeur et équilibres des jeux flous sans paiements latéraux, C. R. Acad. Sci. Paris. 279-A (1974) 963-966.

[3] D. Butnariu, Stability and Shapley value for an n-persons Fuzzy Game, Fuzzy Sets and Systems 4 (1980) 63-72.