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A NOTE ON THE DISTRIBUTION OF THE TIME OF THE FIRST k-RECORD

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We explicitly give the probability mass function and the probability generating function of the first k-record index for a sequence of independent and identically distributed random variables that take on a finite set of possible values. We also compute its factorial moments.

1. INTRODUCTION AND SUMMARY

Let $X_1, X_2, \ldots$ be independent and identically distributed finite-valued random variables with probability mass function

$$p_i = P\{X = i\},\ i = 1, \ldots, m.$$

For a fixed positive integer k, k>1, the random variable

$$T = \min\{n: X_n \le X_i \text{ for exactly k of the values } i, i=1,\ldots,n\}$$

is called the first *k-record index.*

Adler and Ross[1] determined the first two moments of T via a recursive formula for its probability generating function obtained by conditioning on a random variable T_j, which is defined as the first k-record index when the observed random variables have probability mass function

$$P\{X = i\} = \frac{p_i}{p_j + \ldots + p_m}, \quad i=j, \ldots, m.$$

In this note, we derive the probability mass function and the probability generating function of T explicitly and then determine its factorial moments.

THEOREM

Let $a_i = p_i + \ldots + p_m$, $i = 1,\ldots,m$. Then the probability mass function of T is given by

$$P\{T = n\} = \sum_{i=1}^{m} p_i \binom{n-2}{k-2} a_i^{k-1}(1 - a_i)^{n-k}, \quad n \ge k. \tag{1}$$

Moreover the probability generating function $P(s) = E[s^T]$ is given by

$$P(s) = \sum_{i=1}^{m} p_i s\left(\frac{a_i s}{1-(1-a_i)s}\right)^{k-1}. \tag{2}$$

PROOF Let $P_j(s) = E[s^{T_j}]$, j=1,...,m. Then Adler and Ross[1, Theorem 1] give a recursive formula

$$P_j(s) = \frac{p_j}{a_j} s^k + \left(1 - \frac{p_j}{a_j} s\right) P_{j+1}\left(\frac{a_{j+1} s}{a_j - p_j s}\right), \quad j=1,\ldots,m-1$$

which, combined with $P_m(s) = s^k$, can be easily solved to yield

$$P_j(s) = \sum_{i=j}^{m} \frac{p_i s}{a_j}\left(\frac{a_i s}{a_j-(a_j-a_i)s}\right)^{k-1}.$$

Thus, since $a_1=1$, we have

$$P(s) = P_1(s) = \sum_{i=1}^{m} p_i s\left(\frac{a_i s}{1-(1-a_i)s}\right)^{k-1},$$

which yields (2). (1) is immediate from (2), because $\left(\frac{a_i s}{1-(1-a_i)s}\right)^{k-1}$ denotes the probability generating function of the negative binomial random variable with parameters k-1 and a_i, and hence can be expanded into

$$\left(\frac{a_i s}{1-(1-a_i)s}\right)^{k-1} = \sum_{n=k}^{\infty} \binom{n-2}{k-2} a_i^{k-1}(1-a_i)^{n-k}s^{n-1}.$$

Thus the proof is complete.

The following corollary gives the factorial moments of T.

COROLLARY
For j = 0, 1, 2,...,

$$E[(T+j-1)(T+j-2)\ldots(T-1)] = \sum_{i=1}^{m} p_i \left(\frac{k+j-1}{a_i}\right)\left(\frac{k+j-2}{a_i}\right)\cdots\left(\frac{k-1}{a_i}\right).$$

PROOF For j = 0, 1, 2, ..., we have from (1) that

$$\begin{aligned} &E[(T+j-1)(T+j-2)\ldots(T-1)] \\ &= \sum_{n=0}^{\infty} (n+k+j-1)(n+k+j-2)\ldots(n+k-1)P\{T = n+k\} \\ &= (k+j-1)(k+j-2)\ldots(k-1)\sum_{i=1}^{m} p_i a_i^{k-1}\left\{\sum_{n=0}^{\infty}\binom{n+k+j-1}{n}(1-a_i)^n\right\} \\ &= (k+j-1)(k+j-2)\ldots(k-1)\sum_{i=1}^{m} p_i a_i^{k-1} a_i^{-(k+j)}, \end{aligned}$$

which is the desired result. The last equality follows from the well known identity(see, e.g., Ross[2], p. 153)

$$\sum_{n=0}^{\infty}\binom{n+k+j-1}{n}(1-a)^n = a^{-(k+j)}, \qquad 0<a<1,\ k+j\geq 1.$$

References

1. Adler, I., & Ross, S.M. (1997). Distribution of the time of the first k-record. *Probability in the Engineering and Informational Sciences* 11: 273-278.
2. Ross, S.M. (1997). *Introduction to probability models*, 6th ed. New York: Academic Press.
3. Tamaki, M. (1998). A note on the distribution of the time of the first k-record. *Probability in the Engineering and Informational Sciences* 12(to appear).