A GAME THEORETIC APPROACH TO PROBE COMPLEXITY IN QUORUM SYSTEMS

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1. Introduction

Quorum systems have been studied as a coordination method for distributed systems, such as mutual exclusion, data replication and dissemination of information. A quorum system is a collection of sets over an underlying processors, where arbitrary two sets intersect, that is, they share at least one common processor. The sets are called quorums. Each processor tries to access all the processors in any quorum, and if it can get permission from them, it can enter its critical section. However, if other processor has already entered its critical section, there exists some common processors which do not give permission.

There are several criteria to evaluate quorum systems. Availability, the frequently used measure, is the probability that the system is operating when failures occur. Load balancing is the measure of avoiding network congestion. Probe complexity, investigated in this report, is the cost of probing if the system is available or not. More precisely, we have to search for a live quorum, in which all the processors are nonfaulty. If there is no such quorum, we cannot use the system. Thus we may learn the state of the system by probing the processors one by one until finding a live quorum or an evidence that no live quorum exists.

Our approach is a game theoretic method. First, we formulate as a two-person zero-sum game between the person who breaks down the system and the person who probes the system. The game value would mean the minimax cost of probing.

Since Garcia-Molina and Barbara (1985) presented the concept of "*coterie*", a lot of related work has been done in this area. They introduced the concept of "*domination*" and showed several properties of non-dominated coteries. Recently, Peleg and Wool (2002) introduced the probe complexity, the expected overhead of message complexity due to failures. Furthermore, Hassin and Peleg (2001) extended the probe complexity to the case where each processor fails with some fixed probability.

2. Game-theoretic Model

For any finite set X, we denote by $|X|$ the cardinality of X.

First we state the problem in the general case. There are n points, numbered as $1,\ldots,n$. Let $U=\{1,\ldots,n\}$. Let $Q(U)=\{T_1,\ldots,T_k\}$ be a subset of 2^U satisfying

(1) $$T_i \cap T_j \neq \emptyset \quad \text{for all} \quad i\neq j \quad \text{and} \quad T_i \not\subseteq T_j \quad \text{for all} \quad i \neq j.$$

Let Σ is the set of all permutations on U. Let $\sigma\in\Sigma$. Let $I(\sigma,j)\equiv\{\sigma(1),\ldots,\sigma(j)\}$ for all $j=1,\ldots,n$.

Lemma 2.1. Suppose $S\subseteq U$. Then the only one of the two cases holds:
(i) For some j and some ℓ, $I(\sigma,j)\backslash S\supseteq T_\ell$.

(ii) For any $\ell, T_\ell \not\subseteq U \backslash S$.

Furthermore, (ii) is equivalent to :

(iii) For some j , $I(\sigma, j) \cap S \cap T_\ell \neq \emptyset$ for any ℓ.

Now we define a zero-sum two person game related to $Q(U)$. There are two players (Player I and Player II). Player I chooses $S \in 2^U$ and Player II chooses a permutation σ on U. The payoff, $f(S,\sigma)$, to Player I is defined by :

$$f(S,\sigma) = j \quad \text{if} \quad \left\{ \begin{array}{l} \text{either} \\ \text{the case (i) in Lemma 1.1 first occurs at } j \\ \text{or} \\ \text{the case (iii) first occurs at } j \end{array} \right\}. \tag{2}$$

The case (i) in Lemma 1.1 means that after the examinations of j points Player II has found that the system works. The case (iii) in lemma 1.1 means that after the examinations of j points Player II has known that the system is down.

The payoff to Player II is $-f(S,\sigma)$. So Player I is the maximizer of f and Player II is the minimizer. The startegy spaces of Player I and Player II can be expressed by 2^U and Σ respectively. $(f; 2^U, \Sigma)$ is a two-person zero-sum game. The mixed extension of this game is denoted by $(f; P, Q)$ where $p \in P$ is a 2^n-dimensional vector satisfying $\sum_{S \in 2^U} p(S) = 1$and $p(S) \geq 0$ for all $S \in 2^U$ and $q \in Q$ is an $n!$-dimensional vector satisfying $\sum_{\sigma \in \Sigma} q(\sigma) = 1$ and $q(\sigma) \geq 0$ for all $\sigma \in \Sigma$. For $p \in P$ and $q \in Q$, we define $f(p,q) = \sum_{S \in 2^U, \sigma \in \Sigma} f(S,\sigma) p(S) q(\sigma)$.

3. The Wheel

We solve the two-person game when $Q(U)$ is, what is called, the wheel, i.e., $T_j = \{1, j\}$ for all $j = 2, \ldots, n$ and $U_{-1} \equiv U \backslash \{1\}$, and

$$Q(U) = \{T_2, \ldots, T_n, U_{-1}\}.$$

Theorem 3.1. An optimal strategy of Player I is :

$$\left\{ \begin{array}{l} p(\{1\}) = p(U_{-1}) = \frac{n-2}{2n} \text{ and } p(T_j) = p(U \backslash T_j) = \frac{1}{n(n-1)} \text{ for all } j; \\ p(S) = 0 \text{ for all other } S. \end{array} \right.$$

An optimal strategy for Player II is :

$$\left\{ \begin{array}{lll} q(1 i_2, \ldots, i_n) = \frac{2}{n!}, & \text{for any permutation } i_2, \ldots, i_n & \text{of } 2, \ldots, n; \\ q(i_1, \ldots, i_{n-1}, 1) = \frac{n-2}{n!} & \text{for any permutation } i_1, \ldots, i_{n-1} & \text{of } 2, \ldots, n. \end{array} \right.$$

The value of the game is : $n - 1 + \frac{2}{n}$.

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