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Analysis of a Multiclass Polling System with Feedback

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1 Introduction

A polling system with feedback has been analyzed by Sidi et al. [7]. We extend their system so as to include multiple customer classes and (local) priorities. We can easily calculate the mean path time which is the mean amount of time spent by an arbitrary customer traversing a specific path. We extend the model in [9]. Moreover our method takes a different approach from the existing methods. This method has a lot in common with the method for a priority queueing system [10], and can also be applied to a Markovian polling system [8]. The advantage of our method will be its wide applicability to the analysis of mean sojourn times in many types of M/G/1 multiclass queueing systems.

Numerous studies and techniques have been developed for computing the mean waiting times in polling systems with N stations (*the buffer occupancy method* [1, 4], *the station time method* [2] and its variation [6]). The mean sojourn times in polling systems with feedback can also be obtained by the buffer occupancy method [7]. It also gives a set of N^3 linear equations, usually requiring $O(N^9)$ operations to solve it. Although the iterative algorithms requiring $O(N^3 \log_\alpha \epsilon)$ operations have been proposed ([3, 5]), its efficiency will degrade as the resource utilization ρ approaches to 1. Our method requires $O(N^6)$ operations regardless of ρ.

2 Model description

J *groups* of customers arrive at the system. Further group i consists of L_i *classes* of customers. Let $\mathcal{S} \equiv \{(i,\alpha) : i = 1,\ldots,J \text{ and } \alpha = 1,\ldots,L_i\}$. (i,α)-customers arrive from outside the system according to a Poisson process with rate $\lambda_{i\alpha}$. A single server serves customers at these stations. Service times $S_{i\alpha}$ of (i,α)-customers are independently, identically and arbitrarily distributed with mean $E[S_{i\alpha}]$ and second moment $\overline{s^2_{i\alpha}}$. After receiving a service, an (i,α)-customer either returns to the system as a (j,β)-customer with probability $p_{i\alpha,j\beta}$, or departs from the system with probability $1-\sum_{j=1}^{J}\sum_{\beta=1}^{L_j} p_{i\alpha,j\beta}$.

The server selects the stations in a cyclic order: $1 \to 2 \to \cdots \to J \to 1 \to \cdots$. An arbitrarily distributed switchover time S^o_i with mean $\overline{s^o_i}$ and second moment $\overline{s^{o2}_i}$ is incurred when the server switches from station i to station $i+1$ (this period is denoted by i^s). The system is separated into two parts which are called the 'service facility' (s.f.) and the 'waiting rooms' (w.r.s) of the stations. All customers in the stations that are not selected by the server must wait for service at their waiting rooms. Each arriving customer (who arrives exogenously or by feedback) enters the waiting room unless the gate is opened.

Customers in the system are served according to a *scheduling algorithm*. The customer selection rule for each station is either *gated* or *exhaustive*. The service order of customers in the service facility is either an *FCFS order*, or a *fixed priority order*. For the groups with the fixed priority order, class α customers have priority over class β customers if $\alpha < \beta$.

We define the stochastic process $\mathcal{Q} = \{\boldsymbol{Y}(t) = (X(t), \Gamma(t), \kappa(t), a(t), r(t), g(t), n(t), L(t)) : t \geq 0\}$ with state space $\mathcal{E}$ representing the system evolution.

The performance measures.

Let σ^e_l be the time just when the e^{th} customer (c^e) arrives at one of the stations after completing its l^{th} service $(e = 1,2,\ldots;\ l = 0,1,2,\ldots)$. Let

$$C^e_{Wi\alpha}(t) \equiv \begin{cases} 1, & \text{if } c^e \text{ stays in the waiting room} \\ & \text{as an } (i,\alpha)\text{-customer at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

We define the expected times spent in the w.r.s.:

$$W_{i\alpha}(\boldsymbol{Y},e,l) \equiv E[\int_{\sigma^e_l}^{\infty} C^e_{Wi\alpha}(t)dt|\boldsymbol{Y}(\sigma^e_l) = \boldsymbol{Y}], \quad (2.1)$$

$$W^I_{i\alpha}(\boldsymbol{Y},e,l) \equiv E[\int_{\sigma^e_l}^{\sigma^e_{l+1}} C^e_{Wi\alpha}(t)dt|\boldsymbol{Y}(\sigma^e_l) = \boldsymbol{Y}], \quad (2.2)$$

for $\boldsymbol{Y} \in \mathcal{E}$. Then we have the feedback equation:

$$\begin{aligned} W_{i\alpha}(\boldsymbol{Y},e,l) &= W^I_{i\alpha}(\boldsymbol{Y},e,l) \\ &+ E[W_{i\alpha}(\boldsymbol{Y}(\sigma^e_{l+1}),e,l+1)|\boldsymbol{Y}(\sigma^e_l) = \boldsymbol{Y}]. \quad (2.3) \end{aligned}$$

In the same manner, we define $H_{i\alpha}(\boldsymbol{Y},e,l,k)$ and $H^I_{i\alpha}(\boldsymbol{Y},e,l,k)$, the expected waiting times in the w.r.s. during the system is in period k, by replacing $C^e_{Wi\alpha}(t)$ by $C^e_{Wi\alpha}(t)1\{\kappa(t) = k\}$[1]. We also define $F_{i\alpha}(\boldsymbol{Y},e,l)$ and $F^I_{i\alpha}(\boldsymbol{Y},e,l)$, the expected waiting times in the s.f., by replacing $C^e_{Wi\alpha}(t)$ by $C^e_{Fi\alpha}(t)$

[1] $\kappa(t)$ denotes the (service or switchover) period at time t.

($C^e_{Fi\alpha}(t) \equiv 1$ if c^e stays in the s.f. as an (i,α)-customer at time t, or $\equiv 0$ otherwise). Then the feedback equations similar to eq. (2.3) also hold.

3 Polling instants

The vector of the numbers of customers just when the server will select station i (polling instant) for the first time after any specified arrival epoch τ is denoted by $\boldsymbol{\nu}^i(\tau) \equiv (\nu^i_{k\gamma}(\tau) : (k,\gamma) \in \mathcal{S})$ $(i = 1,\ldots,J)$. Let

$$\bar{\boldsymbol{\nu}}^i(\boldsymbol{Y}) \equiv E[\boldsymbol{\nu}^i(\tau)|\boldsymbol{Y}(\tau) = \boldsymbol{Y}]. \quad (\boldsymbol{Y} \in \mathcal{E}) \quad (3.1)$$

Proposition 1: For any $\boldsymbol{Y} = (j,\beta,\kappa,a,r,\boldsymbol{g},\boldsymbol{n},L) \in \mathcal{E}$ and $i = 1,\ldots,J$, the following expressions hold.

$$\begin{aligned}\bar{\boldsymbol{\nu}}^i(\boldsymbol{Y}) = (r, \mathbf{1}(r))\boldsymbol{b}^i(\kappa,a) \\ +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{B}^i(\kappa) + \boldsymbol{b}^i_0(\kappa,j,\beta),\end{aligned} \quad (3.2)$$

where $\mathbf{1}(r) = 1$ if $r > 0$, or $= 0$ otherwise.

4 Quantities for each visit

By using the results in the last section, we have

Proposition 2: Let $\boldsymbol{Y} = (j,\beta,\kappa,a,r,\boldsymbol{g},\boldsymbol{n},L) \in \mathcal{E}$, $e = 1,2,\ldots$, and $l = 0,1,2,\ldots$. Then we have

$$\begin{aligned}W^I_{j\beta}(\boldsymbol{Y},e,l) = (r,\mathbf{1}(r))\boldsymbol{\varphi}^{j\beta}(\kappa,a,0) \\ +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}^{j\beta}(\kappa,0) + w^{j\beta}(\kappa,0),\end{aligned} \quad (4.1)$$

$$\begin{aligned}H^I_{j\beta}(\boldsymbol{Y},e,l,k) = (r,\mathbf{1}(r))\boldsymbol{\varphi}^{j\beta}(\kappa,a,k) \\ +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}^{j\beta}(\kappa,k) + w^{j\beta}(\kappa,k),\end{aligned} \quad (4.2)$$

$$\begin{aligned}F^I_{j\beta}(\boldsymbol{Y},e,l) = (r,\mathbf{1}(r))\boldsymbol{\eta}^{j\beta}(\kappa,a) \\ +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{f}^{j\beta}(\kappa) + f^{j\beta}(\kappa).\end{aligned} \quad (4.3)$$

$$\begin{aligned}E[(\boldsymbol{g}(\sigma^e_{l+1}),\boldsymbol{n}(\sigma^e_{l+1}))|\boldsymbol{Y}(\sigma^e_l) = \boldsymbol{Y}] = \\ (r,\mathbf{1}(r))\boldsymbol{v}^{j\beta}(\kappa,a) + (\boldsymbol{g},\boldsymbol{n})\boldsymbol{U}^{j\beta}(\kappa) + \boldsymbol{u}^{j\beta}(\kappa).\end{aligned} \quad (4.4)$$

These quantities are required for solving the feedback equations obtained in section 2.

5 The performance measures

By solving eq. (2.3) and the similar feedback equations related to $H_{i\alpha}$ and $F_{i\alpha}$, we have

Proposition 3: The performance measures defined in section 2 have the following expressions:

$$\begin{aligned}W_{i\alpha}(\boldsymbol{Y},e,l) = (r,\mathbf{1}(r))\varphi_{i\alpha}(j,\beta,\kappa,a,0) \\ +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}_{i\alpha}(j,\beta,\kappa,0) + w_{i\alpha}(j,\beta,\kappa,0), \quad (5.1)\\ H_{i\alpha}(\boldsymbol{Y},e,l,k) = (r,\mathbf{1}(r))\varphi_{i\alpha}(j,\beta,\kappa,a,k) \\ +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}_{i\alpha}(j,\beta,\kappa,k) + w_{i\alpha}(j,\beta,\kappa,k), \quad (5.2)\\ F_{i\alpha}(\boldsymbol{Y},e,l) = (r,\mathbf{1}(r))\boldsymbol{\eta}_{i\alpha}(j,\beta,\kappa,a) \\ +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{f}_{i\alpha}(j,\beta,\kappa) + f_{i\alpha}(j,\beta,\kappa), \quad (5.3)\end{aligned}$$

for $\boldsymbol{Y} = (j,\beta,\kappa,a,r,\boldsymbol{g},\boldsymbol{n},L) \in \mathcal{E}$, $e = 1,2,\ldots$, $l = 0,1,2,\ldots$ and $(i,\alpha) \in \mathcal{S}$.

6 Steady state values

We evaluate the average sojourn times:

$\bar{w}_{i\alpha}(j,\beta)$ = the average times customers arriving from outside the system as (j,β)-customers spend during they are (i,α)-customers. (6.1)

For $\kappa \in$ {service periods, switchover periods}, let

$$\begin{aligned}\tilde{\boldsymbol{Y}}^\kappa &\equiv (\tilde{X}^\kappa, \tilde{\Gamma}^\kappa, \kappa\tilde{q}^\kappa, \tilde{a}^\kappa, \tilde{r}^\kappa, \tilde{\boldsymbol{g}}^\kappa, \tilde{\boldsymbol{n}}^\kappa, \tilde{L}^\kappa) \\ &\equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t E[\boldsymbol{Y}(s)\mathbf{1}\{\kappa(s) = \kappa\}]ds. \quad (6.2)\end{aligned}$$

From proposition 3, Little's theorem and the PASTA, we have the following equations for the average numbers of customers $\tilde{\boldsymbol{g}}^k$ (in s.f.) and $\tilde{\boldsymbol{n}}^k$ (in w.r.s).

$$\tilde{n}^k_{i\alpha} = \sum_{(j,\beta)\in\mathcal{S}} \lambda_{j\beta}\tilde{\varphi}_{i\alpha}(j,\beta,k) + \sum_\kappa (\tilde{\boldsymbol{g}}^\kappa, \tilde{\boldsymbol{n}}^\kappa)\tilde{\boldsymbol{w}}_{i\alpha}(\kappa,k),$$

$$\tilde{g}^i_{i\alpha} = \sum_{(j,\beta)\in\mathcal{S}} \lambda_{j\beta}\tilde{\eta}_{i\alpha}(j,\beta) - \tilde{q}^{i\alpha} + \sum_\kappa (\tilde{\boldsymbol{g}}^\kappa, \tilde{\boldsymbol{n}}^\kappa)\tilde{\boldsymbol{f}}_{i\alpha}(\kappa),$$

$$\begin{cases}\tilde{\boldsymbol{w}}_{i\alpha}(\kappa,k) = \sum_{(j,\beta)\in\mathcal{S}} \lambda_{j\beta}\boldsymbol{w}_{i\alpha}(j,\beta,\kappa,k), \\ \tilde{\boldsymbol{f}}_{i\alpha}(\kappa) = \sum_{(j,\beta)\in\mathcal{S}} \lambda_{j\beta}\boldsymbol{f}_{i\alpha}(j,\beta,\kappa).\end{cases} \quad (6.3)$$

Proposition 4: For $(i,\alpha),(j,\beta) \in \mathcal{S}$, the average sojourn times are given by

$$\begin{aligned}\bar{w}_{i\alpha}(j,\beta) &= \sum_\kappa (\tilde{\boldsymbol{g}}^\kappa, \tilde{\boldsymbol{n}}^\kappa)(\boldsymbol{w}_{i\alpha}(j,\beta,\kappa,0) + \boldsymbol{f}_{i\alpha}(j,\beta,\kappa)) \\ &\quad +\tilde{\varphi}_{i\alpha}(j,\beta,0) + \tilde{\eta}_{i\alpha}(j,\beta). \quad (6.4)\end{aligned}$$

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