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AN OPTIMAL HOSTAGE RESCUE PROBLEM

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1 Introduction

Although not all the information is available for accurate statistics, it could be said that different hostage scenarios continue to occur all over the world. The most important decision for the person in charge of a crisis settlement is the timing to enact rescue of the hostages, especially after all possible negotiations have broken down. The purpose of this paper is to propose two types of mathematical models of an optimal hostage rescue problem by using the concept of a sequential stochastic decision processes and examine the properties of optimal rescuing rules.

2 Models

Consider the following sequential stochastic decision process with a finite planning horizon. Here, for convenience, let points in time be numbered backward from the final point in time of the planning horizon, time 0, as 0, 1, $\cdots$, and so on. Let the time interval between two successive points, say times t and $t-1$, be called the period t. Here, assume that time 0 is the deadline at which a rescue attempt is considered as the only course of action for some reason, say, the hostage's health condition, the degree of criminal desperation, and so on.

Suppose $i \geq 1$ persons are taken as hostages at a given point in time t, and we have to make a decision on attempting either rescue or no rescue. Let x denote a decision variable of a certain point in time t where $x = 0$ if no rescue is attempted and $x = 1$ if a rescue is attempted, and X_t denote the set of possible decisions of time t, i.e., $X_t = \{0, 1\}$ for $t \geq 1$ and $X_0 = \{1\}$.

Let p $(0 < p < 1)$ be the probability of a hostage being killed if $x = 1$, and let s $(0 \leq s < 1)$ be the probability of criminal(s) surrendering up to the next point in time if $x = 0$, further, let q and r $(0 < q < 1,\ 0 \leq r < 1$, and $0 < q + r < 1)$ be the probabilities of a hostage being, respectively, killed and set free up to the next point in time if $x = 0$ and criminal(s) not surrendering. In this paper we consider the following two different objective functions:

1. The expected number of hostages killed, which is to be minimized.

2. The probability of no hostage being killed, which is to be maximized.

For convenience, let us call the model with the former objective function the *expectation model*, and the one with the later objective function the *probability model*.

3 Definition

For simplicity, by A and W let us denote the decisions of, respectively, "attempting a rescue" and "waiting up to the next point in time". If the decisions are optimal at a given point in time t, let us employ the symbols $\mathtt{A}_t$ and $\mathtt{W}_t$, and if $\mathtt{A}_t$ and $\mathtt{W}_t$ are indifferent, let us use the symbol $\mathtt{A}_t \sim \mathtt{W}_t$. Further, if the decisions are optimal for all times $t \geq 1$, we use $\mathtt{A}_{t\geq 1}$ and $\mathtt{W}_{t\geq 1}$.

Now, let $f_p(m|i)$ be the probability of m hostages being killed among i hostages if a rescue attempt is made $(x = 1)$, given by

$$f_p(m|i) = \binom{i}{m} p^m (1-p)^{i-m}, i \geq 1, 0 \leq m \leq i.$$

Let $f_{qr}(k, \ell|i)$ be the probability of k hostages being killed and ℓ hostages being set free among i hostages if a rescue attempt is not made $(x = 0)$ and criminal(s) does not surrender up to the next point in time, given by

$$f_{qr}(k, \ell|i) = \frac{i!}{k!\ell!(i-k-\ell)!} q^k r^\ell (1-q-r)^{i-k-\ell}, \quad i \geq 1, 0 \leq k + \ell \leq i.$$

Further, let

$$z = q + (1-q-r)p$$

where $0 < z < 1$ due to the assumptions of p, q and r.

4 Expectation Model

By $A(i)$ let us denote the expected number of hostages being killed when i hostages are taken, provided that a rescue attempt is made $(x = 1)$ at any time t. Then, clearly

$$A(i) = \sum_{m=0}^{i} m f_p(m|i) = ip, \quad i \geq 1.$$

Now, let $v_t(i)$ be the minimum expected number of hostages being killed, starting from time $t \geq 0$ with i hostages, expressed as

$$\begin{aligned} v_0(i) &= A(i), & i \geq 1, \\ v_t(i) &= \min\{A(i), W_t(i)\}, & i \geq 1, \quad t \geq 1, \end{aligned}$$

where $W_t(i)$ is the minimum expected number of hostages being killed over the period from time t to 0

Table 6.1: Summary of Optimal Decision Rules

	Expectation Model	Probability Model		
	$s \geq 0, \quad i \geq 1$	$s = 0$	$i = 1$	$s > 0, \quad i \geq 2$
$p < (1-s)z$	$\mathtt{A}_{t\geq 1}$	$\mathtt{A}_{t\geq 1}$	$\mathtt{A}_{t\geq 1}$	$i < i^* \Rightarrow \mathtt{A}_{t\geq 1}$ $i \geq i^* \Rightarrow \mathtt{W}_{t\geq 1}$
$p = (1-s)z$	$\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$	$\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$	$\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$	$\mathtt{W}_{t\geq 1}$
$p > (1-s)z$	$\mathtt{W}_{t\geq 1}$	$\mathtt{W}_{t\geq 1}$	$\mathtt{W}_{t\geq 1}$	$\mathtt{W}_{t\geq 1}$

if no rescue attempt is made ($x = 0$). Noting $k+\ell \leq i$, we can express $W_t(i)$ as

$$W_t(i) = (1-s) \sum_{k+\ell \leq i} \big(k + v_{t-1}(i-k-\ell)\big) f_{qr}(k,\ell|i), \qquad i \geq 1, \quad t \geq 1.$$

Now, let

$$V_t(i) = W_t(i) - A(i), \quad i \geq 1, \quad t \geq 1.$$

Then, the optimal decision rule be stated as follows:

(a) If $V_t(i) > 0$, then $\mathtt{A}_t$.
(b) If $V_t(i) = 0$, then $\mathtt{A}_t \sim \mathtt{W}_t$.
(c) If $V_t(i) < 0$, then $\mathtt{W}_t$.

5 Probability Model

By $A(i)$ let us denote the probability of no hostage being killed if a rescue attempt is made ($x = 1$) at any time t. Then

$$A(i) = f_p(0|i) = (1-p)^i, \quad i \geq 1.$$

Now, let $v_t(i)$ be the maximum probability of no hostage being killed, starting from time $t \geq 0$ with i hostages, expressed as

$$v_0(i) = A(i), \qquad i \geq 1,$$
$$v_t(i) = \max\{A(i), W_t(i)\}, \quad i \geq 1, \quad t \geq 1,$$

where $W_t(i)$ is the maximun probability of no hostage being killed over the period from time t to 0 if no rescue attempt is made ($x = 0$). Noting $k + \ell \leq i$, we can express $W_t(i)$ as

$$W_t(i) = s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i) v_{t-1}(i-\ell)\Big), \qquad i \geq 1, \quad t \geq 1.$$

Now, define

$$V_t(i) = W_t(i) - A(i), \quad i \geq 1, \quad t \geq 1.$$

Then, the optimal decision rule be stated as follows:

(a) If $V_t(i) < 0$, then $\mathtt{A}_t$.
(b) If $V_t(i) = 0$, then $\mathtt{A}_t \sim \mathtt{W}_t$.
(c) If $V_t(i) > 0$, then $\mathtt{W}_t$.

6 Summary of Conclusions

A. We can eventually summarize the optimal decision rules for both models as in Table 6.1. It should be noted in Table 6.1 that

1 $\mathtt{A}_{t\geq 1}$ means that it is optimal to attempt a rescue at the time the hostage event occurs and is detected, and $\mathtt{W}_{t\geq 1}$ means that it is optimal to wait up to the deadline and attempt a rescue at that time.

2 Let $p < (1-s)z$. Then $\mathtt{A}_{t\geq 1}$ except the probability model with $s > 0$ and $i \geq 2$.

3 Let $p = (1-s)z$. Then $\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$ except the probability model with $s \geq 0$ and $i \geq 2$.

4 Let $p > (1-s)z$. Then $\mathtt{W}_{t\geq 1}$ for both models.

5 If empolying $\mathtt{W}_{t\geq 1}$ when $\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$, it follows that if $p \leq (1-s)z$, then $\mathtt{W}_{t\geq 1}$ for both models.

6 In the probability model with $p < (1-s)z$, $s > 0$ and $i \geq 2$, there exists a t-independent $i^* \geq 2$, and the decision is made as in the following scenario: If the number of hostages $i < i^*$ are taken at the time when the hostage event occurs, immediately attempt a rescue, and if $i \geq i^*$, wait up to the time when the number of hostages decreases by i^* (i.e., $i < i^*$) either due to the fact that they are killed or released with time, and attempt a rescue. Here, the i^* is given by a i satisfying the following two inequalities.

$$(1-p)^{i-1} > s + (1-s)(1-z)^{i-1},$$
$$(1-p)^i \leq s + (1-s)(1-z)^i.$$

B. In general, an optimal decision rule of a sequential decision process depends on time t. However, as seen in Table 6.1, one of the most major conclusions of this paper is that the optimal decision rule is independent of time t. This implies that it is optimal to behave always *as if* only a single period of planning horizon remains, i.e., as if the next point in time is a deadline. Usually, this property is called a *myopic property*.

Reference

[1] F. Shi: Optimal hostage rescue problem where action can only be taken once – case where effect vanishes thereafter–. *Discussion Paper Series, No. 915. March, 2001 (Institute of Policy and Planning Sciences)*