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A Simple Example of Weighted Majority Games with Undesirable Deegan-Packel Indices

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1. Introduction

For the weighted majority games, several value concepts for measuring voting power have been proposed by Shapley and Shubik (1954), Banzhaf (1965) and Deegan and Packel (1978). Among these three indices, we shall briefly discuss the validity of the Deegan-Packel index.

Let n be a positive integer which denotes the number of players $P := \{1, \cdots, n\}$. The *weighted majority game* is a sequence of nonnegative integers $G = [q : w_1, w_2, \cdots, w_n]$ satisfying the condition that $w_i \geq 0$ and $\frac{1}{2}\sum_{p \in P} w_p < q \leq \sum_{p \in P} w_p$ where the integer q denotes the quota for winning the game and each w_p $(p \in P)$ denotes the voting weight of player p. A *winning coalition* is a subset S of players P such that $\sum_{p \in S} w_p \geq q$. A winning coalition S is called *minimal* if S is minimal in the sense of set inclusion. We denote the family of all minimal winning coalitions by Ω_{min}. The *Deegan-Packel index* γ_p for player p is defined by

$$\gamma_p = \begin{cases} \dfrac{1}{|\Omega_{min}|} \displaystyle\sum_{p \in S \in \Omega_{min}} \dfrac{1}{|S|} & \text{if } \{S : p \in S \in \Omega_{min}\} \neq \emptyset, \\ 0 & \text{if } \{S : p \in S \in \Omega_{min}\} = \emptyset. \end{cases} \tag{1}$$

The factor $\frac{1}{|\Omega_{min}|}$ is to normalize the voting power, and hence $\sum_{p \in P} \gamma_p = 1$. Note that the Deegan-Packel indices may not preserve the order of the weights, *i.e.*, $w_1 \geq w_2 \geq \cdots \geq w_n$ does not implies $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. To clarify the reason for the undesirability, we give a simple example of the weighted majority games.

2. Simple Example

Let $m \geq 2$. Consider the following weighted majority game with $2m + 1$ players:

$$[q : w_1, w_2, w_3, \cdots w_{2m+1}] = [m^2 + m : m^2, m^2 - m, \overbrace{1, \cdots, 1}^{2m-1}].$$

It is easily verified that each minimal winning coalition in this weighted majority game is of one of the following two types: $\{1, 2\}$ or $\{1, i : i \in I \subset \{3, \cdots, 2m+1\}, |I| = m\}$, *i.e.*, $\Omega_{min} = \{1, 2\} \cup \{1, i : i \in I \subset \{3, \cdots, 2m+1\}, |I| = m\}$ and $|\Omega_{min}| = 1 + {}_{2m-1}C_m$. Therefore, the Deegan-Packel indices of this weighted majority game are as follows:

$$\gamma_1 = \frac{1}{1 + {}_{2m-1}C_m}\left(\frac{1}{2} + \frac{1}{m+1}{}_{2m-1}C_m\right), \qquad \gamma_2 = \frac{1}{1 + {}_{2m-1}C_m} \times \frac{1}{2},$$
$$\gamma_i = \frac{1}{1 + {}_{2m-1}C_m} \times \frac{1}{m+1} \times {}_{2m-2}C_{m-1} \quad (\text{for } i = 3, \cdots, 2m+1).$$

Therefore, $\gamma_1 > \gamma_3 = \cdots = \gamma_{2m+1} > \gamma_2$ though $w_1 > w_2 > w_3 = \cdots = w_{2m+1}$. Moreover, we have the following asymptotic behavior of the Deegan-Packel indices: (for $i = 3, \cdots, 2m+1$)

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$$
\begin{aligned}
\frac{\gamma_2}{\gamma_1} &= \frac{1}{O(2^{2m})} &\to 0 \\
\frac{\gamma_i}{\gamma_1} &= \frac{1}{2} - \frac{1}{O(m)} &\to \frac{1}{2} \\
\frac{\gamma_2}{\gamma_i} &= \frac{1}{O(2^{2m})} &\to 0 \\
\gamma_1 \to 0,\ \gamma_2 \to 0,\ &\sum_{i=3}^{2m+1} \gamma_i \to 1
\end{aligned}
\quad \text{though} \quad
\begin{aligned}
\frac{w_2}{w_1} &= 1 - \frac{1}{O(m)} &\to 1 \\
\frac{w_i}{w_1} &= \frac{1}{O(m^2)} &\to 0 \\
\frac{w_2}{w_i} &= O(m^2) &\to \infty \\
w_1 > w_2 > &\sum_{i=3}^{2m+1} w_i
\end{aligned}
$$

Repeatedly, we note that

- the Deegan-Packel index γ_i of each player i ($i = 3, \cdots, 2m+1$) converges to 0 in the same order as that γ_1 of player 1 as $m \to \infty$ (the ratio γ_i/γ_1 of the indices converges to $1/2$, though the ratio w_i/w_1 of the weights converges to 0). Note that every (minimal) winning coalition contains player 1.
- the Deegan-Packel index γ_2 of player 2 converges to 0 as $m \to \infty$ much (exponentially) faster than that γ_i of each player i for $i = 3, \cdots, 2m+1$, though the ratio w_2/w_i of weights diverges to ∞.
- the sum $\sum_{i=3}^{2m+1} \gamma_i$ of Deegan-Packel indices of players 3 to $2m+1$ converges to 1 though the sum of their weights is less than that of player 2.

Note that only *minimal* winning coalitions are considered in the definition (1) of the Deegan-Packel index. Since player 2 fails to be a member of many *minimal* winning coalitions by its heavy weight, the sum in (1) for the player 2 have only one term of $1/2$, whereas the sum for each player i ($i \in \{3, \cdots, 2m+1\}$) consists of exponentially many ($O(2^m)$) terms of $1/O(m)$. The kind of combinatorial explosion leads to the undesirability. Note that $n = 5$ ($m = 2$ in our example) is enough to explode. Recently, Ogawa (1998) theoretically showed that the smallest number of players in the weighted majority game with undesirable Deegan-Packel indices is equal to 5. Smallest example with undesirable Deegan-Packel indices is $G = [5 : 3, 2, 1, 1, 1]$. Simplest example in the sense of the number (= 3) of minimal winning coalitions is $G = [8 : 5, 3, 2, 1, 1]$.

Remark. The Shapley-Shubik index γ_{SS} and the Banzhaf index γ_B of the game are:

$$
\begin{aligned}
\gamma_{SS} &= \left[\frac{5m+3}{4(2m+1)}, \frac{m+1}{4(2m+1)}, \frac{m}{2(2m-1)(2m+1)}, \cdots\right] &\to \left[\frac{5}{8}, \frac{1}{8}, \frac{1}{O(m)}, \cdots\right], \\
\gamma_B &= \left[\frac{3}{4}, \frac{1}{4}, \frac{{}_{2m-2}C_{m-1}}{2^{2m}}, \cdots\right] &\to \left[\frac{3}{4}, \frac{1}{4}, \frac{1}{O(\sqrt{m})}, \cdots\right].
\end{aligned}
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