

Another Axiomatization of the Shapley Values of Cooperative Fuzzy Games

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1. Introduction

The Shapley value is a well-known solution concept in cooperative game theory. Most of researches on the Shapley value treat games with crisp coalitions. However, in some situations, some agents do not fully participate in a coalition, but to a certain extent. A coalition including some players who participate partially can be treated as a so-called fuzzy coalition.

Butnariu [1] has introduced the Shapley function as a function that gives us the Shapley value of a given fuzzy game for a given fuzzy coalition axiomatically. He has also introduced a class of fuzzy games and investigated the Shapley function on it. However, both the class and the Shapley function defined by him are unnatural.

In our previous study [2], we have defined the Shapley function, which is applicable to any class of fuzzy games. We have introduced more natural class of fuzzy games $G_C(N)$ than Butnariu's and shown a Shapley function on it in an explicit form. The properties of the Shapley function and related concepts on it have been investigated. However, it has not been proven the uniqueness of the Shapley function.

In this study, we prove that the function given by us is the unique Shapley function on $G_C(N)$. Furthermore, we extend another system of axioms for the Shapley value by Young [3] to fuzzy games. Finally, we show that the function is also the unique function satisfying the new system of axioms.

2. Notations and Definitions

In this paper, we consider cooperative fuzzy games with the set of players $N = \{1, \dots, n\}$. A fuzzy coalition is a fuzzy subset of N identified with a function from N to $[0, 1]$. Then for a fuzzy coalition S and a player i , $S(i)$ indicates the i -th player's participation degree to S . For a fuzzy coalition S , the level set is denoted by $[S]_h = \{i \in$

$N \mid S(i) \geq h\}$ for any $h \in [0, 1]$, and the support is denoted by $\text{Supp } S = \{i \in N \mid S(i) > 0\}$. The set of all fuzzy coalitions is denoted by $L(N)$. Particularly, $P(N)$ denotes the set of crisp subsets of N . For the sake of simplicity, let $L(U) = \{S \in L(N) \mid S \subseteq U \text{ for } U \in L(N)\}$ where $S \subseteq U$ if and only if $S(i) \leq U(i)$ for any $i \in N$.

A fuzzy game is a function v from $L(N)$ to $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ such that $v(\emptyset) = 0$. $G(N)$ denotes the set of all fuzzy games. $v(S)$ is often regarded as the least profit when the crisp coalition S is formed. It follows that any crisp game v is superadditive; and hence monotone nondecreasing with regard to set inclusion. This paper follows this interpretation.

In this paper, union and intersection of two fuzzy sets are defined as usual, i.e.,

$$\begin{aligned} (S \cup T)(i) &= \max\{S(i), T(i)\}, \quad \forall i \in N, \\ (S \cap T)(i) &= \min\{S(i), T(i)\}, \quad \forall i \in N. \end{aligned}$$

3. A Shapley Function and A New Class of Fuzzy Games

We have defined C -carrier for U , which is an extension of carrier for $V \in P(N)$, in order to define the Shapley function.

Definition 1 Let $v \in G(N)$ and $U \in L(N)$. $S \in L(U)$ is called a C -carrier for U if S satisfies:

$$v(S \cap T) = v(T), \quad \forall T \in L(U).$$

Moreover, we have made the following definitions to define the Shapley function.

Definition 2 For $S \in L(U)$ and $i, j \in N$, let S_{ij}^U be defined by

$$S_{ij}^U(k) = \begin{cases} \min\{S(i), U(j)\}, & \text{if } k = i, \\ \min\{S(j), U(i)\}, & \text{if } k = j, \\ S(k), & \text{otherwise.} \end{cases}$$

Definition 3 For $S \in L(N)$, let $\mathcal{P}_{ij}[S]$ be a fuzzy coalition defined by

$$\mathcal{P}_{ij}[S](k) = \begin{cases} S(j), & \text{if } k = i, \\ S(i), & \text{if } k = j, \\ S(k), & \text{otherwise.} \end{cases}$$

We have defined the Shapley function as follows.

Definition 4 Let $G'(N)$ be a class of fuzzy games, i.e. $G'(N) \subseteq G(N)$. A function f from $G'(N)$ to $(\mathbb{R}_+^n)^{L(N)}$ is said to be a Shapley function on $G'(N)$ if f satisfies the following four axioms.

Axiom F_1 : If $v \in G'(N)$ and $U \in L(N)$, then

$$\begin{cases} \sum_{i \in N} f_i(v)(U) = v(U), \\ f_i(v)(U) = 0, \quad \forall i \notin \text{Supp } U, \end{cases}$$

where $f_i(v)(U)$ is the i -th element of $f(v)(U)$.

Axiom F_2 : If $v \in G'(N)$, $U \in L(N)$ and T is a C -carrier for U , then

$$f_i(v)(U) = f_i(v)(T), \quad \forall i \in N.$$

Axiom F_3 : Let $v \in G'(N)$ and $U \in L(N)$. If $v(S) = v(S_{ij}^U) = v(\mathcal{P}_{ij}[S_{ij}^U])$ for any $S \in L(U)$, then

$$f_i(v)(U) = f_j(v)(U).$$

Axiom F_4 : Let $U \in L(N)$. For $v_1, v_2 \in G'(N)$, let $v_1 + v_2 \in G'(N)$ be defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for any $S \in L(N)$. Then

$$f_i(v_1 + v_2)(U) = f_i(v_1)(U) + f_i(v_2)(U), \quad \forall i \in N.$$

We have defined a particular class of fuzzy games. In the class, we have obtained a Shapley function in an explicit form.

Definition 5 For $S \in L(N)$, let $Q(S) = \{S(i) \mid S(i) > 0, i \in N\}$ and let $q(S)$ be the cardinality of $Q(S)$. We rewrite the elements of $Q(S)$ in the increasing order as $h_1 < \dots < h_{q(S)}$. Then $v \in G(N)$ is said to be a fuzzy game 'with Choquet integral form' iff the following holds:

$$v(S) = \sum_{l=1}^{q(S)} v([S]_{h_l}) \cdot (h_l - h_{l-1}), \quad \forall S \in L(N),$$

where $h_0 = 0$. We denote by $G_C(N)$ the set of all fuzzy games with Choquet integral forms.

The class defined above can be more natural than Butnariu's, since any $v \in G_C(N)$ is monotone nondecreasing and continuous with regard to players' participation degree [2]. We have obtained a Shapley function on $G_C(N)$.

Theorem 1 [2] Let f be a function from $G_C(N)$ to $(\mathbb{R}_+^n)^{L(N)}$ defined by

$$f_i(v)(U) = \sum_{l=1}^{q(U)} f'_i(v)([U]_{h_l}) \cdot (h_l - h_{l-1}), \quad (1)$$

where $f'_i(v)(V)$ is the Shapley value of the crisp game v for $V \in P(N)$. Then f is a Shapley function on $G_C(N)$.

4. A New Axiomatization of the Shapley Function

First, we prove the uniqueness of the Shapley function.

Theorem 2 The function f defined by (1) is the unique Shapley function on $G_C(N)$, i.e., the unique function satisfying Axioms $F_1 \sim F_4$ on $G_C(N)$.

Next, we shall prepare some notations in order to introduce another system of axioms. For $S \in L(U)$, define $S_{-i} \in L(U)$ by

$$S_{-i}(j) = \begin{cases} 0, & \text{if } j = i, \\ S(j), & \text{otherwise.} \end{cases}$$

For $v \in G$, let us define $\Delta_i v(S) \in G$ by $\Delta_i v(S) = v(S) - v(S_{-i})$.

Now, let us introduce the property which is called 'strong monotonicity.'

Definition 6 Let $G'(N) \subseteq G(N)$. A function f from $G'(N)$ to $(\mathbb{R}_+^n)^{L(N)}$ is said to be strong monotone on $G'(N)$ if f satisfies the following.

Axiom F_5 : Let $v \in G'(N)$, $U \in L(N)$ and $i \in N$. If $(\Delta_i v(S))_{S \in L(U)} \geq (\Delta_i w(S))_{S \in L(U)}$, then $f_i(v)(U) \geq f_i(w)(U)$.

Then we can show that f is the unique function on $G_C(N)$ satisfying another system of axioms as follows.

Theorem 3 The function f defined by (1) is the unique function satisfying F_1, F_3 and F_4 on $G_C(N)$.

References

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