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A Simple Relationship with the Number of Cells on a Coverage Process in the Boolean Model of CDMA Wireless Communications

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1. Introduction

The model stems from the analysis of wireless communication systems where several antennas share the same (or different but interfering) channels, and the good reception of the signal emitted by an antenna depends on the signal to noise or signal to interference ratio. This is for instance the case for the CDMA (Code Division Multiple Access) technology which is one of the basic schemes of the 3rd generation wireless communications.

This paper describes a simple relationship with the number of cells of the Boolean model covering a point [1] . We can say that this relationship gives the classical Little's $L = \lambda W$ formula in the queueing theory. Baccelli and Błaszczyszyn [2] derived this relationship by using moments of a marked point process [3] .

Miyazawa [4] proposed the generalized Mecke's formula, which is called GMF. As special cases of GMF, he derived the Swiss Army formula [5], Mecke's formula for a random measure (MF-M), Mecke's formula for a point process (MF-P) and Little's formula ($L = \lambda W$) and so on.

We derive this relationship with the number of cells of the Boolean model covering the point of origin by using some types of Mecke's formula by means of GMF.

2. Description of the model

Let $\Phi = \{(X_i, Z_i)\}$ be a marked point process on the d-dimensional Euclidean space $\mathbb{R}^d$, where X_i denotes the location of points, where the marks $Z_i = (S_i, A_i)$ are such that S_i belong to some metric space $\mathbb{D}$ and $A_i = (a_i, b_i, c_i)$ in $(\mathbb{R})^3$. In addition to this marked point process, the model is based on a function $L : \mathbb{D} \times \mathbb{R}^d \to \mathbb{R}^+$, which is continuous with respect to its second argument, and such that $L(s, x) \to 0$ when $|x| \to \infty$ (where $|x|$ is the Euclidean norm of x in $\mathbb{R}^d$).

We define the cell C_0 attached to the point X_0 as the following subset of $\mathbb{R}^d$

$$C_0 = C_0(\Phi) = \Big\{y : a_0 L(S_0, y - X_0) \geq b_0 I_\Phi(y) + c_0\Big\}, \quad (1)$$

where $I_\Phi(y)$ denotes the value of the shot noise process at point y for the response function L. The union of all cells

$$\Xi = \Xi(\Phi) = \bigcup_i C_i(\Phi) \quad (2)$$

is the associated coverage process. Denote by N_K the random number of cells C_i that hit a given bounded set K

$$N_K = \sum_i \mathbf{1}\Big(K \cap C_i \neq \emptyset\Big). \quad (3)$$

We want to analyze the cell $C(x; \Phi)$ attached to a point at x of the marked Poisson point process Φ under the Palm distribution P_x. Due to Slivnyak's theorem, the law of this set under P_x is the same as that of the random closed set

$$C\Big(x; \Phi + \delta_{(x,Z)}\Big) \quad (4)$$

under P, where Φ is the original Poisson point process, δ_x is the Dirac measure at x, and $Z = (S, A) = (S, (a, b, c))$ is an additional mark distributed like the other marks and independent of Φ.

3. Factorial moment of N_x

Baccelli and Błaszczyszyn [2] proved the following results.

Theorem 1 *The n-th factorial moment of the number N_x of cells of $\Xi(\Phi)$ covering point x is equal to*

$$E[N_x^{(n)}] = \int_{(\mathbb{R}^d)^n} P\Big(x \in \bigcap_{k=1}^{n} C(x_k; \Phi + \sum_{i=1}^{n} \delta_{(x_i, Z_i)})\Big)\mu(dx_1)\ldots\mu(dx_n), \quad (5)$$

where Φ is the Poisson point process, $\{Z_i\}_{i=1}^n$ is an independent sequence of mutually independent vectors distributed as the generic mark, and $\mu(\cdot)$ is its intensity measure, provided the integral on the right hand side is finite.

If Φ is a homogeneous Poisson point process with intensity $\mu(dx) = \lambda dx$ then for each $x \in \mathbb{R}^d$

$$E[N_x^{(n)}] = E[N_0^{(n)}] = \lambda^n \int_{(\mathbb{R}^d)^n} P\Big(0 \in \bigcap_{k=1}^{n} C(x_k; \Phi + \sum_{i=1}^{n} \delta_{(x_i, Z)})\Big)dx_1 \ldots dx_n, \quad (6)$$

provided the integral is finite. In particular, for $n = 1$

$$E[N_0] = \lambda E\Big[\nu_d\Big(C(0; \Phi + \delta_{(0,Z)})\Big)\Big] \quad (7)$$

where $\nu_d(C(\ldots))$ is the d-dimensional volume of the typical cell.

We can say that the above relationship (7) gives the classical Little's $L = \lambda W$ formula in the queueing theory.

4. Miyazawa's GMF

Miyazawa [4] proposed the generalized Mecke's formula (GMF), which deals with a pair of stationary random measures. Let $(\Omega, \mathscr{F}, P)$ be a probability space with a measurable shift-operator group $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω, and $\Lambda_i (i=1,2)$ be random measures on the line. He assumed

(i) $\{\theta_t\}$ is stationary w.r.t. P.

(ii) $\Lambda_i (i=1,2)$ is consistent with $\{\theta_t\}$.

(iii) $E(\Lambda_i((0,1]))$ is positive and finite, which is denoted by λ_i.

Under assumptions (i), (ii) and (iii), he proved the following theorem.

Theorem 2

$$\begin{aligned} &\lambda_1 E_{\Lambda_1}\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)\right) \\ = \; &\lambda_2 E_{\Lambda_2}\left(\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\right). \end{aligned} \tag{8}$$

From the symmetry with respect to the random measures Λ_1 and Λ_2, he derived

$$\begin{aligned} &\lambda_1 E_{\Lambda_1}\left(\int_{-\infty}^{+\infty} f(-u,\theta_u)\Lambda_2(du)\right) \\ = \; &\lambda_2 E_{\Lambda_2}\left(\int_{-\infty}^{+\infty} f(u,\theta_0)\Lambda_1(du)\right). \end{aligned} \tag{9}$$

Denote Lebesgue measure on the line by m and a stationary point process with an intensity λ by N. Thus, letting $\Lambda_1 = m, \Lambda_2 = N$ and $f = h_0$ in (9), he got $H = \lambda G$ in the stationary framework,

$$E(X(0)) = \lambda E_N\left(\int_{-\infty}^{+\infty} h_0(u,\theta_0)du\right). \tag{10}$$

GMF (8) with $\Lambda_1 = m$ and $\Lambda_2 = \Lambda$ equals the Mecke's formula for the stationary random measure Λ, which is denoted by MF-M,

$$E\Big(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda(du)\Big) = \lambda E_\Lambda\Big(\int_{-\infty}^{+\infty} f(u,\theta_0)du\Big). \tag{11}$$

Consider the queue in which the nth customer arrives at time T_n, his sojourn time in system is W_n and he departs at time D_n. Let N_a and N_d be point processes generated by $\{T_n\}$ and $\{D_n\}$, respectively. Define the queue length $L(t)$ by

$$L(t) = \sum_{n=-\infty}^{+\infty} \mathbf{1}_{\{T_n \le t < T_n + W_n\}}. \tag{12}$$

Putting $\Lambda_1 = \Lambda$ and $\Lambda_2 = N_a$ in (9), he got

$$\lambda E_\Lambda(L(0-)) = \lambda_a E_{N_a}(\Lambda((0,W_0])). \tag{13}$$

Formula (13) is equivalent to (5.3) derived from Swiss Army formula in Brémaud [5]. If $\Lambda = m$, then (13) is identical with Little's formula.

5. Other derivations of the equation (7)

Define a coverage process $\{N_x\}$ by

$$N_x = \sum_k \mathbf{1}\Big(\{x\} \cap C_k \neq \emptyset\Big). \tag{14}$$

We can see that $N_x \circ T_y = N_{x+y}$, where T_y is translation [6] ; i.e. $\{N_x\}$ is a stationary process. The left hand side of (10) is

$$N_0 = \sum_k \mathbf{1}\Big(\{0\} \in C(x_k; \Phi + \delta_{(x_k,Z)})\Big). \tag{15}$$

The right hand side of (10) is

$$\begin{aligned} &\lambda E\Big(\int_{\mathbb{R}^d} \mathbf{1}\Big(x \in C(0;\Phi + \delta_{(0,Z)})\Big)\Big)dx \\ = \; &\lambda \int_{\mathbb{R}^d} P\Big(x \in C(0;\Phi + \delta_{(0,Z)})\Big)dx \\ = \; &\lambda E\Big[\nu_d\Big(C(0;\Phi + \delta_{(0,Z)})\Big)\Big]. \end{aligned} \tag{16}$$

Thus we get the equation (7) from $H = \lambda G$ in the stationary framework. The integrand of the second equation in (16) means the probability that the point of origin is covered by the typical cell located at origin.

We can think of a coverage process as a way of sharing space between the points of a spatial point process with given marks. Note that sharing actually means quite different things: in queues, the sharing of time is implemented by shifting customers in excess to later times, while keeping their service times unchanged. In contrast, for a coverage process, sharing of space is obtained by shrinking the marks. In the same vein, the Boolean model can be seen as a special analogue of the infinite server queue.

Since $\nu_d\Big(C(0;\Phi + \delta_{(0,Z)})\Big)$ means the d-dimensional volume of the typical cell located at origin, which is equal to Lebesgue measure on $[\mathbb{R}^d, \mathcal{B}^d]$, putting in (13)

$$\Lambda = \nu_d\Big(C(0;\Phi + \delta_{(0,Z)})\Big), \tag{17}$$

we get the equation (7).

References

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