2000年度日本オペレーションズ・リサーチ学会
秋季研究発表会

Optimum Requirement Cycle with a Monge-like Property

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1 Introduction

The *optimum requirement spanning tree problem* (ORST problem) studied by Hu [5] is often discussed as a problem of finding a communication network of tree type with the minimum average cost. However, from the viewpoint of reliability, k-connected graphs ($k \geq 2$) are desirable as the topologies of communication networks. To minimizes the cost of construction, we take up in this paper cycle graphs (connected regular graphs of degree 2, just call them *cycles*). Here, we consider a problem of finding a cycle which minimizes an objective function similar to that of the ORST problem.

Before detailed discussion, let us define some basic notation and review the ORST problem. Let $V = \{0, 1, \ldots, n-1\}$ be a set of n vertices, $\binom{V}{2}$ the set of all pairs of distinct vertices in V, $\mathcal{G}$ the whole set of simple graphs with the vertex set V, and $\mathcal{T}(\in \mathcal{G})$ the whole set of spanning trees on V. A graph $G \in \mathcal{G}$ with an edge set E is denoted by $G = (V, E)$, and the edge $e \in E$ connecting two vertices $v, w \in V$ is denoted by $e = (v, w)$. For a graph $G \in \mathcal{G}$, let $d(v, w; G)$ be the distance (the length of the shortest path(s)) between two vertices v and w on G. Assume that a nonnegative value r_{vw} (called *requirement*, representing the frequency of communication between v and w) is given to each pair $\{v, w\} \in \binom{V}{2}$, where $r_{vw} = r_{wv}$ holds. Hu [5] defined an ORST as a tree $T \in \mathcal{T}$ which minimizes

$$f(T) = \sum_{\{v,w\} \in \binom{V}{2}} d(v, w; T) r_{vw},$$

and showed that an ORST is obtained by the Gomory-Hu algorithm [3] when the degrees of vertices are *not* restricted. On the other hand, Anazawa [1] considered a problem of finding a tree $T \in \mathcal{T}$ which minimizes

$$f_g(T) = \sum_{\{v,w\} \in \binom{V}{2}} g(d(v, w; T)) r_{vw}$$

(where $g(x)$ is an arbitrary real-valued function of real variable x such that it is monotone nondecreasing on $[0, n-1]$) under the constraint that, for each vertex v, the degree of v in T denoted by $\deg(v; T)$ cannot exceed a given integer l_v, that is,

$$\deg(v; T) \leq l_v \text{ holds for all } v \in V, \tag{1}$$

where

$$l_0 \geq l_1 \geq \cdots \geq l_{n-1} \geq 1 \quad \text{and} \quad \sum_{v=0}^{n-1} l_v \geq 2(n-1) \tag{2}$$

are assumed. And he showed that if $\{r_{vw}\}$ satisfies

$$r_{vw} + r_{v'w'} \geq r_{vw'} + r_{v'w} \tag{3}$$

for all 4-tuple $\{v, v', w, w'\}$ ($v < v', w < w'$) such that r_{vw}, $r_{v'w'}$, $r_{vw'}$ and $r_{v'w}$ are all defined, then a particular tree $T^* \in \mathcal{T}$ (which is obtained by a sort of greedy algorithm but is explicitly definable) is a solution to the problem. Condition (3) seems a little tight, but there is a case where the condition reflects a practical situation (see [1]). Also, condition (3) is similar to the *Monge property*, which is named after the French mathematician Gaspard Monge and rediscovered by Hoffman [4] (compactly reviewed by Pferschy et al. [6] and Deineko et al. [2]). Monge property is originally discussed in the classical Hitchcock transportation problem, and known to make some NP-hard problems (ex. travelling salesman problem) efficiently solvable (see [6]).

Here, we define the problem to be considered in this paper. Let $\mathcal{C}(\in \mathcal{G})$ be the set of all cycles with the vertex set V. For a cycle $C \in \mathcal{C}$ and two vertices v and w on C, there exist two paths between v and w, say P_1 and P_2. Suppose that the lenght of P_1 is shorter than or equal to that of P_2. Then the lenght of P_1 equals $d(v, w; C)$ and that of P_2 equals $n - d(v, w; C)$. Let p_{vw} ($0 \leq p_{vw} \leq 1$) be the relative frequency of using P_1. Then the *average distance* between v and w on C is defined by

$$d_{\text{AVG}}(v, w; C) = p_{vw} d(v, w; C) + (1 - p_{vw})(n - d(v, w; C)).$$

Assume that p_{vw} is expressed by $p(d(v, w; C))$ for any $\{v, w\} \in \binom{V}{2}$, where $p(d)$ is a monotone nonincreasing function of d defined on $[0, \frac{n}{2}]$ and satisfies $1 \geq p(d) \geq \frac{1}{2}$ for $d \in [0, \frac{n}{2}]$. The problem we want to solve is to find a cycle $C \in \mathcal{C}$ which minimizes a function

$$f_{\text{AVG}}(C) = \sum_{\{v,w\} \in \binom{V}{2}} d_{\text{AVG}}(v, w; C) r_{vw},$$

and we call a cycle minimizing this function an *optimum requirement cycle* (ORC).

The main result of this paper is the following

Main Theorem *Let*

$$e_i^* = \begin{cases} (0, 1) & \text{for } i = 1 \\ (i-2, i) & \text{for } i = 2, 3, \ldots, n-1 \\ (n-2, n-1) & \text{for } i = n \end{cases},$$

and $C^* = (V, E^*)$ *where* $E^* = \{e_1, e_2, \ldots, e_n\}$. *If* $\{r_{vw}\}$ *satisfies condition (3), then* C^* *is an ORC.*

In this paper, after giving some lemmas without proofs in Section 2, we will present the proof of Main Theorem in Section 3.

2 Lemmas

Lemma 1 *For any cycle $C \in \mathcal{C}$ and any vertices v, w, v' and w' on C, if $d(v,w;C) < d(v',w';C)$, then $d_{\text{AVG}}(v,w;C) \le d_{\text{AVG}}(v',w';C)$ holds.*

For a cycle $C = (V,E) \in \mathcal{C}$ and a path $P = (u_1,\dots,u_k)$ ($k = 2$ or 3) of C, let

$$m = \begin{cases} \lfloor n/2 \rfloor & \text{if } k = 2 \\ \lfloor (n-1)/2 \rfloor & \text{if } k = 3 \end{cases},$$

where $\lfloor x \rfloor$ is the maximum integer not exceeding x, and let

$$P(u_i) = (V(u_i), E(u_i)) \quad (i = 1 \text{ or } k),$$

where $V(u_1) \cup V(u_k) \subset V$, $V(u_1) \cap V(u_k) = \emptyset$,

$$V(u_1) = \{s_1(=u_1), s_2, \dots, s_m\},$$

$$E(u_1) = \{(s_i, s_{i+1}) \in E | i = 1,2,\dots,m-1\},$$

$$V(u_k) = \{t_1(=u_k), t_2, \dots, t_m\},$$

$$E(u_k) = \{(t_i, t_{i+1}) \in E | i = 1,2,\dots,m-1\}$$

are satisfied. For the path $P = (u_1,\dots,u_k)$, we define an isomorphism $\sigma_P : V(u_1) \to V(u_k)$ by $\sigma_P(s_i) = t_i$ ($i = 1,2,\dots,m$). Also, we consider the following transformation of C which may reduce the f_{AVG} value: Let $V_x = \{v \in V(u_1) | v > \sigma_P(v)\}$, and exchange v and $\sigma_P(v)$ for all $v \in V_x$. We call such a transformation *biasing* with respect to σ_P. Further, let C' be a cycle obtained from C by biasing with respect to σ_P.

Lemma 2 *If $\{r_{vw}\}$ satisfies condition (3), then*

$$f_{\text{AVG}}(C') \le f_{\text{AVG}}(C)$$

holds.

Next, we show a property of a subgraph of the cycle $C^* = (V,E^*)$ defined in Main Theorem. Let $V_\nu = \{0,1,\dots,\nu-1\}$ ($1 \le \nu \le n$) and $P_\nu^* = (V_\nu, E_\nu^*)$ where $E_\nu^* = \{e_1^*, e_2^*, \dots, e_{\nu-1}^*\}$. Note that C^* is obtained by adding $e_n^* = (n-2, n-1)$ to P_n^*.

Lemma 3 *Suppose that a cycle $C = (V,E) \in \mathcal{C}$ contains a subgraph P_ν^* ($1 \le \nu \le n$), that is, $E_\nu^* \in E$ holds. For an arbitrarily-selected path $P = (u_1,\dots,u_k)$ ($k =$ 2 or 3) of C, let $C' = (V,E') \in \mathcal{C}$ be a cycle obtained from C by biasing with respect to σ_P. Then C' also contains P_ν^*.*

3 Proof of Main Theorem

Let $C^* = (V,E^*) \in \mathcal{C}$ be the cycle defined in Main Theorem. For a cycle $C = (V,E) \in \mathcal{C}$, let

$$v_C = \begin{cases} \min\{v > 0 | e_v^* \notin E\} & \text{if } E \neq E^* \\ n-1 & \text{if } E = E^*. \end{cases}$$

We will show that any ORC can be transformed into C^* with the f_{AVG} value unchanged.

Let $C = (V,E)$ be an ORC with $v_C < n-1$. Note that C contains a subgraph $P_{v_C}^*$. Also, let v^* be a vertex with $e_{v_C}^* = (v^*, v_C)$, and v^{**} a vertex with $v^{**} > v_C$ and $(v^*, v^{**}) \in E$ (such v^{**} obviously exists). We can consider a path $P' = (v^{**}, v^*, \dots, 0, \dots, v_C)$ of C, and let v_1 be a vertex on P' adjacent to v_C. Then it is obvious that $v^* < v_1$ holds. Let $P = (u_1,\dots,u_k)$ ($k =$ 2 or 3) be a subpath of P' satisfying

$$d(u_1, v^*; C) = d(u_k, v_1; C).$$

Defining σ_P for the path P in the same way with that in Section 2, we find that $\sigma_P(v^*) = v_1$ and $\sigma_P(v^{**}) = v_C$ hold. Also, let $C' \in \mathcal{C}$ be a cycle obtained from C by biasing with respect to P. Then we find from Lemmas 2 and 3 that C' is also an ORC and contains $P_{v_C}^*$. Also, C' has an edge $e_{v_C}^* = (v^*, v_C)$, which implies that $v_{C'} > v_C$ holds.

By continuing this process, we find that C^* is an ORC.

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