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Three Methods for Measuring Malmquist Index

01302170 National Graduate Institute for Policy Studies TONE Kaoru

1. Introduction

The concept of Malmquist productivity index was first introduced by Malmquist (1953), and has further been studied and developed in the non-parametric framework by several authors. It is an index representing Total Factor Productivity (TFP) growth of a DMU, in that it reflects progress or regress in efficiency along with progress or regress of the frontier technology over time under the multiple inputs and multiple outputs framework. We present three different approaches for measuring the Malmquist index in the non-parametric DEA framework, along with numerical examples.

2. Dealing with Panel Data

The Malmquist index evaluates the productivity change of a DMU between two time periods. It is defined as the product of "Catch-up" and "Frontier-shift" terms. The catch-up (or recovery) term relates to the degree of efforts that a DMU attains for improving its efficiency, while the frontier-shift (or innovation) term reflects the change in the efficient frontiers surrounding the DMU between the two time periods. The catch-up effect is measured by the following formula.

$$\frac{\text{Efficiency of } (x_o, y_o)^2 \text{ w.r.t. the period 2 frontier}}{\text{Efficiency of } (x_o, y_o)^1 \text{ w.r.t. the period 1 frontier}}$$

The frontier-shift effect is defined by

$$\text{Frontier-shift} = \phi = \sqrt{\phi_1 \phi_2},$$

where

$$\phi_1 = \frac{\text{Efficiency of } (x_o, y_o)^1 \text{ w.r.t. period 1 frontier}}{\text{Efficiency of } (x_o, y_o)^1 \text{ w.r.t. period 2 frontier}}$$

and

$$\phi_2 = \frac{\text{Efficiency of } (x_o, y_o)^2 \text{ w.r.t. period 1 frontier}}{\text{Efficiency of } (x_o, y_o)^2 \text{ w.r.t. period 2 frontier}}.$$

The Malmquist index (MI) is computed as the product of Catch-up and Frontier-shift effects.

$$\text{MI} = (\text{Catch-up}) \times (\text{Frontier-shift}).$$

3. The Radial MI

The input-oriented radial MI measures the *within* and *intertemporal* scores by the linear programs given below.

[*Within* score in input-orientation]

$$\begin{aligned} \delta^s((\boldsymbol{x}_o, \boldsymbol{y}_o)^s) &= \min \theta \\ s.t. \ \theta \boldsymbol{x}_o^s &\geq X^s \boldsymbol{\lambda} \\ \boldsymbol{y}_o^s &\leq Y^s \boldsymbol{\lambda} \\ L &\leq \boldsymbol{e\lambda} \leq U \\ \boldsymbol{\lambda} &\geq \mathbf{0}. \end{aligned}$$

[*Intertemporal* score in input-orientation]

$$\begin{aligned} \delta^s((\boldsymbol{x}_o, \boldsymbol{y}_o)^t) &= \min \theta \\ s.t. \ \theta \boldsymbol{x}_o^t &\geq X^s \boldsymbol{\lambda} \\ \boldsymbol{y}_o^t &\leq Y^s \boldsymbol{\lambda} \\ L &\leq \boldsymbol{e\lambda} \leq U \\ \boldsymbol{\lambda} &\geq \mathbf{0}. \end{aligned}$$

We solve this program for the pairs (s,t)=(1, 2) and (2, 1). If $(\boldsymbol{x}_o, \boldsymbol{y}_o)^t$ is not enveloped by the technology at the period s, the intertemporal score, if it exists, results in the value greater than 1.

4. The Non-Radial and Oriented MI

The radial approaches suffer from one general problem, i.e., the neglect of slacks. In an effort to overcome this problem, we introduce the SBM (slacks-based measure) and Super-SBM.

[SBM-I]

$$\begin{aligned} \delta^t((\boldsymbol{x}_o, \boldsymbol{y}_o)^s) &= \min 1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{io}^s \\ s.t. \ \boldsymbol{x}_o^s &= X^t \boldsymbol{\lambda} + \boldsymbol{s}^- \\ \boldsymbol{y}_o^s &\leq Y^t \boldsymbol{\lambda} \\ L &\leq \boldsymbol{e\lambda} \leq U \\ \boldsymbol{\lambda} &\geq \mathbf{0}, \ \boldsymbol{s}^- \geq \mathbf{0}. \end{aligned}$$

[Super-SBM-I]

$$\begin{aligned} \delta^t((\boldsymbol{x}_o, \boldsymbol{y}_o)^s) &= \min 1 + \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{io}^s \\ s.t. \ \boldsymbol{x}_o^s &\geq X^t \boldsymbol{\lambda} - \boldsymbol{s}^- \\ \boldsymbol{y}_o^s &\leq Y^t \boldsymbol{\lambda} \\ L &\leq \boldsymbol{e\lambda} \leq U \\ \boldsymbol{\lambda} &\geq \mathbf{0}, \ \boldsymbol{s}^- \geq \mathbf{0}. \end{aligned}$$

5. The Non-Radial and Non-Oriented MI

The models in this category deal with input and output slacks simultaneously.

The [SBM] and [Super-SBM] models used for computing $\delta^t((\boldsymbol{x}_o,\boldsymbol{y}_o)^s)$ are represented by the following fractional programs:

[SBM]

$$\begin{aligned}\delta^t((\boldsymbol{x}_o,\boldsymbol{y}_o)^s) &= \min \frac{1-\frac{1}{m}\sum_{i=1}^m s_i^-/x_{io}^s}{1+\frac{1}{s}\sum_{i=1}^s s_i^+/y_{io}^s} \\ s.t.\ \boldsymbol{x}_o^s &= X^t\boldsymbol{\lambda}+\boldsymbol{s}^- \\ \boldsymbol{y}_o^s &= Y^t\boldsymbol{\lambda}-\boldsymbol{s}^+ \\ L &\le \boldsymbol{e\lambda}\le U \\ \boldsymbol{\lambda} &\ge \boldsymbol{0},\ \boldsymbol{s}^-\ge\boldsymbol{0},\ \boldsymbol{s}^+\ge\boldsymbol{0}.\end{aligned}$$

[Super-SBM]

$$\begin{aligned}\delta^t((\boldsymbol{x}_o,\boldsymbol{y}_o)^s) &= \min \frac{1+\frac{1}{m}\sum_{i=1}^m s_i^-/x_{io}^s}{1-\frac{1}{s}\sum_{i=1}^s s_i^+/y_{io}^s} \\ s.t.\ \boldsymbol{x}_o^s &\ge X^t\boldsymbol{\lambda}-\boldsymbol{s}^- \\ \boldsymbol{y}_o^s &\le Y^t\boldsymbol{\lambda}+\boldsymbol{s}^+ \\ L &\le \boldsymbol{e\lambda}\le U \\ \boldsymbol{\lambda} &\ge \boldsymbol{0},\ \boldsymbol{s}^-\ge\boldsymbol{0},\ \boldsymbol{s}^+\ge\boldsymbol{0}.\end{aligned}$$

6. Comparisons of the Methods

We compare the three approaches using numerical examples. (We will show them at the presentation.)

6.1 Radial vs. Non-radial

The radial MI is based upon the radial DEA scores and hence remaining non-zero slacks are not counted in the scores. If slacks are not freely disposal, the radial MI cannot fully characterize the productivity change. In contrast, the input (output) oriented non-radial MI takes into account the input (output) slacks. This results in a smaller objective function value than in the radial model both in the *Within* and *Intertemporal* scores. The Non-radial and Non-oriented MI takes into account the input and output slacks at the same time.

6.2 Inclusive vs. Exclusive

In evaluating the *within* score, there are two schemes: one 'inclusive' and the other 'exclusive.' 'Inclusive' scheme means that when we evaluate $(\boldsymbol{x}_o,\boldsymbol{y}_o)^s$ with respect to the technology $(X,Y)^t$, the DMU $(\boldsymbol{x}_o,\boldsymbol{y}_o)^s$ is always included in the evaluator $(X,Y)^s$, thus resulting in the score not greater than 1. 'Exclusive' scheme employs the method in which the DMU $(\boldsymbol{x}_o,\boldsymbol{y}_o)^s$ is removed from the evaluator group $(X,Y)^s$. This method of evaluation is equivalent to that of super-efficiency evaluation, and the score, if exists, may be greater than 1. The *intertemporal* comparisons naturally utilize this 'exclusive' scheme. So adoption of this scheme even in the *within* evaluations is not unnatural and promotes the discrimination power.

6.4 Infeasible LP issues

Occasionally, the oriented models suffer from infeasible LPs. Actually, in the VRS model $[(L,U)=(1,1)$: variable returns to scale], it may occur that the intertemporal LP has no solution in its input and output orientations. In case of input-oriented model, it has no feasible solution, if there exists i such that $y_{io}^t > \max_j\{y_{ij}^s\}$ whereas in output-oriented case, it has no feasible solution if there exists i such that $x_{io}^t < \min_j\{x_{ij}^s\}$. In the IRS model $[(L,U)=(1,\infty)$: increasing returns to scale], it may occur that the output-oriented intertemporal LP has no solution, while the input-oriented case is always feasible. In case of DRS $[(L,U)=(0,1)$: decreasing returns to scale], it may occur that the input-oriented intertemporal LP has no solution, while the output-oriented case is always feasible. However, the CRS (constant returns-to-scale) model does not suffer from such trouble.

In contrast to the oriented models, it should be emphasized that, in the non-oriented models, [Super-SBM] is always feasible and has a finite minimum in any RTS environment.

References

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