Full-information rank minimization problem in PPP

01303783 愛知大学 玉置光司 TAMAKI Mitsushi

1 Introduction

The full information rank minimization problem(abbreviated to FIRM problem) can be described as follows: n options, represented by n i.i.d. continuous random variables $X_1, \ldots, X_n$ with known distribution, arrive sequentially, and one of them must be chosen. The objective is to minimize the expected rank (among the ranks of $X_1, \ldots, X_n$) of the option chosen, where the best object, i.e., the one with largest X-value, has rank one, etc. The FIRM problem was studied by Bruss and Ferguson(1993), and then by Assaf and Samuel-Cahn(1996). Both Bruss and Ferguson(1993), and Assaf and Samuel-Cahn(1996) are mainly concerned with studying the classes of the threshold rules, because the optimal stopping rule is very complicated in a sense that it depends on the whole sequence of observations. As an important class of threshold rule, Asaaf and Samuel-Cahn refere to the (a, c)-threshold rule defined in Section 2.2, but they do not derive the limiting expression for the expected rank under that rule(see Remark 6.2). In this note, we attempt to derive the explicit form for it as well as the explicit form for the expected rank attainable under the c/(1-t)-threshold rule defined in section 2.2, via PPP(planar Poisson process) model, which greatly facilitates the derivation of the limiting value for some full information problems. See, e.g., Gnedin(1996), (2003), Samuels(2003) and Mazalov and Tamaki(2003).

2 Explicit expression for the expected rank

As in Samuels(2003)(see his Sec.9), we employ a planar Poisson process with unit rate on the space $\mathcal{T} \times \mathcal{Y} = [0,1] \times [0, \infty)$. The options are identified as atoms on the PPP. This setting turns the problem upside down, thus making Best=Smallest. We scan the process from left to right, and the best, second best, etc., arrivals have values which are sums of i.i.d. exponential (with mean 1) random variables, and arrive at i.i.d. uniform (0,1) times which are independent of the values.

2.1 $c/(1-t)$-threshold rule

We shifts an infinite vertical detector in the positive direction of t and choose the first atom encountered that is located under $c/(1-t)$ thresholds. Let R denote the (absolute) rank of the atom chosen in this way and (T, Y) denote the coordinates of this atom. It is easy to see that the density function of T is given by

$$f_T(t) = c(1-t)^{c-1}, \quad 0 < t < 1, \tag{1}$$

and conditional on $T = t$, Y is uniformly distributed on $(0, c/(1-t))$. Let $R(t, y)$ denote the rank of the atom chosen at state $(T, Y) = (t, y)$. Then we have

$$E[R] = \int_0^1 \left\{ \int_0^{\frac{c}{1-t}} E[R(t,y)] \frac{1-t}{c} dy \right\} f_T(t) dt. \tag{2}$$

The following lemma yields $E[R(t, y)]$.

Lemma 2.1

$$E[R(t,y)] = \begin{cases} 1 + (1-t)y, & \text{if } 0 \le y < c \\ 1 + (1-t)y + (y-c) + c\log\left(\frac{c}{y}\right), & \text{if } c \le y \le \frac{c}{1-t} \end{cases}$$

Hence, applying this and (1) to (2) yields

$$E[R]=\int_0^1\left[\frac{c+2}{2}+c\left\{\frac{(2-t)t}{2(1-t)}+\log(1-t)\right\}\right]c(1-t)^{c-1}dt=1+\frac{c}{2}+\frac{1}{c^2-1},$$

which coincides with Eq.(4.14) of Assaf and Samuel-Cahn.

Remark: In a similar way, the expected rank under $c/(1-t)^2$-threshold rule can be calculated to yield

$$E(R)=\frac{7}{6}+c^{-1}+\frac{2}{3}c-\frac{c^2}{6}+\frac{1}{6}c^3e^c\int_1^\infty\frac{e^{-cu}}{u}du,$$

2.2 (a,c)-threshold rule

The (a,c)-threshold rule, $0\le a\le 1$, is the same as the $c/(1-t)$-threshold rule expect that this rule only chooses a relatively-best atom if it appears before time a. As in section 2.1, we denote by R the rank of the atom chosen under the (a,c)-threshold rule. Then we have the following result.

Theorem 2.2

$$\begin{aligned}E[R] &= \frac{c+1}{2}+\left[\frac{1}{2}+\frac{1+(c-1)a}{(c^2-1)(1-a)}\right]e^{-\frac{ca}{1-a}}\\ &+\frac{1}{2(c^2-1)a^2(1-a)}\left[\left\{c^2(c+1)-(c^3+c^2+c-1)(1-a)-c(c-1)(1-a)^3\right\}e^{-ca}\right.\\ &\left.-\left\{2c+(c^3+2c^2-6c+1)(1-a)-c(c^2+3c-4)(1-a)^2\right\}e^{-\frac{ca}{1-a}}\right]\\ &+\frac{c}{2}\left[e^cI\left(\frac{c}{1-a},c\right)-\frac{c^2(c+1)+(2c^2-c-1)(1-a)}{(c^2-1)(1-a)}I\left(\frac{ca}{1-a},ca\right)\right],\end{aligned}$$

where

$$I(\beta,\alpha)=\int_\alpha^\beta\frac{e^{-u}}{u}du$$

Unfortunately, numerical experiences show that (a,c)-threshold rule gives no significant improvement over $(0,c)$-stopping rule, for example, $E[R]=2.33044$ for $a=0.42, c=1.95$, while $E[R]=2.33182$ for $a=0, c=1.95$.

参考文献

[1] Assf,D. and Samuels-Cahn,E. (1996). The secretary problem : minimizing the expected rank with i.i.d. random variables, *Adv. Appl. Prob.* **28**, 828-852.

[2] Bruss, F.T. and Ferguson, T.S. (1993). Minimizing the expected rank with full information, *J. Appl. Prob.* **30**, 612-626.

[3] Gnedin, A. V. (1996). On the full information best choice problem. *J. Appl. Prob.* **33**, 678-687.

[4] Gnedin, A.V. (2003). Best choice from the planar Poisson process. *Stochastic Processes and Applications* (to appear).

[5] Mazalov, V.V. and Tamaki,M (2003). An explicit formula for the optimal gain in the full-information probleming owning a relatively-best subject (submitted).

[6] Samuels, S.M. (2003). Why do these quite different best-choice problems have the same solutions? (submitted).