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Estimation of Long Run Effect by Dynamic Regression Model with I(1) Variables and Its Applications in OR

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I. Introduction

Recent ideas of unit root, cointegration and error correction representation developed in time series analysis area in the decade have important impacts on the nonstationary time series analysis in OR.

In the present paper, we treat (1) dynamic equation model and (2) rational transfer function model as a dynamic regression model with I(1) variables and derive asymptotic properties of OLS of long run effect coefficients in these models. Some suggestions are also given for the applications in OR.

II. Dynamic Equation Model and Its Estimation

2.1 Dynamic Equation Model and Error Correction Representation

Dynamic Equation Model is defined as follows;

(1) $a(L)y_t = b(L)x_t + \dfrac{\theta(L)}{\phi(L)}u_t, \qquad t = 1,2,\dots,T,$

where;

L = lag operator,

$$a(L) = 1 - \sum_{i=1}^{r} a_i L^i, \qquad b(L) = b_0 - \sum_{i=1}^{r} b_i L^i,$$

$$\phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i, \qquad \theta(L) = 1 - \sum_{i=1}^{q} \theta_i L^i.$$

We set following Assumptions 1 ～ 4.

Assumption 1. All roots of $a(z) = 0$, $\phi(Z) = 0$, $\theta(Z) = 0$ and $\psi(Z) = 0$ are outside the unit circle.

Assumption 2. $(1-L)x_t = v_t = \psi(L)\varepsilon_t,$

$$\psi(L) = \sum_{i=0}^{\infty} \psi_i L^i,\ (\psi_0 = 1), \quad \sum_{i=0}^{\infty} i|\psi_i| < \infty.$$

Assumption 3. $\begin{pmatrix} u_t \\ \varepsilon_t \end{pmatrix} \sim IID\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{pmatrix}\right).$

Assumption 4. p, q and r are known a priori.

Using the relation (Phillips and Solo [1992]),

$$a(L) = a(1) + (1-L)a^*(L),$$

$$a^*(L) = \sum_{i=0}^{r-1} a_i^* L^i, \quad a_i^* = \sum_{j=1}^{r-i} a_{i+j},$$

$$b(L) = b(1) + (1-L)b^*(L),$$

$$b^*(L) = \sum_{i=0}^{r-1} b_i^* L^i, \quad b_i^* = \sum_{j=1}^{r-i} b_{i+j},$$

we can show the error correction representation of (1) as in (2) or (3).

(2) $a(L)(y_t - \gamma x_t) = \sum_{i=1}^{r} (b_i - \gamma a_i)(x_t - x_{t-i}) + \dfrac{\theta(L)}{\phi(L)}u_t,$

(3) $y_t = \gamma x_t - \dfrac{1}{a(1)}\sum_{i=1}^{r} a_i(y_t - y_{t-i}) + \dfrac{1}{a(1)}\sum_{i=1}^{r} b_i(x_t - x_{t-i}) + \dfrac{1}{a(1)}\dfrac{\theta(L)}{\phi(L)}u_t,$

where $\gamma = \dfrac{b(1)}{a(1)}$ is the coefficient of long run effect.

2.2 Least Squares Estimation and Its Asymptotic Properties

Parameters to be estimated are denoted as

$$a = (a_1, a_2, \dots, a_r)', \quad b = (b_1, b_2, \dots, b_r)', \quad c = (a', b')',$$

$$\phi = (\phi_1, \phi_2, \dots, \phi_p)', \quad \theta = (\theta_1, \theta_2, \dots, \theta_q)', \quad \pi = (\phi', \theta')',$$

$$\mu = (\gamma, c', \pi')' = (\gamma, \lambda')'.$$

Let $X_T(r)$ be

$$X_T(r) = \frac{1}{\sqrt{T}} x_{[Tr]} + (Tr - [Tr]) \frac{1}{\sqrt{T}} v_{[Tr]+1},$$

$$\left(\frac{t-1}{T} \le r \le \frac{t}{T}, t = 1,2,\dots,T\right).$$

Following lemmas are useful in the subsequent sequel.

Lemma 1 (FCLT for I(1) process.)

$X_T(r) \Rightarrow \psi(1)\sigma_\varepsilon W_\varepsilon(r),$

where $\Rightarrow$ signifies weak convergence of the associated probability measure as $T \to \infty$ and $W_\varepsilon(r)$ is standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$.

Lemma 2

(i) $\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} x_t \Rightarrow \psi(1)\sigma_\varepsilon \int_0^1 W_\varepsilon(r)dr.$

(ii) $\frac{1}{T^2}\sum_{t=1}^{T} x_t^2 \Rightarrow (\psi(1)\sigma_\varepsilon)^2 \int_0^1 W_\varepsilon^2(r)dr.$

(iii) $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\Delta x_t \Rightarrow \frac{1}{2}(\psi(1)\sigma_\varepsilon)^2 W_\varepsilon^2(1) - \frac{1}{2}V(\Delta x_t).$

(iv) $\frac{1}{T}\sum_{t=1}^{T} x_t\Delta x_t \Rightarrow \frac{1}{2}(\psi(1)\sigma_\varepsilon)^2 W_\varepsilon^2(1) + \frac{1}{2}V(\Delta x_t).$

(v) $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}u_t \Rightarrow \psi(1)\sigma_\varepsilon\sigma_u \int_0^1 W_\varepsilon(r)dW_u(r).$

Here, $\Delta \equiv 1-L$ and $W_\varepsilon(r)$ and $W_u(r)$ are independent standard Brownian motions.

Lemma 2 can be proved using Lemma 1 (FCLT) and CMT (Continuous Mapping Theorem, Billingsley [1968].)

Denote the OLS of μ as $\hat{\mu}$, which minimizes $S = \sum_{t=1}^{T} u_t^2(\mu)$,

where

$$u_t(\mu) \equiv \frac{\phi(L)}{\theta(L)}\left(a(L)(y_t - \gamma x_t) - \sum_{i=1}^{r}(b_i - \gamma a_i)(x_t - x_{t-i})\right).$$

Using Lemma 1 and 2, we can show

Theorem 1.

$$T(\hat{\gamma} - \gamma) \Rightarrow \frac{\theta(1)\sigma_u}{a(1)\phi(1)\psi(1)\sigma_\varepsilon}\frac{\int_0^1 W_\varepsilon(r)dW_u(r)}{\int_0^1 W_\varepsilon^2(r)dr},$$

$$\sqrt{T}(\hat{\lambda} - \lambda) \Rightarrow N(0, \sigma_u^2\Omega^{-1}),$$

$$\Omega = \operatorname*{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{\partial u_t}{\partial\lambda}\frac{\partial u_t}{\partial\lambda'}.$$

Remark 1.

(1) Limiting distribution of $T(\hat{\gamma} - \gamma)$ is mixed Gaussian.

(2) $\hat{\gamma}$ and $\hat{\lambda}$ are asymptotically uncorrelated.

III. Rational Transfer Function Model and Its Estimation

3.1 Rational Transfer Function Model and Error Correction Representation

Rational Transfer Function Model is defined as in (4), and we can show its error correction representation as in (5) or (6).

(4) $y_t = \frac{b(L)}{a(L)}x_t + \frac{\theta(L)}{\phi(L)}u_t.$

(5) $a(L)(y_t - \gamma x_t) = \sum_{i=1}^{r}(b_i - \gamma a_i)(x_t - x_{t-i}) + \frac{a(L)\theta(L)}{\phi(L)}u_t.$

(6) $y_t = \gamma x_t - \frac{1}{a(1)}\sum_{i=1}^{r}a_i(y_t - y_{t-i}) + \frac{1}{a(1)}\sum_{i=1}^{r}b_i(x_t - x_{t-i}) + \frac{1}{a(1)}\frac{a(L)\theta(L)}{\phi(L)}u_t.$

3.2 Least Squares Estimation and Its Asymptotic Properties

Let $u_t(\mu)$ be

$$u_t(\mu) \equiv \frac{\phi(L)}{\theta(L)}\left((y_t - \gamma x_t) - \frac{1}{a(L)}\sum_{i=1}^{r}(b_i - \gamma a_i)(x_t - x_{t-i})\right).$$

We can show the following Theorem for OLS $\hat{\mu}$.

Theorem 2.

$$T(\hat{\gamma} - \gamma) \Rightarrow \frac{\theta(1)\sigma_u}{\phi(1)\psi(1)\sigma_\varepsilon}\frac{\int_0^1 W_\varepsilon(r)dW_u(r)}{\int_0^1 W_\varepsilon^2(r)dr},$$

$$\sqrt{T}(\hat{c} - c) \Rightarrow N(0, \sigma_u^2\Omega_c^{-1}),$$

$$\Omega_c = \operatorname*{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{\partial u_t}{\partial c}\frac{\partial u_t}{\partial c'},$$

$$\sqrt{T}(\hat{\pi} - \pi) \Rightarrow N(0, \sigma_u^2\Omega_\pi^{-1}),$$

$$\Omega_\pi = \operatorname*{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{\partial u_t}{\partial\pi}\frac{\partial u_t}{\partial\pi'}.$$

Remark 2.

(1) Limiting distribution of $T(\hat{\gamma} - \gamma)$ is mixed Gaussian.

(2) $\hat{\gamma}, \hat{c}$ and $\hat{\pi}$ are asymptotically uncorrelated.

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