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Applications of the Conti-Traverso Algorithm for Traveling Salesman Problems

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1 Backgrounds and objectives

The Groebner bases of polynomial ideal were introduced by Buchberger in 1965. He also developed an algorithm to compute a Groebner base. It is known that this algorithm is extremely time-consuming. In 1991, Conti and Traverso proposed an algorithm for integer programming problems (IP) using the Groebner base. This algorithm makes good use of toric ideals, a special class of polynomial ideal. Our objectives are;

(i) to measure the efficiency of Groebner bases for combinatorial optimization problems, and

(ii) to analyze the structure of combinatorial optimization problems from the viwe of toric ideals and Groebner bases.

In this paper, we developed an algorithm for traveling salesman problem (TSP), in which we combined the Conti-Traverso algorithm and branch and bound method.

2 The TSP

We consider the TSP on complete undirected graphs with d nodes, denoted by K_d. Let $N = \{1, \ldots, n\}$ be the node set, $E = \{\{i,j\} : 1 \leq i < j \leq n\}$ be the edge set, and $c_{ij} \in \mathbb{R}^n_{\geq 0}$ be the cost of edge $\{i,j\}$. In our formulation, there are 3 type of constraints;

(i) degree constraints,

(ii) subtour eliminations, and

(iii) $0-1$ constraints.

For all node $k \in N$, let $\delta(k)$ be the set of all edges which are connected to k. Then, TSP can be formulated as following IP;

$$\begin{array}{lll} \text{min.} & \displaystyle\sum_{\{i,j\}\in E} c_{ij}x_{ij}, & \\ \text{s.t.} & \displaystyle\sum_{\{k,l\}\in\delta(i)} x_{kl} = 2, & \forall i \in N, \\ & \displaystyle\sum_{i\in W, j\in W} x_{ij} < |W|, & \forall W \subsetneq N, \\ & x_{ij} \in \{0,1\}, & \forall \{i,j\} \in E. \end{array}$$

3 The Conti-Traverso algorithm

We explain the Conti-Traverso algorithm briefly. Basic references are [1], [2]. Let k be an arbitrary *field*, and $k[y] \stackrel{\text{def}}{=} k[y_1, \ldots, y_n]$ be the *polynomial ring* where $y_1, \ldots, y_n$ are variables. Furthermore, let $\succ$ be a term order on $\mathbb{N}^n$ and $I \subset k[y]$ be an ideal. Denoted a Groebner basis of I *w.r.t.* $\succ$ by $G_\succ(I)$.

Consider following IP;

$$\mathrm{IP}_{A,c}(b) \overset{\mathrm{def}}{=} \min\{c \cdot x \;:\; Ax = b,\; x \in \mathbb{N}^n\},$$

where $A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$ and $c \in \mathbb{R}^n$. In this paper, assume $\mathrm{Ker}(A) \cap \mathbb{R}^n_{\geq 0} = \{0\}$. Let $k[t^{\pm 1}]$ be the Lawrent polynomial ring and $a_1, \ldots, a_n$ be column vectors of A. Consider a homomorphism

$$\phi \;:\; k[y] \rightarrow k[t^{\pm 1}],\; y_i \mapsto t^{a_i}.$$

The kernel of ϕ is called the toric ideal of A, denoted by I_A.

The Conti-Traverso algorithm is very simple.

(1) Find a feasible solution u.

(2) Compute a Groebner base of I_A *w.r.t.* the order $(c|\succ)$.

(3) Compute the normal form of y^u and set it to y^v. Then, v is a optimal.

4 Our algrithms

Our algorithm is based on the branch and bound method. Details are in [3]. In our algorithm, the branch phase is done by splitting the feasible resion along a subtour $(l_1, \ldots, l_r)$. The bound phase is done by solving the relaxation problem in which (i) degree constraints, (iii) $0-1$ constrains, and following additional constraints are considered;

fix $l_1, \ldots, l_{r-1}$ to use, l_r not to use,
...
fix l_1 to use, l_2 not to use,
fix l_1 not to use.

Let A be the incidence matrix of K_d, and M be a sufficient big number. Then additional constraits "to use" and "not to use" can be formulated by replacing cost of the edges to M, $-M$ respectively, and denote the renewed cost vector by $\hat{c}$. Hence, relaxation problems can be represented as $\mathrm{IP}_{\tilde{A},\tilde{c}}(b)$, where $\tilde{A} = \begin{pmatrix} A & 0 \\ I & I \end{pmatrix}$, $\tilde{c} = (\hat{c}, 0)$, and I is $n \times n$ identity matrix. It is known that the reduced Groebner base of $I_{\tilde{A}}$ is a universal Groebner bases, *i.e.,* it is a Groebner base for each term order [4]. We have to solve problems in this form many time in branch and bound framework, but we proved that if once the reduced Groebner base of $I_{\tilde{A}}$ computed, this relaxation problems can be solve by only changing the order and applying the step (3) of the Conti-Traverso algorithm.

5 Conclusion

We developed an algorithm for TSP using the Groebner bases. By combining with the Branch and bound method and several fact of universal Groebner bases, the Conti-Traverso algorithm was certainly improved.

参考文献

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