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A model-based AHP with incomplete pairwise comparisons

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1. Introduction

A model-based justification for the eigenvalue method (EM) of AHP is proposed by Sekitani and Yamaki [2]. Let a_i and w_i be the i^{th} row vector of a comparison matrix A of order n and the i^{th} component of a weight vector w for n objects, respectively, they call w_i *the* i^{th} *self-evaluation value* and $(a_iw - w_i)/(n-1)$ *the* i^{th} *non-self-evaluation value.* Let $I = \{1, \ldots, n\}$, they formulate the following two discrepancy minimization problems with n ratios of the self-evaluation value to the non-self-evaluation value:

$$P_1 : \max_{w>0} \min_{i \in I} \frac{a_iw - w_i}{(n-1)w_i} \text{and} P_2 : \min_{w>0} \max_{i \in I} \frac{a_iw - w_i}{(n-1)w_i}.$$

Both a optimal solution of P_1 and P_2 are a principal eigenvector of A, that is the weight vector of EM.

In order to combine the two optimization models P_1 and P_2, this study develops a new discrepancy-minimization problem that evaluates the ratios of w_i to $(a_iw - w_i)/(n-1)$ and their reciprocals. Typical variation of AHP is the incomplete information case, that is, some entry of A are missing. This study shows that above three models (P_1, P_2 and the combined one) can be applied to such case of AHP as a natural extension of the complete case.

2. Generalized non-self-evaluation value

For AHP with the incomplete pairwise comparisons, let a_{ij} be the pairwise comparison value when the pair of the alternatives i and j is evaluated by a decision maker, and let a_{ij} be 0 when the pair of the alternatives i and j is not evaluated. Let $a_{ii} = 1$ for $i \in I$ and $a_{ji} = 1/a_{ij}$ for $a_{ij} > 0$. Then the nonnegative matrix $A = (a_{ij})$ is well defined. We call $A = (a_{ij})$ an incomplete pairwise comparison matrix. We then define K_i as the number of the positive off-diagonal element a_{ij} for $i \in I$.

As in the case of complete information, w_i is the i^{th} self-evaluation value and $(a_iw - w_i)/K_i$ is called the i^{th} non-self-evaluation value. This definition of non-self-evaluation value is a natural extension of the complete information case, because in this case we have $K_i = n-1$ for $i \in I$.

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3. Discrepancy models

We discusses the incomplete information case in the model-based AHP which is based on the self-evaluation and the non-self-evaluation. The complete information case is dealt as a special case of the incomplete information case.

By introducing the generalized definitions of the self-evaluation value w_i and the non-self evaluation value $(a_iw - w_i)/K_i$ into P_1 and P_2, we formulate the following discrepancy minimization problems with the ratios of the self-evaluation value to the non-self-evaluation value:

$$Q_1 : \max_{w>0} \min_{i \in I} \frac{a_iw - w_i}{K_iw_i} \text{and } Q_2 : \min_{w>0} \max_{i \in I} \frac{a_iw - w_i}{K_iw_i}.$$

P_1/P_2 is identical to Q_1/Q_2 with the complete information, that is $K_i = n-1$ for $i \in I$. For the matrix A, we define the i^{th} row vector $\hat{a}_i = (a_i - e_i)/K_i$ for $i \in I$, where e_i is the i^{th} unit row vector. The matrix $\hat{A}$ consists of the i^{th} row vector $i \in I$.

Lemma 1 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Then $\hat{A}$ is also nonnegative and irreducible.*

Lemma 2 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Then the principal eigenvalue of $\hat{A}$ is a single root of its characteristic equation and there exists a positive principal eigenvector of $\hat{A}$.*

The following two theorems state the relationship between a principal eigenvector of $\hat{A}$ and an optimal solution of Q_1 or Q_2.

Theorem 3 *Suppose that A is nonnegative and irreducible. Let v be any positive n-dimensional vector other than a principal eigenvector of A, then $\min_{i\in I} \frac{a_i v}{v_i} < \lambda_{\max} < \max_{i\in I} \frac{a_i v}{v_i}$, where $\lambda_{\max}$ is the principal eigenvalue of A.*

Theorem 4 *Suppose that an incomplete pairwise comparison matrix A is irreducible. An optimal solution of Q_1 is equal to a positive principal eigenvector of $\hat{A}$, and vice versa. An optimal solution of Q_2 is also equal to a positive principal eigenvector of $\hat{A}$, and vice versa.*

In order to combine the two optimization problems Q_1 and Q_2, we propose the following discrepancy minimization problem that evaluates the ratios of the self-evaluation value to non-self-evaluation value and their reciprocal:

$$Q_3 : \min_{w>0} \max_{i\in I} \left\{ \frac{a_i w - w_i}{K_i w_i}, \frac{K_i w_i}{a_i w - w_i} \right\}$$

Lemma 5 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Then Q_3 has an optimal solution.*

Theorem 6 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Let $\hat{\lambda}_{\max}$ be the principal eigenvalue of $\hat{A}$, then the optimal values of Q_3 is $\max\left\{\hat{\lambda}_{\max}, \hat{\lambda}_{\max}^{-1}\right\}$. Furthermore an optimal solution of Q_3 is equal to a positive principal eigenvector of $\hat{A}$, and vice versa.*

Theorem 6 asserts that Q_1, Q_2 and Q_3 have the same optimal solutions.

4. Some properties of discrepancy models

In order to describe the structure of the incomplete pairwise comparisons for n alternatives, we consider the following undirected graph with n nodes: If a pair (i,j) of alternatives i and j is compared by a decision maker, the arc (i,j) between the node i and the node j is defined. We denote the graph corresponding to the incomplete pairwise comparison matrix A by $G(A)$. In the case of the incomplete information, the graph is not complete.

Harker method [1] is available for evaluating the weight vector from an irreducible incomplete pairwise matrix A of order n and the weight vector of Harker method is a principal eigenvector of A with the diagonal entry a_{ii} replaced by $n - K_i$. Therefore we formulate the following optimization problem corresponding to Harker method:

$$Q_4 : \min_{w>0} \max_{i\in I} \frac{a_i w}{w_i} + n - K_i - 1.$$

Lemma 7 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. An optimal solution of Q_4 is equal to a principal eigenvector of A with the diagonal entry a_{ii} replaced by $n - K_i$, and vice versa.*

Theorem 8 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. Assume that $K_1 = \cdots = K_n$. An optimal solution of Q_1, Q_2 and Q_3 is equal to an optimal solution of Q_4, and vice versa.*

All nodes of the graph $G(A)$ have the same degree if and only if $K_1 = \cdots = K_n$. Such a graph is called regular. The above theorem can be also expressed in terms of graphs:

Corollary 9 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that $G(A)$ is connected and regular. An optimal solution Q_4 is equal to an optimal solution of Q_1, Q_2 and Q_3, respectively, and vice versa.*

The following theorem guarantees that both Q_3 and Q_4 provide non-biased weights for the consistent pairwise comparison values.

Theorem 10 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. Assume that the optimal value of Q_4 is n. An optimal solution of Q_4 is equal to an optimal solution of Q_3.*

The above two assertions means from Theorem 6 that an optimal solution of Q_4 is an optimal solution of Q_i for $i = 1, 2, 3$, respectively.

Corollary 11 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. Assume that the optimal value of Q_4 is n. An optimal solution of Q_4 is equal to an optimal solution of Q_i for $i = 1, 2, 3$, respectively.*

Here, we consider the special structure of $G(A)$, a spanning tree.

Corollary 12 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that $G(A)$ is a spanning tree. An optimal solution of Q_4 is equal to an optimal solution of Q_i for $i = 1, 2, 3$, respectively.*

References

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[2] Sekitani, K and Yamaki, N. (1999), "A logical interpretation for the eigenvalue method in AHP", *Journal of the Operations Research Society of Japan*, 42, 219-232.